Quantum Schubert Calculus

Lecture 1 QH of Grassmannian

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Classical Schubert Calculus

Recall

$$Gr(k,n) = \{ \text{subspaces of dim } k \text{ in } \mathbb{C}^n \}$$

We have

$$H^\bullet(Gr(k,n)) = \bigoplus_\lambda \mathbb{Q}\sigma_\lambda$$

sum over λ inside $k\{\underbrace{\boxed{\cdots}}_{n-k}$

$$\tfrac{1}{2}\deg \sigma_\lambda = |\lambda|$$

with multiplication

$$\sigma_\lambda \sigma_\mu = \sum_\nu \underbrace{c_{\lambda\mu}^{\nu}} \sigma_\nu \quad (\text{usual product})$$

inside $(n-k)^k$

LR coefficient

Example (Pieri Rule)

$$\sigma_\square \cdot \sigma_\lambda = \sum_\mu \sigma_\mu$$

$\mu = \lambda + \square$ inside $k\{\underbrace{\boxed{\cdots}}_{n-k}$

More general,

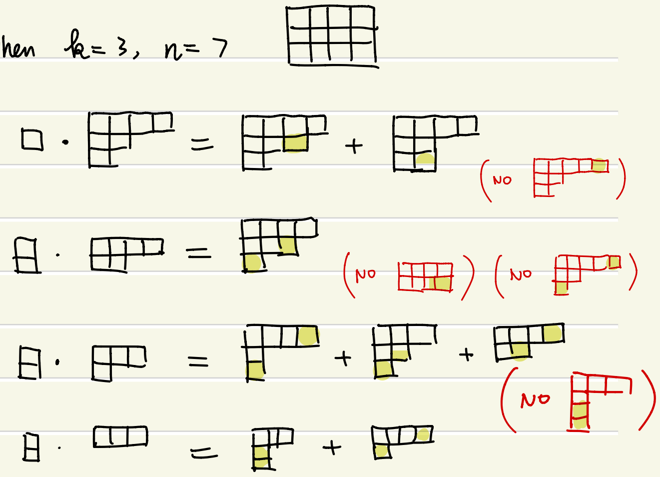
$$e_r \cdot \sigma_\lambda = \sum_\mu \sigma_\mu$$

$$e_r = \sigma_{\boxed{\vdots}\}r}$$

$\mu = \lambda + r$ many row-different $\square$'s

(inside $k\{\underbrace{\boxed{\cdots}}_{n-k}$)

When $k=3$, $n=7$



Quantum Schubert Calculus

As a graded vector space

$$\tfrac{1}{2}\deg q = n$$

$$QH^\bullet(Gr(k,n)) = H^\bullet(Gr(k,n))[q]$$
$$= \bigoplus_\lambda \mathbb{Q}[q]\sigma_\lambda$$

Theorem

quantum product $\quad 1 = \sigma_\emptyset$

$(QH^\bullet(Gr(k,n)), *, 1)$ is a deformation of usual cohomology ring $H^\bullet(Gr(k,n))$.

For $\gamma_1, \gamma_2, \gamma_3 \in H^\bullet(Gr(k,n))$

● $(\gamma_1 * \gamma_2) * \gamma_3 = \gamma_1 * (\gamma_2 * \gamma_3)$

$\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$

$\gamma_1 * 1 = \gamma_1$

● $(\gamma_1 + \gamma_2) * \gamma_3 = \gamma_1 * \gamma_3 + \gamma_2 * \gamma_3$

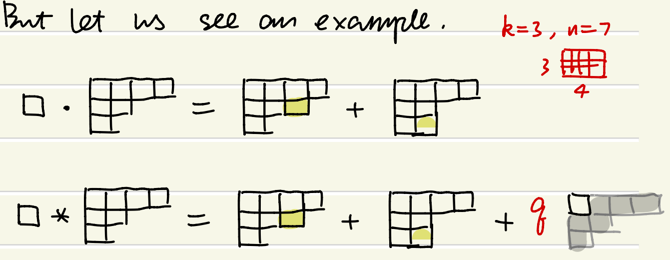
$(q^k \gamma_1) * (q^l \gamma_2) = q^{k+l}(\gamma_1 * \gamma_2)$

● $\gamma_1 * \gamma_2 \equiv \gamma_1 \cdot \gamma_2 \ \cancel{\bmod q}$ $+q(\cdots)$

Example

We will characterize $*$ soon.

But let us see an example.



deleting a ribbon hook of length $(n-1)$

Quantum Pieri formula (Bertram)

$$e_r * \sigma_\lambda = \sum_\mu \sigma_\mu + q \sum_\mu \sigma_\mu$$

described above

$\mu = \lambda + r$ many row different $\square$'s (inside $k\{\underbrace{\boxed{\cdots}}_{n-k+1}$)

$-$ hook of length n (inside $k\{\underbrace{\boxed{\cdots}}_{n-k}$)

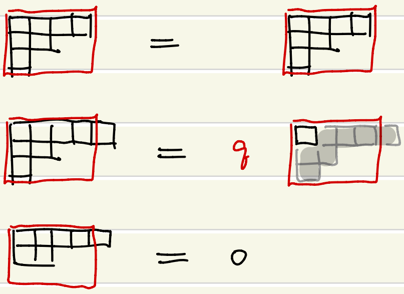
More precisely, for a partition μ inside $k\{\underbrace{\boxed{\cdots}}_{n-k+1}$

we define

$$[\mu] = \begin{cases} \sigma_\mu & \text{if } \mu \text{ inside } k\{\underbrace{\boxed{\cdots}}_{n-k} \\ q\,\sigma_{\bar\mu} & \text{if } \mu = \bar\mu + \text{ribbon hook of length } n \\ 0 & \text{otherwise} \end{cases}$$

$\begin{cases} \mu_1 > n-k \\ \mu_k > 0 \end{cases}$

Example (n=7, k=3)



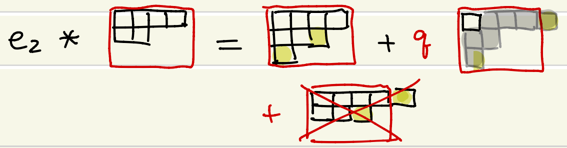
Quantum Pieri formula

$$e_r * \sigma_\lambda = \Sigma_\mu [\mu]$$

$s_\lambda(x_1,\cdots,x_k)$

$\mu = \lambda + r$ many row-different □'s

(inside $k\{\underbrace{\boxed{\cdots}}_{n-k+1}$)



Exercise (Quantum Chevalley)

$$\sigma_\square * \sigma_\lambda = \Sigma_\mu \sigma_\mu + q \Sigma_\mu \sigma_\mu$$

$\mu = \lambda + \square$

(inside $k\{\underbrace{\boxed{\cdots}}_{n-k+1}$)

$\mu = \lambda -$ ribbon of length $(n-1)$

Exercise (Vanishing of quantum terms)

For $d \le n-k$,

$$e_{r_1} * \cdots * e_{r_d} = e_{r_1} \cdot \cdots \cdot e_{r_d}$$

for any $r_1, \cdots, r_d \le k$.

Exercise (Quantum Giambelli (JC))

If $\lambda' = \mu = \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-k} \geqslant 0$, then

$$\sigma_\lambda = \det(e_{\mu_i+j-i})_{1 \le i,j \le n-k}$$

(evaluated by *)

An algebraic approach

$$\Lambda_k = \mathbb{Q}[e_1, \cdots, e_k] = \bigoplus_{\text{length}(\lambda) \le k} \mathbb{Q} s_\lambda(x_1,\cdots,x_k)$$

$e_1(x_1,\cdots,x_k)$

$e_k(x_1,\cdots,x_k)$

Let $I = \text{span}(S_\lambda \mid \lambda \text{ exceeds } (n-k)^k>$

Exercise Show that $e_r I \subseteq I$, thus $\Lambda_k I \subseteq I$ i.e. I is an ideal.

i.e. $\lambda_1 = n-k+1$

Theorem

● $I = < s_\lambda \mid \lambda$ exceeds by one column $>$

● $\Lambda_k / I \cong H^*(\text{Gr}(k,n))$ (as a ring)

$$s_\lambda \longmapsto \begin{cases} \sigma_\lambda & \lambda \le (n-k)^k \\ 0 & \text{otherwise} \end{cases}$$

Let $I' = < s_\lambda \mid \lambda$ exceeds by one column $>$, and $M = \text{span}(s_\lambda : \lambda \subseteq (n-k)^k)$, then

1) $M \cap I' = \{0\}$ $(I' \subseteq I)$

2) $M + I' = \Lambda_k$ $\left(\begin{array}{l} e_r M \subseteq M + I' \\ \Rightarrow \Lambda_k M \subseteq M + I' \\ \Rightarrow \Lambda_k \subseteq M + I' \ (1 \in M) \end{array} \right)$

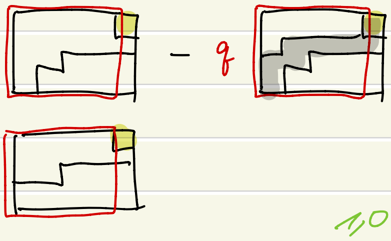
3) Thus $I = I'$ and $\Lambda_k = M \oplus I$

$$\Lambda_k / I = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q}(s_\lambda \bmod I) \cong H^*(Gr(k,n))$$

by comparison of LR-coefficients.

Quantum version

Let $I_q = \left\langle \begin{array}{l} s_\lambda - q s_{\bar\lambda} \ \left(\begin{array}{l}\lambda = \bar\lambda + \text{a ribbon} \\ \text{of length } n\end{array}\right) \\ \text{or } s_\lambda \ (\text{otherwise}) \end{array} \middle| \begin{array}{l} \text{if } \lambda \text{ exceeds} \\ \text{by one column} \end{array} \right\rangle$



$$\sigma_\lambda * \sigma_\mu = \sum_{\nu,k} \overset{1,0}{N_{\lambda,\mu}^{\nu,k}} q^k \sigma_\nu$$

$\sigma_\lambda * \sigma_\mu$ $\begin{array}{l} s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu \\ \sigma_\lambda * \sigma_\mu = \sum c_{\lambda\mu}^{\nu} \pm q^{?} \sigma_{core(\nu)} \end{array}$

- $\Lambda_k[q]/I_q \cong QH^*(Gr(k,n))$ (as a ring)

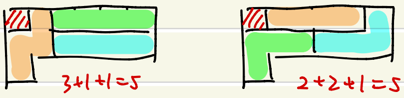
$$q \longmapsto q$$

$$s_\lambda \longmapsto \begin{cases} \overset{q^s(-1)^{ks-\sum h_i}}{(-1)^s q^{ks - \sum h_i}} \sigma_{core(\lambda)} \\ 0 \end{cases}$$

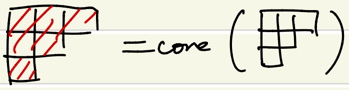
see below

I will give a different proof of this fact next time.

where $core(\lambda) \subseteq (n-k)^k$ is the partion by deleting as many n-ribbons as we can, with s ribbons of height $h_1, \dots, h_s$.



(Independent of choices)

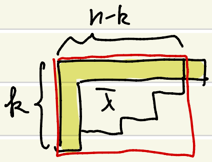


Exercise

The condition $\lambda \subseteq (n-k+1)^k$

$\lambda = \bar\lambda +$ n-ribbon for $\bar\lambda \subseteq (n-k)^k$

$\Longleftrightarrow$



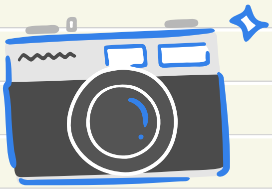
$= \lambda$

References

- Betram, Quantum Schubert Calculus
- Betream, Fontanine, Fulton, Quantum multiplication of Schur polynomials.

Next time

More about $QH^*(Gr(k,n))$



THANKS

Quantum Schubert Calculus

Lecture 2 More on Grassmannian

Rui Xiong

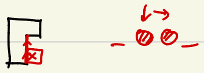
Recall

Quantum Pieri rule

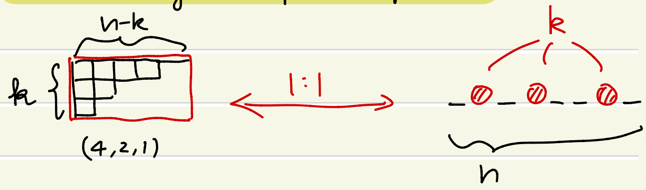
$$e_r * \sigma_\lambda = ??$$

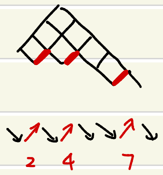
polynomial

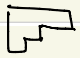
- Compute $e_r \cdot s_\lambda = \sum s_\mu$
- If $\mu \subseteq (n-k)^k$, change s_μ to σ_μ
- If μ exceeds, replace s_μ by $q\, s_{\bar\mu}$ where $\bar\mu = \mu \setminus$ n-ribbon (by 0 if impossible)

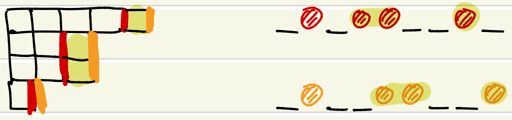


Another way to represent partition





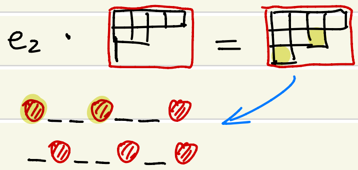
	
adding a box	move ● to its right nhood (if empty)
adding boxes not in the same row	move a subset of ● to right by 1 step.



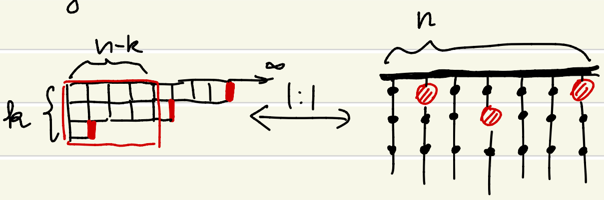
Pieri Rule restated

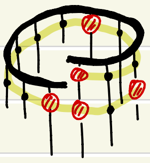
$$e_r \sigma_\lambda = \sum_\mu \sigma_\mu$$

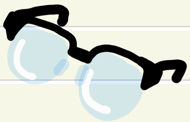
μ = Move an r-subset of ● of λ to right by 1 step

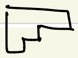


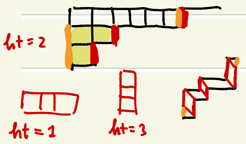
In general,

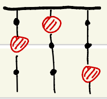


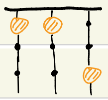


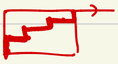


	
adding a box	move ● to its "cyclic" right nhood (if empty)
adding an n-ribbon	move ● down by 1 step (if empty)





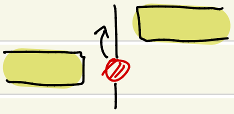




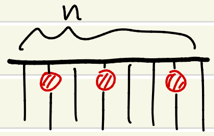
Exercise core(λ) ⊆ $(n-k)^k$

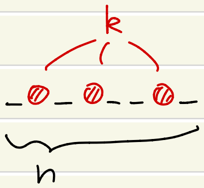
⟺ each T has at most one ●

Exercise height = # ● in 🟨



Note that if $\lambda \subseteq (n-k)^k$, then we identify

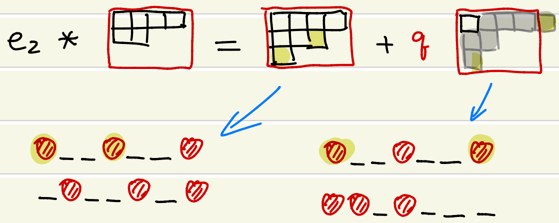


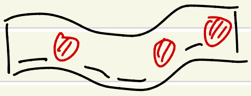


Quantum Pieri Rule restated

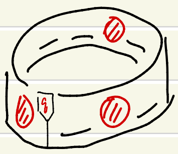
$$e_r * \sigma_\lambda = \sum_\mu \sigma_\mu$$

μ = Move an r-subset of ⊘ of λ to right by 1 step modulo n





usual product



quantum product

Exercise For $\lambda \subseteq (n-k)^k$, denote "$\lambda + \mathbb{1}$"

$$\theta(\sigma_\lambda) = \begin{cases} \sigma_\mu & \text{if } \lambda_1 < n-k, \\ & \mu_i = \lambda_i + 1 \\ q\,\sigma_\mu & \text{if } \lambda_1 = n-k \\ & \mu_i = \lambda_{i+1} \end{cases}$$

Show that

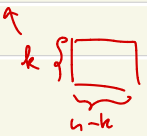
1) $\theta^n(\sigma_\lambda) = q^k \sigma_\lambda$,

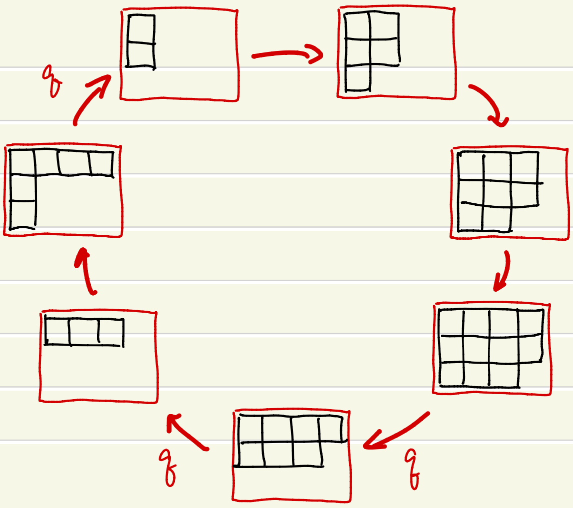
2) $\sigma_\lambda * \sigma_\mu = \sum c^\nu_{\lambda\mu}(q)\, \sigma_\nu$

$\Rightarrow \sigma_\lambda * \theta(\sigma_\mu) = \sum c^\nu_{\lambda\mu}(q)\, \theta(\sigma_\nu)$

Hint: suffices to show $\sigma_\lambda = e_r$

3) $\theta(\sigma_\lambda) = e_k * \sigma_\lambda$





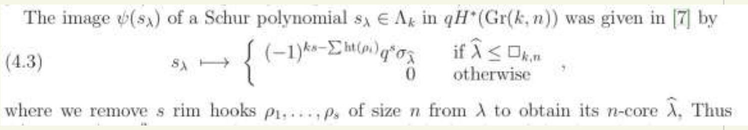
Turn to Schur polynomials

Theorem There is a ring homomorphism

$$\varphi: \Lambda_k[q] \longrightarrow QH^*(Gr(k,n))$$

$$s_\lambda \longmapsto \begin{cases} q^s (-1)^{ks - \sum h_i} \sigma_{core(\lambda)} \\ 0 \end{cases}$$

$core(\lambda) \subseteq \underbrace{\square}_{n-k}\}k$



The image $\psi(s_\lambda)$ of a Schur polynomial $s_\lambda \in \Lambda_k$ in $qH^*(\mathrm{Gr}(k,n))$ was given in [7] by

$$s_\lambda \longmapsto \begin{cases} (-1)^{ks-\sum \mathrm{ht}(\rho_i)} q^s \sigma_{\widehat{\lambda}} & \text{if } \widehat{\lambda} \leq \square_{k,n} \\ 0 & \text{otherwise} \end{cases}, \tag{4.3}$$

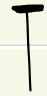
where we remove s rim hooks $\rho_1, \ldots, \rho_s$ of size n from λ to obtain its n-core $\widehat{\lambda}$. Thus

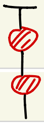
Note that

1) We have (Note that $k\{\overbrace{\square}^{n-k}$)

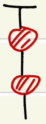
$$\varphi(\underbrace{e_k s_\lambda}_{s_{\lambda+\mathbb{1}}}) = \underbrace{e_k * \varphi(s_\lambda)}_{\theta(\varphi(s_\lambda))}$$

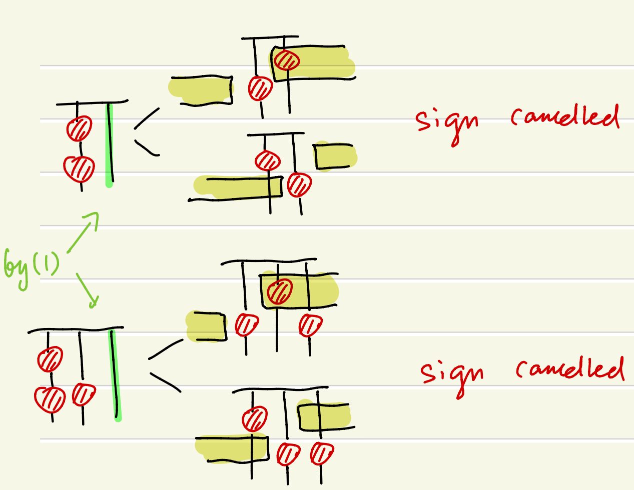
$\Longrightarrow$ Reduce to the case no ⊘ on the last



2) If $\varphi(\mathcal{S}_\lambda) = 0$, pick the last 

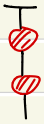
$$\varphi(\underbrace{e_r \mathcal{S}_\lambda}_{=\sum \mathcal{S}_\mu}) = e_r * \varphi(\mathcal{S}_\lambda) = 0$$

It suffices to consider when μ has no , they are cancelled by sign.



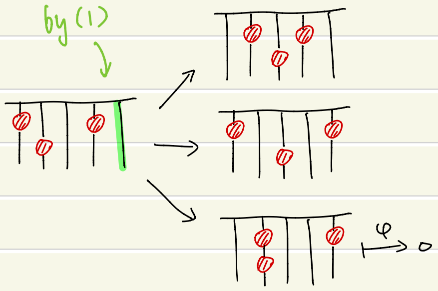
3) If $\varphi(\mathcal{S}_\lambda) \neq 0$, i.e. only 

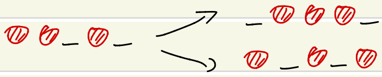
$$\varphi(\underbrace{e_r \mathcal{S}_\lambda}_{=\sum \mathcal{S}_\mu}) = e_r * \varphi(\mathcal{S}_\lambda)$$

It suffices to consider when μ has no 

① They are in bijection with RHS

② Their sign coincides with λ





Since $\varphi(1) = 1$ and $\Lambda_k = \mathbb{Q}[e_1, \cdots, e_k]$, we can conclude φ is a ring homomaplism

Exercise Show $\ker \varphi = I_q$.

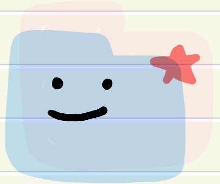
Question Think about QK?

References

- Changzheng Li etc.
  On Seidal representation in quantum K-theory of Grassmennians
- Betrem, Fontanine, Fulton,
  Quantum multiplication of Schur polynomials.

Next time

Equivariant QH

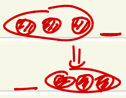


MATH

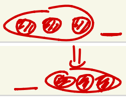
THANKS

$$\overset{(x_1 + \cdots + x_k)}{\overset{\parallel}{e_1}} \cdot \mathcal{G}_\lambda = \sum_\mu \mathcal{G}_\mu$$

Note that e_1 is not a Grothendieck polynomial

$\mu =$ move 

$$e_r * \mathcal{G}_\lambda = \sum_\mu \mathcal{G}_\mu$$

$\mu =$ move  modulo n.

Quantum Schubert Calculus

Lecture 3 Equivariant QH

Rui Xiong

Classical EH

Let $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \subseteq GL_n$.

We have

$$H_T^\bullet(Gr(k,n)) = \bigoplus_{\lambda} \mathbb{Q}[\overset{\frac{1}{2}\deg t_i = 1}{t_1,\cdots,t_n}]\,\sigma_\lambda$$

λ inside $(n-k)^k$

The product is given by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu} \underbrace{c_{\lambda\mu}^{\nu}}(t)\,\sigma_\nu$$

ν inside $(n-k)^k$

LR coefficients for double schur polynomials

$t = 0$

$\Rightarrow$ non-equivant (in L1)

More precisely, we have a ring homomorphism

$$\Lambda_k[t_i]_{i=1}^{\infty} \longrightarrow H_T^\bullet(Gr(k,n))$$

double schur polynomial

$$s_\lambda(x,t) \longmapsto \begin{cases} \sigma_\lambda, & \lambda \subseteq (n-k)^k, \\ 0, & \lambda \text{ exceeds}. \end{cases}$$

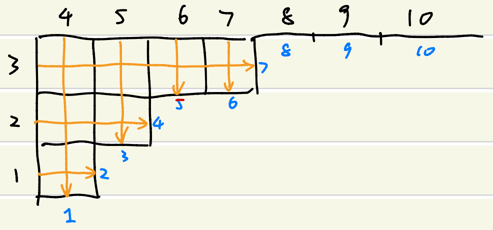
$$t_i \longmapsto \begin{cases} t_i, & 1 \le i \le n \\ 0, & \text{otherwise} \end{cases}$$

Localization (Specialization)

For each λ of parts $\le k$, we define a permutation w_λ such that

$$\begin{array}{llll} 1 \longmapsto \lambda_k + 1 & & k+1 \longmapsto w_\lambda(k+1) \\ \quad\quad\ \ \wedge & & \quad\quad\quad\ \wedge \\ 2 \longmapsto \lambda_{k-1} + 2 & & k+2 \longmapsto w_\lambda(k+2) \\ \vdots \quad\quad \vdots & & \vdots \quad\quad\quad \vdots \\ \quad\quad\ \ \wedge & & \\ k \longmapsto \lambda_1 + k - 1 & & \vdots \quad\quad\quad \vdots \end{array}$$

Example (k=3)



For any $f(x,t) \in \Lambda_k[t_i]_{i=1}^{\infty}$, we define

$$f|_\lambda = f(\underset{\uparrow}{w_\lambda t}, t) \in \mathbb{Q}[t_i]_{i=1}^{\infty}$$

$x_i = t_{w_\lambda(i)}$

		x_i	t_i	σ_λ	q
combinatorial convention →	deg	1	1	$\lvert\lambda\rvert$	n
our convention →	$\frac{1}{2}$deg	1	1	$\lvert\lambda\rvert$	n

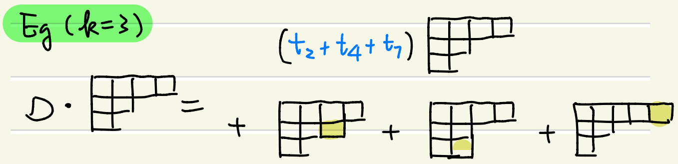
Chevalley formula

Let $D = x_1 + \cdots + x_k$, then $(s_\lambda = s_\lambda(x,t))$

$$D \cdot s_\lambda = D|_\lambda\, s_\lambda + \sum_{\mu = \lambda + \square} s_\mu$$

Note: $s_\square = (x_1 - t_1) + \cdots + (x_k - t_k) \ne D$

Eg (k=3)



$D \cdot$ $=$ $(t_2 + t_4 + t_7)$ $+$ $+$ $+$

$$\begin{tikzcd} & H \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ EH \arrow[dr, Rightarrow] & & QH \arrow[dl, Rightarrow] \\ & EQH & \end{tikzcd}$$

E quantum H

We have

$$QH_T^\bullet(Gr(k,n)) = \bigoplus_{\lambda} \mathbb{Q}[t_1,\cdots,t_n][\overset{\frac{1}{2}\deg = n}{q}]\,\sigma_\lambda$$

Theorem $(QH_T^\bullet(Gr(k,n)), \overset{\text{quantum product}}{*}, \overset{1 = \sigma_\emptyset}{1})$ is a deformation of usual cohomology ring $H_T(Gr(k,n))$.

For $\gamma_1, \gamma_2, \gamma_3 \in H_T(Gr(k,n))$

- $(\gamma_1 * \gamma_2) * \gamma_3 = \gamma_1 * (\gamma_2 * \gamma_3)$

  $\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$

  $\gamma_1 * 1 = \gamma_1$

  $\mathbb{Q}[t_1,\cdots,t_n][q] \rightarrow H_T(Gr(k,n))$ ring homo.

- $(\gamma_1 + \gamma_2) * \gamma_3 = \gamma_1 * \gamma_3 + \gamma_2 * \gamma_3$

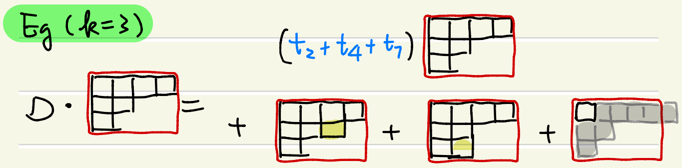
  $(t^m q^k \gamma_1) * (t^l q^h \gamma_2) = t^{m+l} q^{k+h} (\gamma_1 * \gamma_2)$

- $\gamma_1 * \gamma_2 \equiv \gamma_1 \cdot \gamma_2 \mod q$

EQ Chevalley formula (Mihalcea)

$$D * \sigma_\lambda = D|_\lambda \cdot \sigma_\lambda + \sum_{\substack{\mu=\lambda+\square \\ \subseteq (n-k)^k}} \sigma_\mu + q \sum_{\mu=\lambda\setminus(n-1)\text{-ribbon}} \sigma_\mu$$

Eg ($k=3$)



New phenomenon — Chevalley tells everything

Assume

$$\sigma_\lambda * \sigma_\mu = \sum_\nu N^{\nu k}_{\lambda\mu}(t)\, q^k \sigma_\nu$$

Theorem (Mihalcea)

We can completely determine $N^{\nu k}_{\lambda\mu}$ using facts

① $\{\sigma_\lambda\}_{\lambda\in(n-k)^k}$ forms a basis ;

② EQ Chevalley formula.

Observation

$$(D*\sigma_\lambda)*\sigma_\mu = \Big(D|_\lambda\, \sigma_\lambda + \sum_{\lambda'=\lambda+\square} \sigma_{\lambda'} + \sum_{\lambda'=\lambda-\ulcorner} q\, \sigma_{\lambda'}\Big) * \sigma_\mu$$

$$= \sum_\nu \boxed{\Big(D|_\lambda\, N^{\nu k}_{\lambda\mu} + \sum_{\lambda'=\lambda+\square} N^{\nu k}_{\lambda'\mu} + \sum_{\lambda'=\lambda-\ulcorner} N^{\nu k-1}_{\lambda'\mu}\Big)}\, q^k \sigma_\nu$$

$$D*(\sigma_\lambda * \sigma_\mu) = D*\Big(\sum_{\nu'} N^{\nu' k}_{\lambda\mu}\, q^k \sigma_{\nu'}\Big)$$

$$= \sum_\nu \boxed{\Big(D|_\nu\, N^{\nu k}_{\lambda\mu} + \sum_{\nu'=\nu\setminus\square} N^{\nu' k}_{\lambda\mu} + \sum_{\nu'=\nu+\ulcorner} N^{\nu' k-1}_{\lambda\mu}\Big)}\, q^k \sigma_\nu$$

$$\uparrow \qquad\qquad \uparrow$$

$$\nu=\nu'+\square \qquad \nu=\nu'\setminus\ulcorner$$

So ▭ = ▭ i.e.

$$(D|_\nu - D|_\lambda)\, N^{\nu k}_{\lambda\mu}$$

$$= \Big(\sum_{\lambda'=\lambda+\square} N^{\nu k}_{\lambda'\mu} + \sum_{\lambda'=\lambda-\ulcorner} N^{\nu k-1}_{\lambda'\mu}\Big)$$

$$- \Big(\sum_{\nu'=\nu\setminus\square} N^{\nu' k}_{\lambda\mu} + \sum_{\nu'=\nu+\ulcorner} N^{\nu' k-1}_{\lambda\mu}\Big)$$

The main induction is on k.

i.e. we assume $N^{\nu k'}_{\lambda\mu} \in \boxed{\text{known}}$ for $k'<k$.

Exercise Prove that

$$D|_\lambda = D|_\mu \iff \lambda=\mu$$

Thus if $\nu\neq\lambda$, we have

$$N^{\nu k}_{\lambda\mu} = \frac{\sum_{\lambda'=\lambda+\square} N^{\nu k}_{\lambda'\mu} - \sum_{\nu'=\nu\setminus\square} N^{\nu' k}_{\lambda\mu}}{D|_\nu - D|_\lambda} + \boxed{\text{known}}$$

Part A $N^{\nu k}_{\lambda\mu}$ is known for $\lambda \not\subseteq \nu$

● $N^{\phi k}_{▨\,\mu} \in \boxed{\text{known}}$ ▨ $=(n-k)^k$

$(=0,$ if $k=0)$

$[\phi \neq ▨,\ \nexists\ ▨+\square,\ \nexists\ \phi\setminus\square\ ]$

● $N^{\nu k}_{\lambda\mu} \in \boxed{\text{known}}$ if $\lambda\not\subseteq\nu$

$(=0$ if $k=0)$

$$\left[\left.\begin{array}{l}\lambda\not\subseteq\nu'=\nu\setminus\square \\ \text{and } \lambda+\square\not\subseteq\nu\end{array}\right\} \xRightarrow{\text{reduce}} \ ●\ \right]$$

Part B $N^{\nu k}_{\lambda\mu}$ is known for $\lambda\subseteq\nu$

● $N^{\nu k}_{\phi,\mu} \in \boxed{\text{known}}$ $1\cdot\sigma_\mu = \sum N^{\nu k}_{\phi\mu}\, q^k \sigma_\nu$

$(=\delta^\nu_\mu\,\delta_{k=0})$

[Since $1=\sigma_\phi$ is the unit]

The rest — by induction on ν.

(i.e. $N^{\nu' k}_{\lambda\mu} \in \boxed{\text{known}}$ if $\nu'<\nu$)

Observation

$$N^{\nu k}_{\lambda\mu} = \frac{\sum_{\lambda'=\lambda+\square} N^{\nu k}_{\lambda'\mu} - \sum_{\nu'=\nu\setminus\square} N^{\nu' k}_{\lambda\mu}}{D|_\nu - D|_\lambda} + \boxed{\text{known}}$$

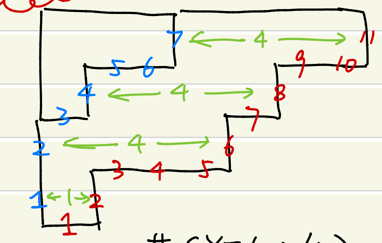
$$= \sum_{\lambda'=\lambda+\square\subseteq\nu} \frac{1}{D|_\nu - D|_\lambda} N^{\nu k}_{\lambda'\mu} + \boxed{\text{known}}$$

$$= \sum_{\substack{\lambda'=\lambda+\square \\ \lambda''=\lambda'+\square\subseteq\nu}} \frac{1}{D|_\nu - D|_\lambda}\,\frac{1}{D|_\nu - D|_{\lambda'}}\, N^{\nu k}_{\lambda''\mu} + \boxed{\text{known}}$$

$$= \cdots = \underbrace{\Big(\sum_{SYT(\nu/\lambda)} \frac{1}{\cdots}\cdot\frac{1}{\cdots}\cdots\frac{1}{\cdots}\Big)}_{\text{nonzero}} N^{\nu k}_{\nu\mu} + \boxed{\text{known}}$$

Here (nonzero) since if we set

$t_i = i$ then $D|_{\nu} - D|_{\lambda} = |\nu| - |\lambda|$



$t_{11} + t_8 + t_6 + t_2$

$11 + 8 + 6 + 2$

$t_7 + t_4 + t_2 + t_1$

$7 + 4 + 2 + 1$

Thus

$$(\text{nonzero}) = \frac{\# SYT(\nu/\lambda)}{(|\nu| - |\lambda|)!} > 0$$

We know $N^{\nu k}_{\lambda\mu}$ for $\lambda = \nu$.

[By observation:

$$\boxed{\text{known}} \Rightarrow N^{\nu k}_{\phi\mu} = (\text{nonzero})\, N^{\nu k}_{\nu\mu} + \boxed{\text{known}}$$

We know $N^{\nu k}_{\lambda\mu}$ for $\lambda \subseteq \nu$

[By observation]

$$N^{\nu k}_{\lambda\mu}$$

$$\begin{tikzcd}
\boxed{k=0,\ \lambda \not\subseteq \nu} \arrow[d, Rightarrow] & \boxed{k=1,\ \lambda \not\subseteq \nu} \arrow[d, Rightarrow] & \cdots \\
\boxed{k=0,\ \lambda \subseteq \nu} \arrow[ur, Rightarrow] & \boxed{k=1,\ \lambda \subseteq \nu} \arrow[ur, Rightarrow] & \cdots
\end{tikzcd}$$

Problem For λ, denote

$$\theta(\sigma_\lambda) = \begin{cases} \sigma_\mu & \text{if } \lambda_1 < n-k, \ \underbrace{\mu_i = \lambda_i + 1}_{} \ \text{“}\lambda+1\text{”} \\ q\,\sigma_\mu & \text{if } \lambda_1 = n-k \\ & \mu_i = \lambda_{i+1} \end{cases}$$

$$\theta(t_i) = t_{i+1 \bmod n}$$

Show that

$$\alpha * \sigma_\mu = \sum (\quad)\, \sigma_\nu$$

$$\Rightarrow \alpha * \theta(\sigma_\mu) = \sum \theta(\quad)\, \theta(\sigma_\nu)$$

Hint: it is true for $\alpha = D$;

Let $\alpha = \sigma_\lambda$, apply Mihalcea's trick.

Problem There exists a ring homomorphism

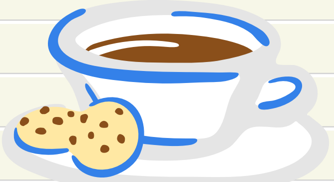
$$\begin{aligned} \Lambda_k[q][t_i]_{i=1}^{\infty} &\longrightarrow QH^{\bullet}_{T}(Gr(k,n)) \\ q &\longmapsto q \\ s_\lambda(x,t) &\longmapsto \begin{cases} (-1)^s q^{ks - \sum h_i}\, \sigma_{core(\lambda)} \\ 0 \end{cases} \\ t_i &\longmapsto t_{i \bmod n} \end{aligned}$$

References

- Mihalcea. On equivariant cohomology of homogenous spaces: Chevalley formulae and algorithms.
- Buch, Quantum cohomology of partial flag variety.

Next time

QH of flag varieties



Thanks

Quantum Cohomology of Fl_n

1. Usual cohomology

$$Fl_n = \{0 \subseteq \phi_1 \subseteq \phi_2 \subseteq \cdots \subseteq \phi_{n-1} \subseteq \mathbb{C}^n\}$$

$\dim \phi_i = i$

deg = 1 ele. sym poly in n variables

$$H^*(Fl_n) = \mathbb{Q}[x_1, \cdots, x_n] / \langle e_1(x), \cdots, e_n(x) \rangle$$

$$= \bigoplus_{w \in S_n} \mathbb{Q}\sigma_w \quad \text{(Schubert classes)}$$

$\deg \sigma_w = \ell(w)$

represented by Schubert polynomials.

Chevalley formula

adding a reflection on the right such that degrees match

$$\varpi_k \cdot \sigma_w = \sum_{\substack{a \le k < b \\ \ell(wt_{ab}) = \ell(w)+1}} \sigma_{wt_{ab}}$$

here $\varpi_k = x_1 + \cdots + x_k = \sigma_{s_k}$.

Reformulation

- $\sum_{\alpha^\vee > 0} \Leftrightarrow \sum_{1 \le a < b \le n}$ and $\alpha^\vee = x_a - x_b$

for linear form $\lambda(x) \in \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$, denote

- $\langle \alpha^\vee, \lambda \rangle$ = natural pairing, e.g.

$$\langle x_1 - x_3, x_i \rangle = \begin{cases} 1, & i=1, \\ 0, & i=2, \\ -1, & i=3. \end{cases}$$

- $r_{\alpha^\vee} = t_{ab} \in S_n$

Chevalley formula

adding a reflection on the right such that degrees match

$$\boxed{\lambda(x) \cdot \sigma_w = \sum_{\substack{\alpha^\vee > 0 \\ \ell(wr_\alpha) = \ell(w)+1}} \langle \lambda, \alpha^\vee \rangle \sigma_{wr_{\alpha^\vee}}}$$

2. Quantum Cohomology

$$QH^*(Fl_n) = \bigoplus_{w \in S_n} \mathbb{Q}[q_1, \cdots, q_{n-1}] \sigma_w$$

deg = 2 deg = $\ell(w)$

with quantum product $*$.

General properties of $*$

1) $(\gamma_1 * \gamma_2) * \gamma_3 = \gamma_1 * (\gamma_2 * \gamma_3)$

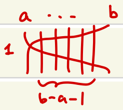
2) $\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$

3) $\gamma_1 * 1 = \gamma_1 \qquad 1 = \sigma_{id}$

4) $(q^{d_1}\gamma_1) * (q^{d_2}\gamma_2) = q^{d_1+d_2} \gamma_1 * \gamma_2$

5) $\gamma_1 * \gamma_2 = \gamma_1 \cdot \gamma_2 \mod q$.

Chevalley formula



- If $\alpha^\vee = x_a - x_b$, we denote

$$q^{\alpha^\vee} = q_{ab} = q_a \cdots q_{b-1}$$

Note that $\deg q^{\alpha^\vee} = 2(b-a) = \ell(r_\alpha) + 1$

adding a reflection on the right such that degrees match

$$\boxed{\lambda(x) * \sigma_w = \sum_{\substack{\alpha^\vee > 0 \\ \ell(wr_\alpha) = \ell(w)+1}} \langle \lambda, \alpha^\vee \rangle \sigma_{wr_{\alpha^\vee}} + \sum_{\substack{\alpha^\vee > 0 \\ \ell(wr_\alpha) = \ell(w)+1-\deg q^{\alpha^\vee}}} q^{\alpha^\vee} \langle \lambda, \alpha^\vee \rangle \sigma_{wr_{\alpha^\vee}}}$$

Note that the condition

$$\ell(wr_\alpha) = \ell(w) + 1 - \deg q^{\alpha^\vee}$$

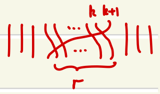
$$\Longleftrightarrow \ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$$

So, we can rewrite

$$\boxed{\lambda(x) * \gamma = \lambda(x) \cdot \gamma + \sum_{\alpha > 0} q^{\alpha^\vee} \langle \lambda, \alpha^\vee \rangle \partial_{r_\alpha} \gamma}$$

i.e. quantum terms is given by a Demazure operator.

3. Questions to answer in this talk



① express σ_w for w special i.e.

$$w = s_{k-r+1} s_{k-r+2} \cdots s_{k-1} s_k$$

recall $\sigma_w = e_r(x_1, \cdots, x_k)$

(evaluated by usual product)

② find presentation of $QH^*(Fl_n)$, i.e.

$$QH^*(Fl_n) = \frac{\mathcal{O}[x_1, \cdots, x_n]}{\langle ???\rangle}$$

$$\mathcal{O} = \mathbb{Q}[q_1, \cdots, q_{n-1}]$$

Rmk on ②.

1. firstly QH^* is generated by $x_1, \cdots, x_n$ over $\mathcal{O}$.

2. secondly it suffices to find relations deforming $e_1(x), \cdots, e_n(x) = 0$.

Assume

:= E(k)

$$\det\left(y I_k + \begin{bmatrix} x_1 & -1 & & & \\ q_1 & x_2 & -1 & & \\ & q_2 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & q_{k-1} & x_k \end{bmatrix} \right)$$

$$= y^k + e_1^q(x_1, \cdots, x_k) y^{k-1} + \cdots + e_k^q(x_1, \cdots, x_k)$$

Answer:

① $\mathfrak{S}_w = e_r^q(x_1, \cdots, x_k)$

(evaluated by quantum product)

② $QH^*(Fl_n) = \mathbb{Q}[x_1, \cdots, x_n] / \langle e_1^q, \cdots, e_n^q \rangle$

n variables

4. Proofs

Denote $\sigma_w = e_r(k)$ for $w = s_{k-r+1} \cdots s_k$.

we want to show $e_r(k) = e_r^q(x_1, \cdots, x_k)$ (eva by $*$)

Recall Chevalley formula:

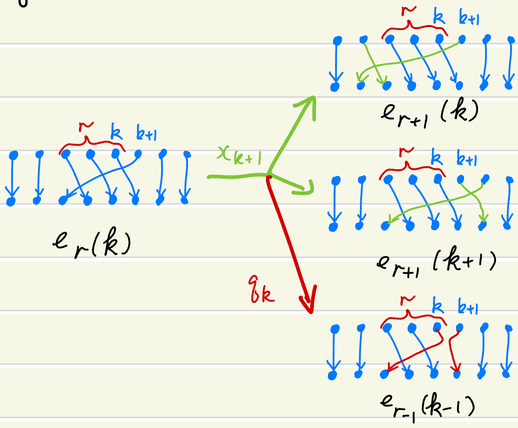
$$x_k * \sigma_w = \Big(-\sum_{i<k} + \sum_{i>k}\Big) \sigma_{wt_{ki}} \text{ or } q_{\dots} \sigma_{wt_{ki}}$$

So we have

understood as 0 when it does not make sense

$$x_k * e_r(k) = e_{r+1}(k) - e_{r+1}(k+1) - q_k e_{r-1}(k-1)$$

diagram:



It is a good exercise to show

$e_r^q(x_1, \cdots, x_k)$ has the same inductive formula

and initial condition.

Actually, it suffices to show

$$x_{k+1} * E(k) = E(k+1) - y E(k) - q_k E(k+1)$$

which follows from expansion of determinant.

By the same reason, we see

$e_k^q(x_1, \cdots, x_n) = 0$ for $k = 1, \cdots, n$.

Thus we get the presentation.

Examples

$e_1^q(x_1, \cdots, x_k) = tr = x_1 + \cdots + x_k$

$e_2^q(x_1, \cdots, x_k) = e_2(x_1, \cdots, x_k) + q_1 + q_2 + \cdots + q_{k-1}$

$$\det \begin{bmatrix} x_1 & -a_1 & & & \\ b_1 & x_2 & -a_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & -a_{n-1} \\ & & & b_{n-1} & x_n \end{bmatrix} = \sum_{\substack{w \in S_n \\ w^2 = id \\ |w(i) - w(i+1)| \leq 1}} \prod_{w(i)=i} x_i \prod_{\substack{w(i) \\ = w(i)+1}} a_i b_i$$

5. Further questions

for general $w \in S_n$, how to find $\sigma_w = ?$.

Answer: quantum Schubert polynomials.

The question is basically

"how to recover everything from Chevalley formula"

$$H^*(Fl_n) = \bigoplus_{w \in S_n} \mathbb{Q} \sigma_w$$

$$= \bigoplus_{\underline{d} \leq (n-1, \cdots, 0)} \mathbb{Q} x^{\underline{d}}$$

$$= \bigoplus_{\underline{d} \leq (1, \cdots, n-1, 0)} \mathbb{Q} e_{\underline{d}}$$

where $e_{\underline{i}} = e_{i_1}(x_1) e_{i_2}(x_1, x_2) \cdots e_{i_{n-1}}(x_1, \cdots, x_{n-1})$.

Denote $e_{\underline{i}}^q = e_{i_1}^q(x_1) e_{i_2}^q(x_1, x_2) \cdots e_{i_{n-1}}^q(x_1, \cdots, x_{n-1})$.

Quantum Schubert polynomial

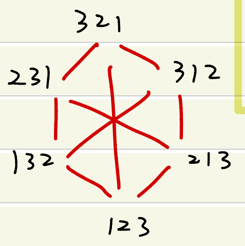
$\mathfrak{S}_w^q = \sum k_{w\underline{i}} e_{\underline{i}}^q$

if $\mathfrak{S}_w = \sum k_{w\underline{i}} e_{\underline{i}}$

Eg (n=3)

$$\boxed{\begin{aligned} x_1^2 x_2 &= x_1(x_1 x_2) \\ &= x_1 * (x_1 * x_2 + q_1) \end{aligned}}$$

$$\boxed{\begin{aligned} &x_1 x_2 \\ &= x_1 * x_2 + q_1 \end{aligned}}$$



$$\boxed{\begin{aligned} x_1^2 &= x_1(x_1 + x_2) - x_1 x_2 \\ &= x_1 * (x_1 + x_2) - (x_1 * x_2 + q_1) \\ &= x_1 * x_1 - q_1 \end{aligned}}$$

$\boxed{x_1 + x_2}$

$\boxed{x_1}$

$\boxed{1}$

6. Proof

may be expressed as polynomial of $x_1, \cdots, x_n$ using usual product

For any $\gamma \in H^*(Fl_n)$, how to express it as

different in general

a polynomial of $x_1, \cdots, x_n$ using quantum product "$*$".

1) We know for σ_w when w is special

2) If we know for γ_1 and γ_2, then so is $\gamma_1 + \gamma_2$

We need extra knowledge ...

Lemma If the first descent of w is $> k$. then

$$x_k * \sigma_w = x_k \cdot \sigma_w$$

For $i < k$, $\ell(w r_{ik}) > \ell(w)$

For $i > k$, $\ell(w r_{ki}) > \ell(w) - \ell(r_{ki})$

i.e. $\partial_{r_{ki}} \sigma_w = 0$

Coro If the first descent of w is $> k$. and $\gamma \in H^*(Fl_n)$ can be expressed as a polynomial in $x_1, \cdots, x_k$ evaluted via "$*$", then

$$\gamma * \sigma_w = \gamma \cdot \sigma_w$$

From the lemma, we have

$$e_{i_1}(1)\, e_{i_2}(2) \cdots e_{i_{n-1}}(n-1)$$

$$= \left(e_{i_1}(1) * e_{i_2}(2)\right) \cdots e_{i_{n-1}}(n-1)$$

since $e_{i_1}(1) = e_{i_1}^q(x_1)$ only involves x_1

$$= \left(\left(e_{i_1}(1) * e_{i_2}(2)\right) * e_{i_3}(3)\right) \cdots e_{i_{n-1}}(n-1)$$

since $e_{i_1}(1) * e_{i_2}(2) = e_{i_1}^q(x_1) e_{i_2}^q(x_1, x_2)$ only involves x_1, x_2

$$= \cdots$$

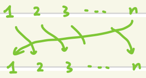
$$= e_{i_1}(1) * e_{i_2}(2) * \cdots * e_{i_{n-1}}(n-1)$$

References

- Fulton & Woodward, On the quantum product of Schubert classes.
- S. Fomin, S. Gelfand and A. Postnikov quantum Schubert polynomials.
- A. Givental and B. Kim, quantum cohomology of flag manifolds and Toda lattices

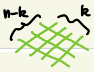
Symmetries of $QH^*(Fl_n)$

1. Seidal symmetry



Let $\eta = (123\cdots n)$.

Clearly $\eta^n = id$

$\eta^k =$ 

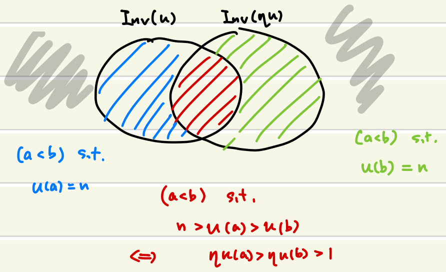
Define

$$O : QH^*(Fl_n) \longrightarrow QH^*(Fl_n)$$
$$\sigma_w \longmapsto q_{w^{-1}(n)\,n}\,\sigma_{\eta w}$$

and extends Θ-linearly.

Recall $\ell(w) = \#\mathrm{Inv}(w)$

$$\mathrm{Inv}(w) = \{(i,j) : i<j \text{ and } w(i) > w(j)\}$$



As a result

$$\ell(\eta u) - \ell(u) = \#\{j > u^{-1}(n)\} - \#\{i < u^{-1}(n)\}$$
$$= (n - u^{-1}(n) - 1) - (u^{-1}(n) - 1)$$

$$\deg q_{u^{-1}(n),n} = 2(n - u^{-1}(n))$$

Thus O is homogenous of degree $n-1$.

Prop If $\gamma * \sigma_u = \Sigma(\cdots)\sigma_w$

Then $\gamma * O(\sigma_u) = \Sigma(\cdots)\,O(\sigma_w)$

We firstly show when γ is linear.

	$ut_{ab} > u$	$ut_{ab} < u$
$\eta u t_{ab} > \eta u$	⬛	🟦
$\eta u t_{ab} < \eta u$	🟩	🟥

LHS = (adding t_{ab}) ∘ (adding η)

RHS = (adding η) ∘ (adding t_{ab})

	LHS		RHS	
🟥		=		$u^{-1}(n) = (ut_{ab})^{-1}(n)$
rest		=		
🟦	q_{an}	=	$q_{ab}q_{bn}$	$u^{-1}(n) = a$
🟩	$q_{ab}q_{bn}$	=	q_{an}	$u^{-1}(n) = b$

As $QH^*(Fl_n)$ is generated by $x_1, \cdots, x_n$, the assertion is true.

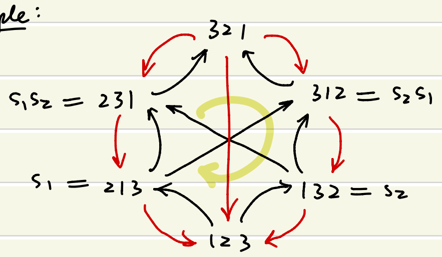
2. Quantum Bruhat graph

Put

$u \longrightarrow ut_{ab}$ if $\ell(ut_{ab}) = \ell(u) + 1$

$u \longrightarrow ut_{ab}$ if $\ell(ut_{ab}) = \ell(u) - \ell(t_{ab})$

Example:



By above, if we forget colors,

$u \longmapsto \eta u$ is a graph automorphism.

Let us denote

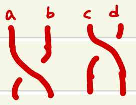
$$Q_{ab}\,u = \begin{cases} ut_{ab} & u \longrightarrow ut_{ab} \\ ut_{ab} & u \longrightarrow ut_{ab} \\ 0 & \text{otherwise} \end{cases}$$

$$R_{ab} = 1 + \varepsilon Q_{ab}$$

$$R_{ba} = R_{ab}^{-1}$$

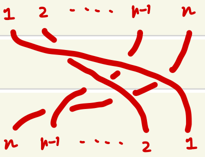
Then R_{ab} satisfies YBE

1) $R_{ab}R_{cd} = R_{cd}R_{ab}$ if $a<b$, $c<d$ distinct



2) $R_{ab}R_{ac}R_{bc} = R_{bc}R_{ac}R_{ac}$

Thus it is well-defined to denote

$R^{w_0} =$ 

Thm $R^{w_0}(u) = \sum_{v\in S_n} \varepsilon^{\ell(u,v)} v$

$\ell(u,v)$ = shortest path in QBG from u to v.

First show when $\varepsilon=1$. Note that

$$Q_{i,i+1}\, u = \begin{cases} us_i & \text{if } \ell(us_i)=\ell(u)+1 \\ us_i & \text{if } \ell(us_i)=\ell(u)-1 \end{cases}$$

$$= us_i$$

$$R_{i,i+1} = 1+(\text{Right } s_i)$$

$\Rightarrow$ $(\text{Right } s_i)\, R_{i,i+1} = R_{i,i+1} = R_{i,i+1}\, (\text{Right } s_i)$

$\Rightarrow$ $R^{w_0}(us_i) = R^{w_0}(u) = R^{w_0}(u)s_i$

$\Rightarrow$ $R^{w_0}(u) = R^{w_0}(id) = \underset{\substack{\| \\ 1}}{(\text{const})} \sum_{v\in S_n} v$

since coefficient of w_0 is 1

The case for general ε follows from a choice of ordering of $(a<b)$ by lexicographical order.