Bumpless pipe dreams meet Puzzles arXiv:2309.00467

(joint with Neil J.Y. Fan and Peter L. Guo)

Rui Xiong

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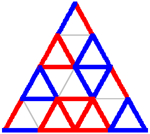
meet



Linear algebra

Denote the *standard opposite flag*

$$F^0 = \mathbb{C}^n \geq \cdots \geq F^{n-2} = \langle \mathbf{e}_n, \mathbf{e}_{n-1} \rangle \geq F^{n-1} = \langle \mathbf{e}_n \rangle \geq F^n = 0.$$

For each $V \leq \mathbb{C}^n$ of dimension k, we have a decreasing flag

$$V = F^0 \cap V \geq \cdots \geq F^{n-2} \cap V \geq F^{n-1} \cap V \geq F^n \cap V = 0.$$

We can assign the set of "jumping indices" λ, i.e.

$$\lambda_i = 1 \iff \dim(F^{i-1} \cap V) > \dim(F^i \cap V)$$
$$\lambda_i = 0 \iff \dim(F^{i-1} \cap V) = \dim(F^i \cap V)$$

Grassmannians

Denote

$$\mathrm{Gr}(k,n) = \{ V \leq \mathbb{C}^n \mid \dim V = k \}.$$

Let us denote *Schubert cell* for $\lambda \in \binom{[n]}{k}$

$$\Sigma_\lambda^\circ = \left\{ V \in \mathrm{Gr}(k,n) \,\middle|\, \text{jumping indices of } V = \lambda \right\}.$$

$$\Sigma_\lambda = \text{closure of } \Sigma_\lambda^\circ, \qquad \sigma_\lambda = [\Sigma_\lambda] \in H^\bullet(\mathrm{Gr}(k,n)).$$

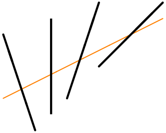
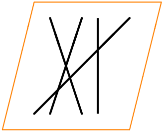
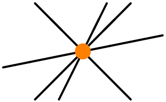
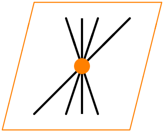
It is known that

$$H^\bullet(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q} \cdot \sigma_\lambda \qquad \text{(as a vector space)}$$

Example

Let us identify

$$\mathrm{Gr}(2,4) = \left\{ \text{lines in } \mathbb{P}^3 \right\}.$$

ALL LINES 1100 	 1010 	 0110 
 1001 	 0101 	A LINE 0011 

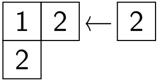
Littlewood–Richardson coefficients

Assume

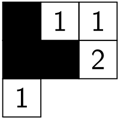
$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\mu \in \binom{[n]}{k}} c_{\lambda\mu}^{\nu} \cdot \sigma_\nu .$$

The coefficients $c_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ are known as *Littlewood–Richardson (LR) coefficients*.

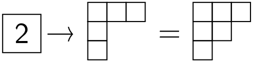
It also appears in the study of representation theory and symemtric functions. These coefficients admit a lot of combinatorial models like



Robinson–Schensted correspondence



jeu de taquin



crystal
Schur operators

Geometric meaning

Let us denote

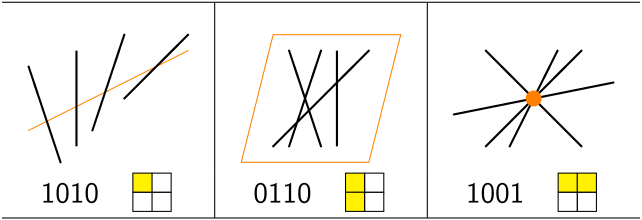
$$c_{\lambda\mu\nu} = c_{\lambda\mu}^{\nu^{op}}$$

Then for generic $x, y, z \in \mathrm{GL}_n$

$$c_{\lambda\mu\nu} = \# \left\{ v \in \mathrm{Gr}(k, n) \,\middle|\, xV \in \Sigma_\lambda,\ yV \in \Sigma_\mu,\ zV \in \Sigma_\nu \right\}.$$

If it is empty or infinite, then it is understood as zero.

Examples



We can compute:

$c_{1010,1010,0110} = 1 \quad c_{1010,1010,1001} = 1$

As a result,

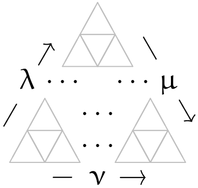
$$\sigma_{1010} \cdot \sigma_{1010} = \sigma_{0110} + \sigma_{1001}.$$

Puzzles

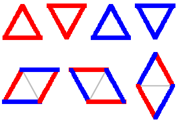
Let us use the following convention

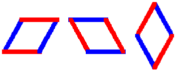
$$\text{red} = 1, \qquad \text{blue} = 0.$$

Theorem (A. Knutson, T. Tao, and C. Woodward)
The number $c^{\nu}_{\lambda\mu}$ is the number of puzzles

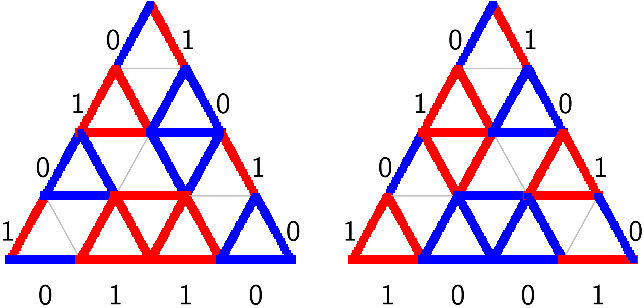


using



Warning:  are not allowed (we cannot reflect puzzles).

Examples



$$\sigma_{1010} \cdot \sigma_{1010} = \sigma_{0110} + \sigma_{1001}.$$

Generalization A $\quad H^\bullet(\mathrm{Gr}(k,n)) \rightsquigarrow K(\mathrm{Gr}(k,n))$

Let us denote

$$\mathcal{O}_\lambda = [\mathcal{O}_{\Sigma_\lambda}] = \text{structure sheaf for } \Sigma_\lambda.$$

$$\mathcal{I}_\lambda = [\mathcal{O}_{\Sigma_\lambda}(-\partial\Sigma_\lambda)] = \text{ideal sheaf for } \partial\Sigma_\lambda = \Sigma_\lambda \setminus \Sigma_\lambda^\circ.$$

It is known that they are dual basis under the Poincaré pairing.

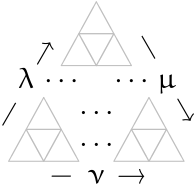
Similarly, we have

$$K(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q} \cdot \mathcal{O}_\lambda = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q} \cdot \mathcal{I}_\lambda.$$

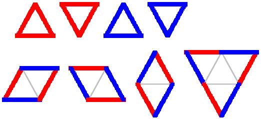
We call the coefficients of their expansion the *structure constants*.

Theorem (Vakil)

The structure constant for $\mathcal{O}_\lambda$ is the number of puzzles

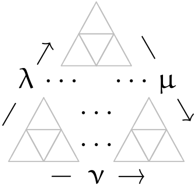


using



Theorem (Wheeler and Zinn-Justin)

The structure constant for $\mathcal{I}_\lambda$ is the number of puzzles



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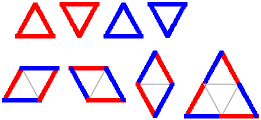
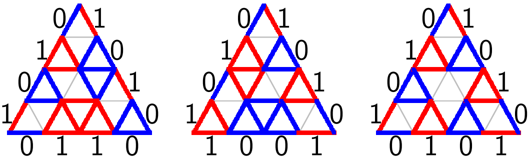
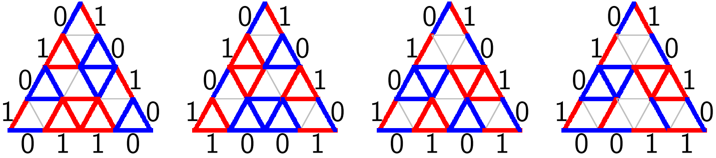


Examples



$$\mathcal{O}_{1010} \cdot \mathcal{O}_{1010} = \mathcal{O}_{0110} + \mathcal{O}_{1001} + \mathcal{O}_{0101}$$



$$\mathcal{I}_{1010} \cdot \mathcal{I}_{1010} = \mathcal{I}_{0110} + \mathcal{I}_{1001} + \mathcal{I}_{0101} + \mathcal{I}_{0011}.$$

Generalization B $\quad H^\bullet(\mathrm{Gr}(k,n)) \rightsquigarrow H_T^\bullet(\mathrm{Gr}(k,n))$

Here we are considering the *toric equivariant cohomology*. We have

$$H_T^\bullet(\mathrm{Gr}(k,n))) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[t_1, \ldots, t_n] \cdot \sigma_\lambda$$

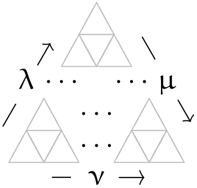
Similarly, we have *toric equivariant K-theory*

$$K_T(\mathrm{Gr}(k,n))) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[\tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1}] \cdot \mathcal{O}_\lambda,$$

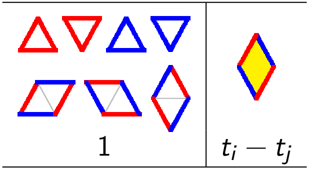
$$= \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[\tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1}] \cdot \mathcal{I}_\lambda.$$

Theorem (Knutson and Tao)

The structure constant for $H_T^\bullet(\mathrm{Gr}(k,n))$ can be computed by

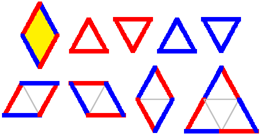


using

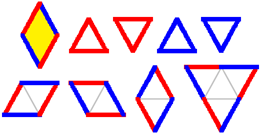


Theorem (Pechenik and Yong, Wheeler and Zinn-Justin)

The structure constant for $K_T(\mathrm{Gr}(k,n))$ can be computed by

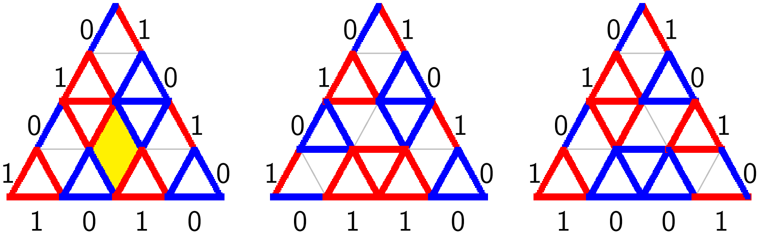


for $\mathcal{O}_\lambda$



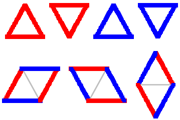
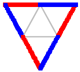
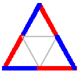
for $\mathcal{I}_\lambda$

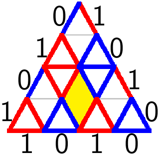
Examples

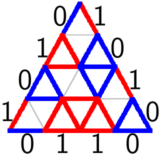


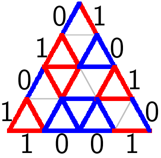
$$\sigma_{1010} \cdot \sigma_{1010} = (t_3 - t_2) \cdot \sigma_{1010} + \sigma_{0110} + \sigma_{1001}.$$

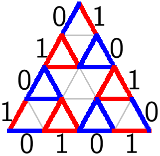
Summary

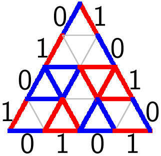
tiles	K-tiles	equivariant tiles
	 or 	

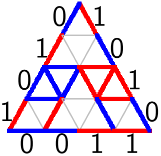












Flag varieties

Now we turn to *flag varieties*

$$\mathrm{Fl}(n) = \{0{=}V_0 < V_1 < \cdots < V_n{=}\mathbb{C}^n\}.$$

For each flag $V_\bullet \in \mathrm{Fl}(n)$, we can similarly assign a permutation w such that

$$w(i) = j \iff \dim \frac{F^{i-1} \cap V_j + F^i}{F^{i-1} \cap V_{j-1} + F^i} = 1.$$

We can similarly define

$$\Sigma_w^\circ = \{V_\bullet \in \mathrm{Fl}(k,n) \,|\, \text{permutations of } V = w\}.$$

$$\Sigma_w = \text{closure of } \Sigma_w^\circ, \qquad \sigma_w = [\Sigma_w] \in H^\bullet(\mathrm{Fl}(n)).$$

Littlewood–Richardson coefficients

It is known that

$$H^\bullet(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q} \cdot \sigma_w \qquad \text{(as a vector space)}$$

The central problem in Schubert calculus is to compute the coefficients c_{uv}^w in the expression

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{uv}^w \cdot \sigma_w.$$

There is no general combinaotrial model for c_{uv}^w up to now.

Schubert poylnomials

To study it, we define *Schubert polynomials*. For $w \in S_\infty$

$$\mathfrak{S}_{n\cdots 21} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

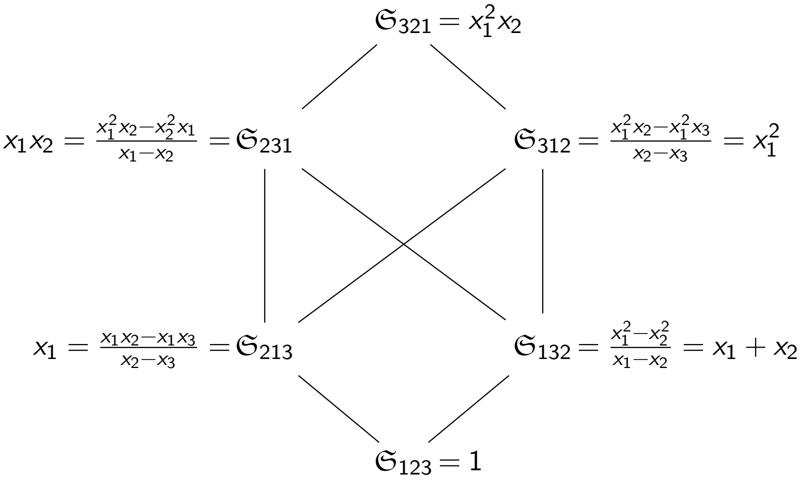
$$\mathfrak{S}_{w(i,i+1)} = \frac{\mathfrak{S}_w - \mathfrak{S}_w|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}, \qquad w_i < w_{i+1}.$$

It turns out the structure constant can be computed by

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c_{uv}^w \cdot \mathfrak{S}_w.$$

Thus we translate a geometric problem to an algebraic problem.

Examples



We have

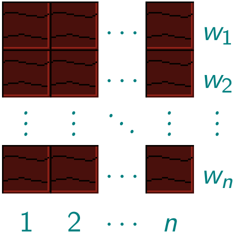
$$\sigma_{213} \cdot \sigma_{132} = \sigma_{231} + \sigma_{312}.$$

Bumpless pipe dream

There is an amazing combinatorial model for Schubert polynomials called *bumpless pipe dream*.

Theorem (Lam, Lee, and Shimozono)

Schubert polynomial $\mathfrak{S}_w$ *is the weighted sum of*

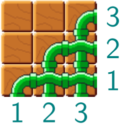


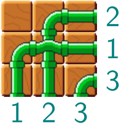
using

	
1	x_i

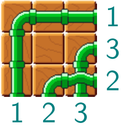
such that each pair of pipes crosses at most once

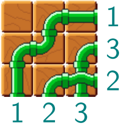
Examples

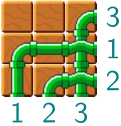
$\mathfrak{S}_{321} = x_1^2 x_2$ 

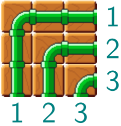
$\mathfrak{S}_{213} = x_1$ 

$\mathfrak{S}_{231} = x_1 x_2$ 

$\mathfrak{S}_{132} = x_1 + x_2$ 



$\mathfrak{S}_{312} = x_1^2$ 

$\mathfrak{S}_{123} = 1$ 

Generalization A $H^\bullet(\mathrm{Fl}(n)) \rightsquigarrow K(\mathrm{Fl}(n))$

Similarly,

$$K(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q} \cdot \mathcal{O}_w \qquad \text{(as a vector space)}.$$

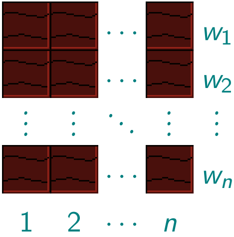
The structure constant of $\mathcal{O}_w$ is the same as the the structure constant of *Grothendieck polynomials*:

$$\mathfrak{G}_{n\cdots 21} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

$$\mathfrak{G}_{w(i,i+1)} = \frac{(1+\beta x_{i+1})\mathfrak{G}_w - (1+\beta x_i)\mathfrak{G}_w|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}, \qquad w_i < w_{i+1}.$$

Theorem (Weigandt)

Grothendieck polynomial $\mathfrak{G}_w$ is the weighted sum of



using

			
1	β	$1+\beta x_i$	x_i

such that

each pair of pipes crosses at most once

in each *, the J-pipe > the Γ-pipe*

Generalization B $\quad H^\bullet(\mathrm{Fl}(k,n)) \rightsquigarrow H_T^\bullet(\mathrm{Fl}(n))$

We have

$$H_T^\bullet(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q}[t_1, \ldots, t_n] \cdot \sigma_w$$

$$K_T(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q}[\tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1}] \cdot \mathcal{O}_w$$

The corresponding polynomial is known as *double Schubert/Grothendieck polynomial*.

Theorem (Lam, Lee, and Shimozono)

Double Schubert polynomial $\mathfrak{S}_w$ *is the weighted sum of bumpless pipe dreams but with double weight:*

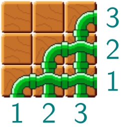
	
1	$x_i - t_j$

Theorem (Weigandt)

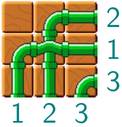
Double Grothendieck polynomial $\mathfrak{G}_w$ *is the weighted sum of bumpless pipe dreams but with double weight:*

			
1	β	$1 + \beta(x_i - t_j)$	$x_i - t_j$

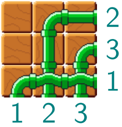
Examples



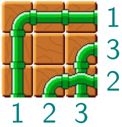
$(x_1 - t_1)(x_1 - t_2)(x_2 - t_1)$

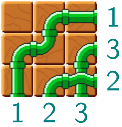


$(x_1 - t_1)$

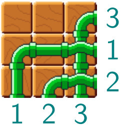


$(x_1 - t_1)(x_2 - t_1)$





$(x_2 - t_2) + (x_1 - t_1)$



$(x_1 - t_1)(x_1 - t_2)$

1

Seperated descents

Assume $u, v \in S_n$ have seperated descents

$$\max(\mathrm{des}(u)) \leq k \leq \min(\mathrm{des}(v)).$$

There is a very recent combinatorial rule by Knutson and Zinn-Justin for the expansion of

$$\mathcal{O}_u \cdot \mathcal{O}_v = \sum_w c_{uv}^w(t) \cdot \mathcal{O}_w,$$

We generalize it to the *triple version*.

Our main result

single Schubert calculus	double Schubert calculus	triple Schubert calculus
non-equivariant $\mathfrak{S}_u(x)\mathfrak{S}_v(x)$	equivariant $\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t)$	★ $\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,y)$

We can view triple Schubert calculus as the *universal rule* for

$$\mathfrak{S}_u(x,t) \cdot \mathfrak{S}_v(x,wt)$$

which geometrically corresponds to the intersection of Schubert varieties of different transversality.

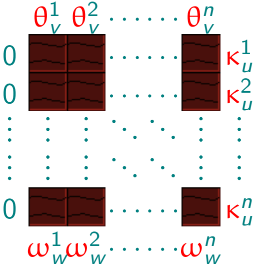
Theorem (FGX)

There is a combinatorial rule for $c_{uv}^w(y,t)$ in the expansion

$$\mathfrak{S}_u(x,y) \cdot \mathfrak{S}_v(x,t) = \sum_{w \in S_\infty} c_{uv}^w(y,t) \cdot \mathfrak{S}_w(x,t).$$

Pipe Puzzles

Let us first state the rule for cohomology, i.e. $\beta = 0$.



using

	
1	$t_j - y_i$

in each , the hori. pipe < the vert. pipe

For K-theory, it can be computed by using one more piece .

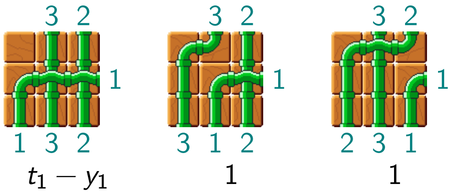
Example

Recall

$$\mathfrak{S}_{213}(x, y) = x_1 - y_1 \qquad \mathfrak{S}_{132}(x, t) = x_1 + x_2 - t_1 - t_2$$

$$\mathfrak{S}_{231}(x, t) = (x_1 - t_1)(x_2 - t_1) \qquad \mathfrak{S}_{312}(x, t) = (x_1 - t_1)(x_1 - t_2)$$

$$k = 1, \qquad u = 2 \mid 13, \qquad v = 1 \mid 32.$$



$$\mathfrak{S}_{213}(x, y) \cdot \mathfrak{S}_{132}(x, t) = (t_1 - y_1)\mathfrak{S}_{132}(x, t) + \mathfrak{S}_{231}(x, t) + \mathfrak{S}_{312}(x, t).$$

On the proof

Our proof is based on the classical *6-vertex model*, and is significantly simple! What we need is to prove

I. induction on y II. induction on t III. initial cases.

Historically, people realized that equivariant cohomology is usually easier than usual cohomology.

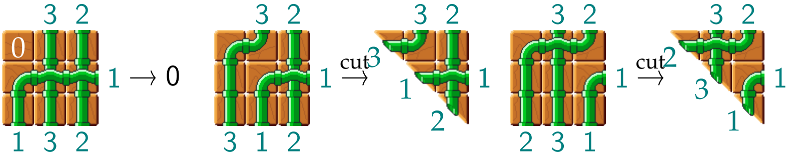
$$\boxed{\text{single}} \Longrightarrow \boxed{\text{double}}$$

It turns out the same happens for

$$\boxed{\text{double}} \Longrightarrow \boxed{\text{triple}}$$

Specialization A — seperated descents puzzles

If we set $y_i = t_i$, then on the diagonal  has weight 0. So it suffices to count those with  or  on the diagonal; so all pipes must go straight down under the diagonal. So we only need the upper triangle. This specializes to Knutson and Zinn-Justin's puzzle.



$$\mathfrak{S}_{213}(x,t) \cdot \mathfrak{S}_{132}(x,t) = \mathfrak{S}_{231}(x,t) + \mathfrak{S}_{312}(x,t).$$

Specialization B — bumpless pipe dream

If we set $k = n$, then $v = \mathrm{id}$. Taking $x = t$ on both sides of

$$\mathfrak{S}_u(x, {\color{red}y}) \cdot \mathfrak{S}_v(x, t) = \sum_{w \in S_\infty} c_{uv}^{w}({\color{red}y}, t) \cdot \mathfrak{S}_w(x, t),$$

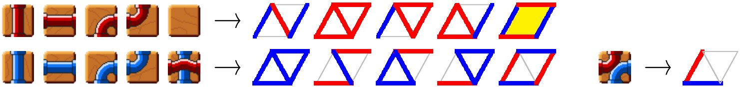
we will get

$$\mathfrak{S}_u(t, {\color{red}y}) = c_{u\,\mathrm{id}}^{\mathrm{id}}({\color{red}y}, t).$$

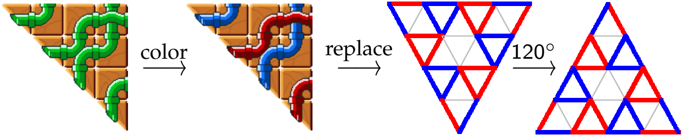
By reflecting against the diagonal and changing the labels, we recover the Weigandt's model of bumpless pipe dream.

Specialization C — classical puzzles

When u and v are both k-Grassmannian (i.e. at most one descent at k), we can recover the Grassmannian puzzles introduced in the first part. First, let us color pipes $\leq k$ by red and $\geq k$ by blue. Then we replace



Then rotate $120°$ anticlockwise.



Algebraic version

Let

$$V(z) = \mathbb{Q}(z)\boldsymbol{e}_0 \oplus \cdots \oplus \mathbb{Q}(z)\boldsymbol{e}_n.$$

We can define a linear map

$$R(t-y) : V(t) \otimes V(y) \to V(y) \otimes V(t)$$
$$\boldsymbol{e}_a \otimes \boldsymbol{e}_b \mapsto \textstyle\sum_{p,q} c_{ab}^{pq}(t-y) \cdot \boldsymbol{e}_p \otimes \boldsymbol{e}_q$$

where $c_{ab}^{pq}(t-y) =$ weight of tile p  b (with a above and q below)

For example,

$\boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \mapsto \text{weight}($  $) \cdot \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \text{weight}($  $) \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_0.$

Quantized loop algebra

So our pipe puzzle is computing the matrix coefficients of

$$\begin{array}{c} V(y_1) \otimes \cdots \otimes V(y_n) \otimes V(t_1) \otimes \cdots \otimes V(t_n) \\ \downarrow \\ V(t_1) \otimes \cdots \otimes V(t_n) \otimes V(y_1) \otimes \cdots \otimes V(y_n). \end{array}$$

Following Knutson and Zinn-Justin, this matrix could be viewed as a shadow of representation of quantized loop algebra $U_q(L\mathfrak{g})$ of type $\mathfrak{a}_n$ (i.e. $\mathfrak{g} = \mathfrak{sl}_{n+1}$).

But why the matrix coefficients computes the triple Schubert coefficients?

Nakajima quiver varieties

By Nakajima, there is an action of $U_q(L\mathfrak{g})$ on the equivariant K-(co)homology of Nakajima's quiver varieties.

We will need the special cases like

$$\mathfrak{M}\left(\begin{array}{l} \boxed{5} \\ \ | \\ 4 - 4 - 3 - 2 - 2 \end{array}\right) \cong T^*\mathcal{F}\ell(2,3,4;5),$$

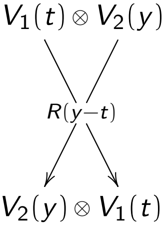
a cotangent bundle of partial flag varieties. Its equivariant K-(co)homology is the component of

$$V(z_1) \otimes \cdots \otimes V(z_5)$$

of weight $5\Lambda_1 - 4\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - 2\alpha_5$.

Stable envelope

By Okounkov and Maulik, and Okounkov, R-matrices can be realized geometrically by *stable envelope*.



$$\begin{array}{c} K_{\mathbb{G}_{w_1} \times \mathbb{G}_{w_2}}(\mathfrak{M}(\boldsymbol{w}_1) \times \mathfrak{M}(\boldsymbol{w}_2)) \\ \big\downarrow \mathbf{Stab} \\ K_{\mathbb{G}_w}(\mathfrak{M}(\boldsymbol{w})) \\ \big\downarrow \mathbf{Stab}^{-1} \\ K_{\mathbb{G}_{w_2} \times \mathbb{G}_{w_1}}(\mathfrak{M}(\boldsymbol{w}_2) \times \mathfrak{M}(\boldsymbol{w}_1)) \end{array}$$

In particular, the coefficients of R-matrices compute the coefficients of two sets of *stable basis* for different alcoves.

Example

$$\mathfrak{M}\left(\begin{array}{l}\boxed{5}\\ \mid\\ 4\ -3-2-0-0\end{array}\right)\times\mathfrak{M}\left(\begin{array}{l}\boxed{5}\\ \mid\\ 5\ -5-5-2-1\end{array}\right)$$

$$\downarrow$$

$$\mathfrak{M}\left(\begin{array}{l}\boxed{10}\\ \mid\\ 9\ -8-7-2-1\end{array}\right)$$

$$\downarrow$$

$$\mathfrak{M}\left(\begin{array}{l}\boxed{5}\\ \mid\\ 5\ -5-5-0-0\end{array}\right)\times\mathfrak{M}\left(\begin{array}{l}\boxed{5}\\ \mid\\ 4\ -3-2-2-1\end{array}\right)$$

Conclusion

As a result, in our case, the R-matrices are computing

$$\text{the coefficients of } \mathsf{Stab}^{\nabla}(u \oplus v) \text{ in } \mathsf{Stab}^{\Delta}(1_n \oplus w).$$

It turns out

- When $w = 1_n$, the coefficients coincide with triple Schubert coefficients.
- These coefficients have the inductive formula we want.

As a result, by induction,

$$R\text{-matrix coefficients} = \text{triple Schubert coefficients.}$$

If we translate everything above in terms of 6-vertex model, this is our combinatorial proof in the paper.

Thanks