Chern Classes of Positroid Varieties

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1 Introduction

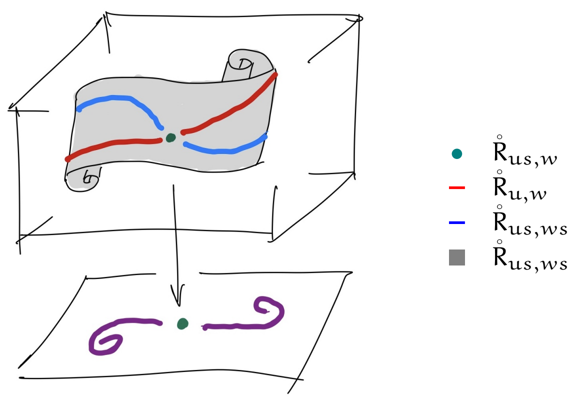
The **open projected Richardson varieties** form a stratification of the partial flag variety G/P. We establish a connection between the Segre–MacPherson classes of an **open projected Richardson variety** and of the corresponding **affine Schubert cell** by pushing or pulling these classes to the affine Grassmannian. In the Grassmannian case, we find polynomial representatives of the Segre–MacPherson classes of open projected Richardson varieties (also known as open **positroid varieties**), constructed in terms of **pipe dreams** for affine permutations.

2 Projected Richardson varieties

It is well-known G/B and G/P are stratified by **Schubert cells**. Under the natural projection $\pi : G/B \to G/P$, a Schubert cell is always mapped to a parabolic Schubert cell. The same is true for opposite Schubert cells. An intersection of Schubert and opposite Schubert cells are called an **open Richardson variety**. Both G/B and G/P are stratified by open Richardson varieties. But now, an open Richardson variety is not always mapped to a parabolic open Richardson variety.

$$\begin{array}{cccc} \text{(opposite)} & \text{Schubert} & G/B & \text{Richardson} \\ \downarrow & \downarrow & \downarrow \pi & \downarrow \times \\ \text{(opposite)} & \text{Schubert} & G/P & \text{Richardson} \end{array}$$

Actually, the projection of open Richardson varieties, known as **open projected Richardson varieties**, form a finer stratification then the paraboloc Richardson varieties. A typical picture is the following



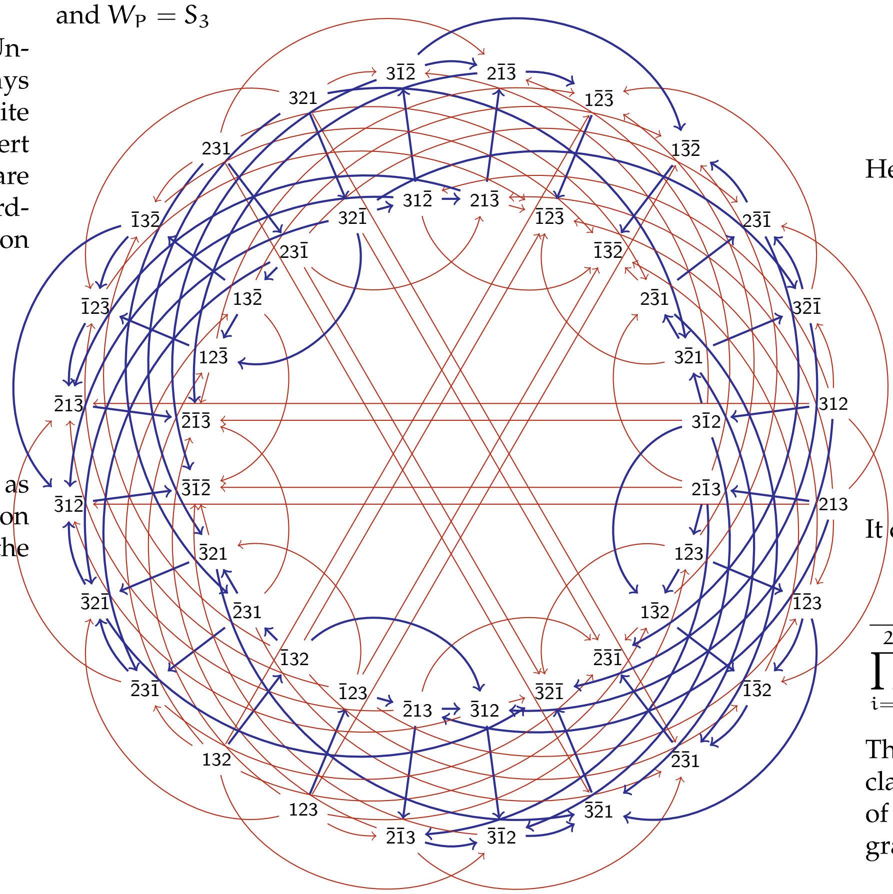
3 Extended P-Bruhat order

The **extended P-Bruhat order** arises naturally from the study of open projected Richardson varieties. By definintion, the **extended P-Bruhat order** is

$$u \le_P w \iff \text{there exists a chain } u \xrightarrow{P} u_1 \xrightarrow{P} \cdots \xrightarrow{P} u_{k-1} \xrightarrow{P} w$$

$$u \xrightarrow{P} w \iff w = ut > u \text{ for some reflection } t \in W \text{ such that } wW_P \neq uW_P.$$

This definition is motivated by the Chevalley formula of CSM classes of Schubert cells. The following is the example $W = B_3$ and $W_P = S_3$



We characterize the **extende P-Bruhat order** to be the strongest W_P-invariant partial order on W weaker than the Bruhat order. Moreover, it can be described by the **geometry of projected Richardson varieties**, and **Bruhat order in the affine Weyl group**.

4 Periodic Pipe dreams

The (open) projected Richardson varieties over Grassmannians are known as **positroid varieties**. They are indexed by **affine bounded permutations**

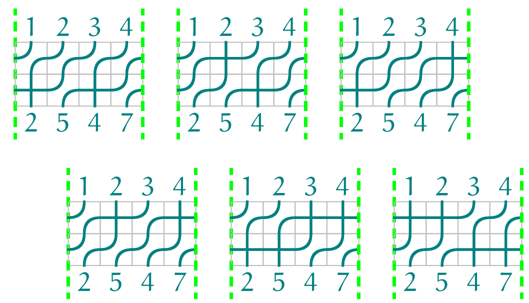
$$\left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijective} \end{array} : \begin{array}{c} f(i+n) = f(i) + n \\ \frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k \\ i \le f(i) \le i + n \end{array} \right\}.$$

It is well-known that a class of $H_T^*(\mathrm{Gr}_k(\mathbb{C}^n))$ can be represented by a symmetric polynomial. We proved the SSM class can be represented by

$$\boxed{s_{SM}(\mathring{\Pi}_f) = \begin{array}{c}\text{weighted sum of certain}\\ \text{"periodic pipe dreams"}\end{array} \in H_T^*(\mathrm{Gr}_k(\mathbb{C}^n)).}$$

Here is an example

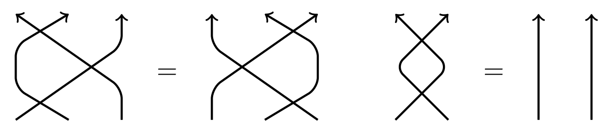
$$f : \begin{array}{c} \cdots \\ 1 \mapsto 2 \\ 2 \mapsto 5 \\ 3 \mapsto 4 \\ 4 \mapsto 7 \\ \cdots \end{array}$$



It contributes

$$\frac{1}{\prod_{i=1}^{2}\prod_{j=1}^{4}(1 + x_i - y_j)} \begin{pmatrix} (x_2 - y_1)(x_2 - y_3) + (x_1 - y_2)(x_2 - y_3) \\ +(x_1 - y_4)(x_2 - y_1) + (x_1 - y_2)(x_1 - y_4) \\ +(x_1 - y_3)(x_1 - y_4)(x_2 - y_1)(x_2 - y_2) \\ +(x_1 - y_1)(x_1 - y_2)(x_2 - y_3)(x_2 - y_4) \end{pmatrix}.$$

The proof is based on our main theorem which relates the SSM classes of open projected Richardson varieties and the SSM classes of **affine Schubert cells**. The combinatorial part is based on a diagrammatic calculation of **R-matrices**.



Yang–Baxter equation (**YBE**) unitary equation (**UE**)