Structure algebras, Hopf algebroids and oriented cohomology of a group

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1 Introduction

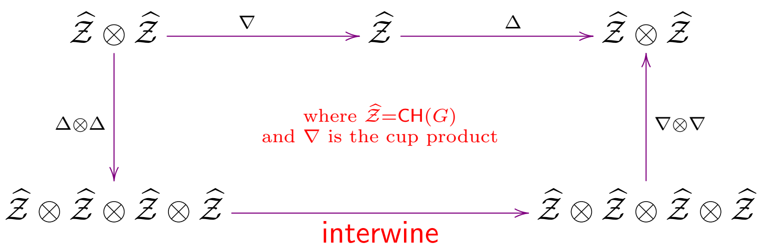
We prove that the structure algebra of a Bruhat **moment graph** of a finite real root system is a Hopf algebroid with respect to the Hecke and the Weyl actions. We introduce new techniques (**reconstruction** and **push-forward formula of a product**, **twisted coproduct**, **double quotients of bimodules**) and apply them together with our main result to linear algebraic groups, to generalized Schubert calculus, to combinatorics of Coxeter groups and **finite real root systems**. As for groups, it implies that the natural **Hopf-algebra structure** on the **algebraic oriented cohomology** $\mathtt{h}(G)$ of Levine–Morel of a split semi-simple linear algebraic group G can be lifted to a **'bi-Hopf' structure** on the T-equivariant algebraic oriented cohomology of the complete flag variety. As for Schubert calculus, we prove several new identities involving (double) generalized equivariant Schubert classes. As for finite real root systems, we compute the Hopf-algebra structure of 'virtual cohomology' of **dihedral groups** $I_2(p)$, where p is an odd prime.

2 Geometric Background

Let G be a split semi-simple linear **algebraic group**. The **Chow ring** $\mathsf{CH}(G)$ is the most celebrated geometric invariant in the theory of linear algebraic groups. The product

$$G\times G\longrightarrow G \quad \text{induces}$$
$$\mathsf{CH}(G)\otimes\mathsf{CH}(G)\xleftarrow{\Delta}\mathsf{CH}(G)$$

which equips $\mathsf{CH}(G)$ a **Hopf algebra structure**. That is, we have the following commutative diagram



and so on.

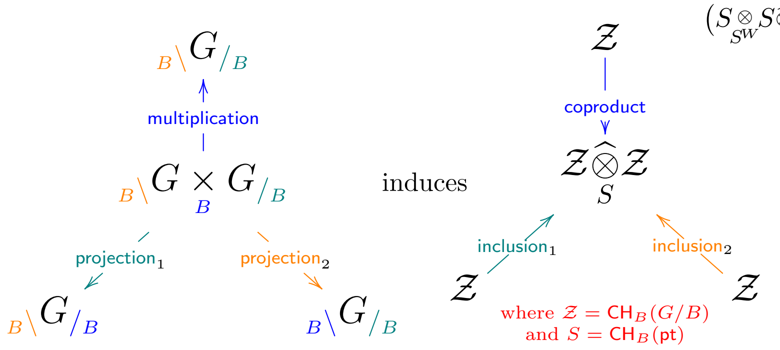
Let $T\subset B\subset G$ be a maximal torus and a Borel subgroup of G. We can lift the multiplication at the level of the equivariant Chow ring as follows. The product (marked the B-equivariance)

$${}_B\backslash G\underset{B}{\times}G_{/B}\longrightarrow {}_B\backslash G_{/B} \quad\text{induces}$$
$$\mathsf{CH}_T(G_{/B})\underset{\mathsf{CH}_T(\mathrm{pt})}{\otimes}\mathsf{CH}_T(G_{/B})\longleftarrow\mathsf{CH}_T(G_{/B}).$$

Now the question is how to compute this **"coproduct"**.

3 Coproduct

We can consider



We can show that

$$\Delta(\zeta(u)_{I_z})=\textstyle\sum_{x,y}\zeta(u)_{I_x}\,\widehat{\otimes}\,\zeta(u)_{I_y}(\cdots),$$

where the coefficients $(\cdots)$ are determined by the structure constants of **formal Hecke algebra**.

4 Duoidal Category

The category of bimodules over a commutative ring is a **duoidal category** since it has **two compatible monoidal structures**. For example, we have an intertwine in the axiom

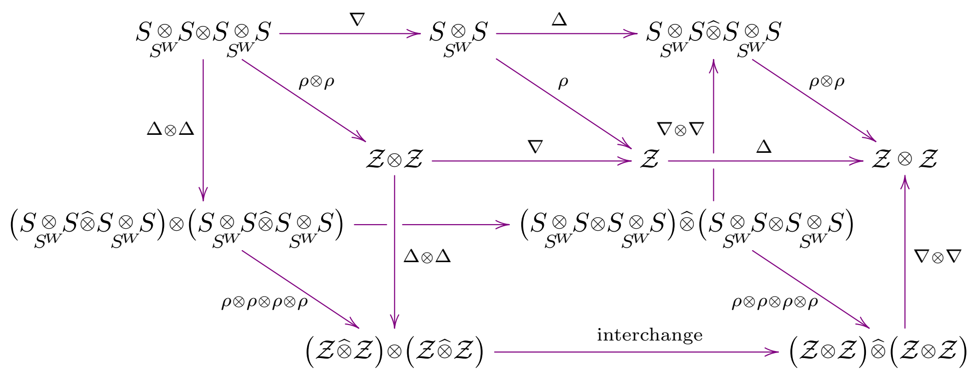
$$(A\,\widehat{\otimes}\,B)\otimes(C\,\widehat{\otimes}\,D)\xrightarrow[\text{intertwine}]{}(A\otimes C)\,\widehat{\otimes}(B\otimes D)$$

We can define Hopf algebroids in this category. For example, the intertwine above is to replace the usual intertwine in a monoidal category.

Let

$$\rho: S\otimes_{S^W}S\longrightarrow\mathcal{Z}$$

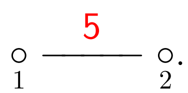
be the **Borel map**. By [CZZ1] the Borel map ρ then becomes an isomorphism after inverting the **torsion index** τ. On the other hand, nearly by definition, $S\otimes_{S^W}S$ is a Hopf algebroid in this duoidal category. We showed that $\mathcal{Z}=\mathsf{CH}_T(G/B)$ is also a Hopf algebroid, and ρ is a homomorphism as follows.



and other diagrams.

5 Dihedral Groups

All the theories developed above are purely algebraic, thus we can move to **non-crystallographic** cases. For the **dihedral group** $I_2(5)$ whose Dynkin diagram is



Note that $I_2(5)$ is the group of symmetries of a regular pentagon. We showed the cohomology of "adjoint algebraic group of type $I_2(5)$" is

$$\mathbb{Z}\Big[\tfrac{\sqrt{5}-1}{2}\Big][x]\,/\,\big\langle x^5,\sqrt{5}x\big\rangle\,.$$

References

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