LECTURES ON AFFINE WEYL GROUPS

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CONTENTS

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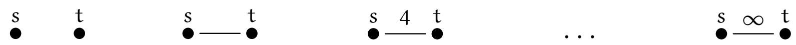
1. REVIEW OF FINITE THEORY

Coxeter groups.

1.1. Definition. A **Coxeter system** (W, S) is a group W and $S \subset W$ such that

$$W = \left\langle s \in S : \begin{array}{c} s^2 = \mathrm{id} \\ \underbrace{st\cdots}_{m_{st}} = \underbrace{ts\cdots}_{m_{st}} \end{array} \right\rangle \qquad \begin{array}{c} \text{where for each } s \neq t \in S \\ m_{st} \in \{2, 3, \cdots\} \cup \{\infty\}. \end{array}$$

We define **Coxeter diagram**



$m_{st} = 2$	$m_{st} = 3$	$m_{st} = 4$	$\cdots$	$m_{st} = \infty$
$st = ts$	$sts = tst$	$stst = tsts$	$\cdots$	no relation

Usually, we reparametrize S by $\{s_i : i \in I\}$ and $m_{ij} = m_{s_i s_j}$.

1.2. Geometric representation. We define

$$\mathfrak{h}^*_{\mathbb{R}} = \bigoplus_{i \in I} \mathbb{R}\alpha_i.$$

We equip a symmetric bilinear form such that

$$\text{length of } \alpha_i \neq 0, \qquad \text{angle of } \alpha_i \text{ and } \alpha_j \text{ is } \pi - \frac{\pi}{m_{ij}}.$$

This form is unique up to a positive rescalar of α_i. We define the geometric representation of W on $\mathfrak{h}^*_{\mathbb{R}}$ by

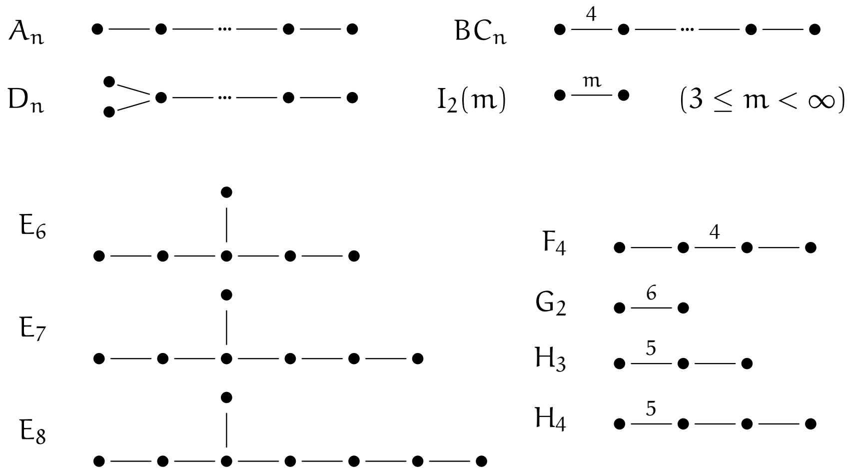
$$S \ni s_i \longmapsto (\text{reflection with respect to } \alpha_i^{\perp}) \in \mathsf{GL}(\mathfrak{h}^*).$$

That is,

$$s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda)\alpha_i, \qquad \text{where } \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

For any Coxeter group, its geometric representation is faithful.

1.3. Finite Coxeter groups. A Coxeter group W is finite if and only if the bilinear form defined above is positive definite. The corresponding Coxeter diagram is a disjoint union of the following diagrams.

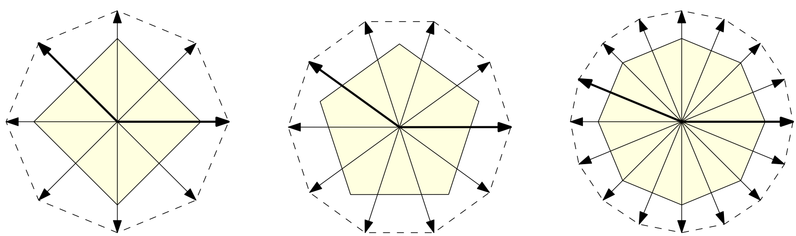


We have

$$D_2 = A_1 \times A_1, \quad D_3 = A_2, \quad A_2 = I_2(3), \quad BC_2 = I_2(4), \quad G_2 = I_2(6).$$

1.4. Example. The dihedral group D_m of order $2m$ is the Coxeter group of type $I_2(m)$. We take $\mathfrak{h}_{\mathbb{R}}^*$ to be the complex plane $\mathbb{C}$, and

$$\alpha_1 = 1, \qquad \alpha_2 = -e^{-\frac{2\pi\sqrt{-1}}{m}}.$$



Weyl groups.

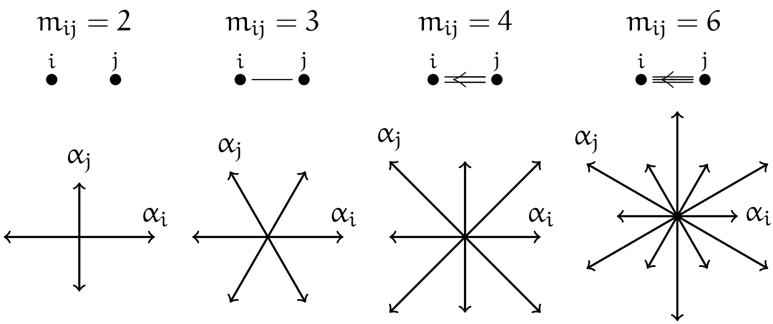
1.5. Weyl group. If a finite Coxeter group W stablizes the **root lattice**

$$Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_{\mathbb{R}}^*,$$

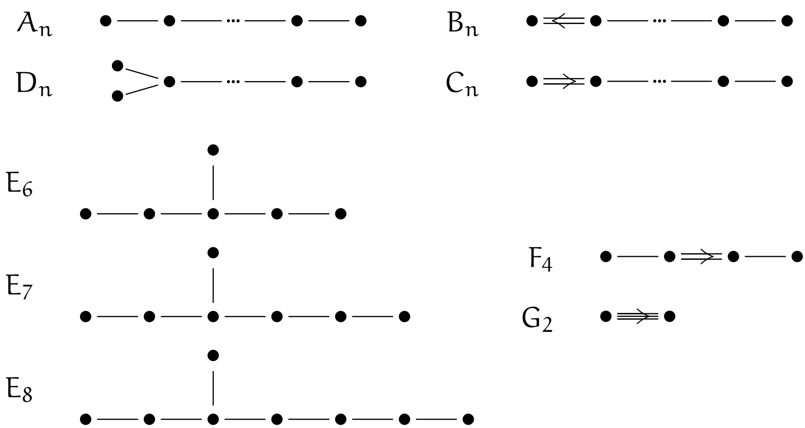
we call W a **Weyl group** and define the **root system**

$$R = \{w\alpha_i : w \in W, i \in I\}.$$

A Weyl group could only have $m_{ij} \in \{2, 3, 4, 6\}$. We define the **Dynkin diagram**



1.6. Finite Weyl group. Up to graph isomorphism, here is the classification of irreducible Weyl groups



We have

$$D_2 = A_1 \times A_1, \qquad D_3 = A_3, \qquad B_2 = C_2.$$

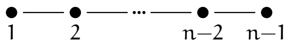
1.7. Example. The symmetric group

$$\mathfrak{S}_n = \left\{ \text{bijections } \{1, \dots, n\} \xrightarrow{w} \{1, \dots, n\} \right\}$$

is the Coxeter group of type A_{n-1}. The Coxeter generator

$$s_i = \left[\begin{array}{c} \text{the permutation exchanging } i \text{ and} \\ i{+}1 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{S}_n$$

labaled as



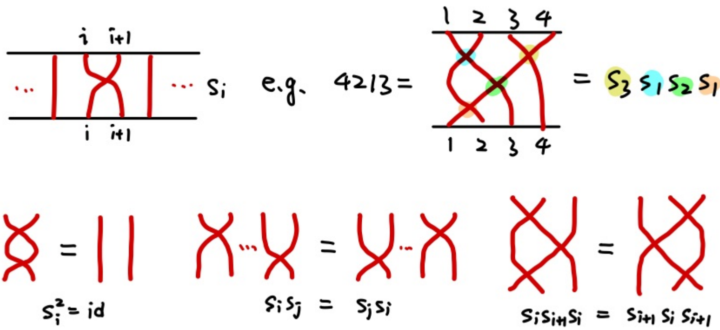
The geometric representation

$$\mathfrak{h}^*_{\mathbb{R}} = \{(a_1, \dots, a_n) : a_1 + \cdots + a_n = 0\} \subset \mathbb{R}^n.$$

The natural pairing over $\mathbb{R}^n$ restricts to $\mathfrak{h}^*_{\mathbb{R}}$. We define

$$\alpha_i = e_i - e_{i+1}, \qquad 1 \le i \le n-1.$$

We have a diagram notation



1.8. Example. The Coxeter group of type BC_n is known as the **signed symmetric group**

$$\mathfrak{BC}_n = \left\{ \text{bijections } \{\pm 1, \dots, \pm n\} \xrightarrow{w} \{\pm 1, \dots, \pm n\} : w(-i) = -w(i) \right\}.$$

Using the monotone bijection

$$\{\pm 1, \dots, \pm n\} \cong \{1, \dots, 2n\}$$

We can describe it as the subgroup of S_{2n} generated by

$$s_o = s_n, \qquad s_i = s_{n-i}s_{n+i} \quad (1 \le i \le n-1).$$

That is,

$$s_o = \left[\begin{array}{c} \text{the permutation exchanging 1 and} \\ \text{-1 with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

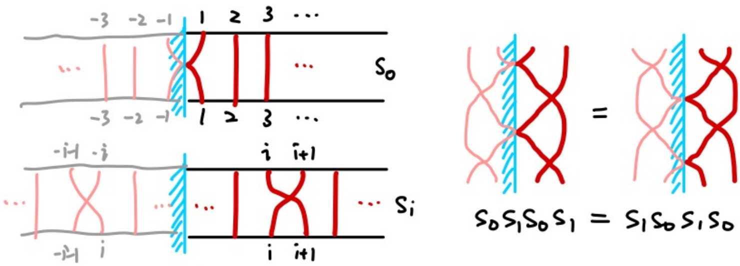
$$s_i = \left[\begin{array}{c} \text{the permutation exchanging } \pm i \text{ and} \\ \pm(i+1) \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

The label is like this

The geometric representation $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ with natural pairing and

$$\alpha_o = \begin{cases} e_1, & \text{type B,} \\ 2e_1, & \text{type C,} \end{cases} \qquad \alpha_i = e_{i+1} - e_i \quad (1 \le i \le n-1).$$

We have a diagram notation



1.9. Example. The Coxeter group of type D_n is known as the **even-signed symmetric group**.

$$\mathfrak{D}_n = \Big\{ w \in \mathfrak{BC}_n : \#\{i < 0 : w(i) > 0\} \text{ is even} \Big\}.$$

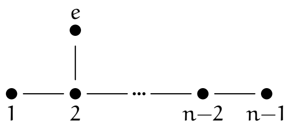
Note that the $s_0 \notin \mathfrak{D}_n$ while the $s_i \in \mathfrak{D}_n$ for $1 \le i \le n-1$. We define

$$s_e = s_0 s_1 s_0 \in \mathfrak{D}_n.$$

That is,

$$s_e = \left[\begin{array}{c} \text{the permutation exchanging } \pm 1 \text{ and} \\ \mp 2 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

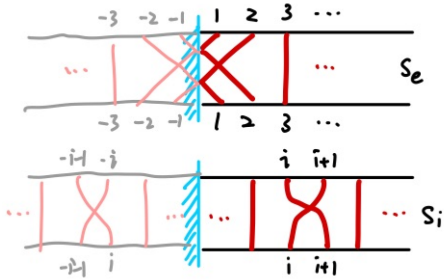
The label is



The geometric representation $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n$ with natural pairing and

$$\alpha_e = e_1 + e_2 \qquad \alpha_i = e_{i+1} - e_i \quad (1 \le i \le n-1).$$

We have a diagram notation



Miscellany.

1.10. Remark. From now, we will assume W is a Weyl group, i.e. we are equipped with a underlying root system. The same result holds for any Coxeter group if we replace R by the set of **root directions**

$$\vec{R} = \left\{ \frac{w\alpha_i}{\|w\alpha_i\|} : w \in W, i \in I \right\} \subset \mathfrak{h}^*_{\mathbb{R}}.$$

1.11. Reflections. For $\alpha \in R$, denote

$$r_\alpha = \text{the reflection with respect to } \alpha \in W.$$

If $\alpha = w\alpha_i$, then $r_\alpha = ws_iw^{-1}$. We define **reflections** by

$$\{\text{reflections}\} = \{ws_iw^{-1} : w \in W, i \in I\} = \{r_\alpha : \alpha \in R\}.$$

We call s_i ($i \in I$) a **simple reflection**.

1.12. Positive roots. The set of **positive/negative roots**

$$R^{\pm} = \{\alpha \in R : \pm\alpha \in \mathrm{span}_{\geq 0}(\alpha_i)_{i \in I}\}.$$

We have $R = R^+ \sqcup R^-$. For $\alpha \in R$, we denote $\alpha > 0$ if $\alpha \in R^+$ and $\alpha < 0$ otherwise. We call α_i $(i \in I)$ a **simple root**.

1.13. Hyperplanes. Let us consider

$$\mathfrak{h}_{\mathbb{R}} = \text{dual space of } \mathfrak{h}^*_{\mathbb{R}} \cong \mathfrak{h}^*_{\mathbb{R}}.$$

For any $\alpha \in R$, we denote

$$H_\alpha = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

1.14. Fundamental coweights. Denote **fundamental (co)weight** $\varpi_i \in \mathfrak{h}^*_{\mathbb{R}}$ $(\varpi_i^\vee \in \mathfrak{h}_{\mathbb{R}})$ be such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}.$$

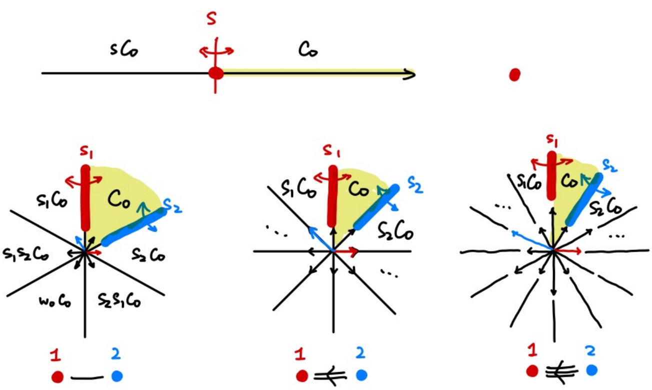
1.15. Chamber. We define **chambers** by

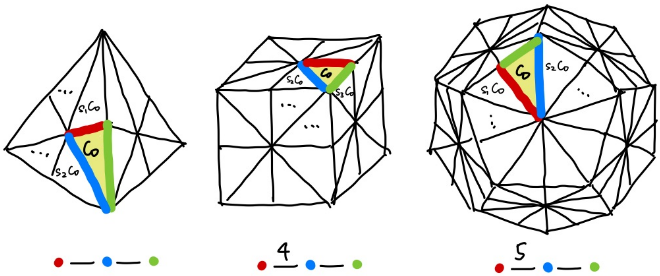
$$\{\text{chambers}\} = \text{connected components of } \left(\mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha \in R} H_\alpha \right).$$

We define the **dominant chamber** to be the cone

$$C_0 = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle \alpha_i, x \rangle > 0\} = \mathrm{span}_{\geq 0}(\varpi_i^\vee : i \in I).$$

Here we colloect example in samll dimensions.





1.16. Theorem. We have a bijection

$$W \longrightarrow \{\text{chambers}\}, \qquad w \longmapsto wC_0.$$

Under this bijection,

$$\text{the chamber of } s_i w = \begin{array}{c}\text{reflection of the chamber}\\ \text{of } w \text{ with respect to } \alpha_i\end{array}$$

$$\text{the chamber of } ws_i = \begin{array}{c}\text{the chamber sharing the wall}\\ wH_{\alpha_i} \text{ with the chamber of } w\end{array}$$

1.17. Length. For any $w \in W$, we define

$$\ell(w) = \begin{array}{c}\text{minimal length of writing } w \text{ as}\\ \text{a product of simple reflections}\end{array}$$

If

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \qquad \ell = \ell(w).$$

We call $(i_1, i_2, \ldots, i_\ell)$ is a **reduced word** of w.

1.18. Length formula. In terms of chambers,

$$\ell(w) = \#\{\text{hyperplanes separating } C_0 \text{ and } wC_0\}$$

In terms of roots,

$$\ell(w) = \#\,\mathrm{Inv}(w), \qquad \mathrm{Inv}(w) = \{\alpha \in R^+ : w\alpha \in R^-\}.$$

There is a bijection between hyperplanes and $\mathrm{Inv}(w^{-1})$.

1.19. Bruhat order. We define the **Bruhat order** over W to be the following equivalent order

- the order generated by
$$u < w \text{ if } w = ur_\alpha \text{ and } \ell(w) = \ell(u) + 1.$$
- the order generated by
$$u < w \text{ if } w = ur_\alpha \text{ and } \ell(w) > \ell(u).$$
- $u \leq w$ if there is a subword of u in a reduced word of w.
- $u \leq w$ if there is a subword of u in any reduced word of w.

We remark that for $\alpha \in R^+$,
$$ur_\alpha > u \iff u\alpha > 0 \iff \alpha \in \mathrm{Inv}(u).$$

2. Two realizations

Realization A. Let W be a finite Weyl group with root system R. Let $\{\alpha_i : i \in I\} \subset R$ be the set of simple roots.

2.1. Root lattice. Recall the definition of $\alpha^\vee$ for $\alpha \in R$. Let us denote the **(co)root lattice**

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_{\mathbb{R}}^* \qquad Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_{\mathbb{R}}.$$

2.2. Definition. The **affine Weyl group** is

$$W_a = W \ltimes Q^\vee.$$

For $\lambda \in Q^\vee$, we define $t_\lambda \in W_a$ the corresponding element. That is,

$$t_\lambda t_\mu = t_{\lambda+\mu}, \qquad t_\lambda^{-1} = t_{-\lambda}, \qquad t_0 = \mathrm{id}, \qquad w t_\lambda w^{-1} = t_{w(\lambda)}.$$

2.3. Example. For type A_1,

$$\text{the Weyl group } W = \{\mathrm{id}, s\} = \mathfrak{S}_2$$
$$\text{the coroot lattice } Q^\vee = \mathbb{Z}\alpha^\vee.$$

Let us denote $t = t_{\alpha^\vee}$. Then we have

$$W_a = \left\langle s, t : \begin{array}{c} s^2 = \mathrm{id} \\ sts = t^{-1} \end{array} \right\rangle \overset{s_0 = ts}{=\!=\!=} \left\langle s, s_0 : s^2 = s_0^2 = \mathrm{id} \right\rangle$$
$$= \text{the Coxeter group of } \left[\ \overset{s}{\bullet} \overset{\infty}{\text{———}} \overset{s_0}{\circ}\ \right]$$

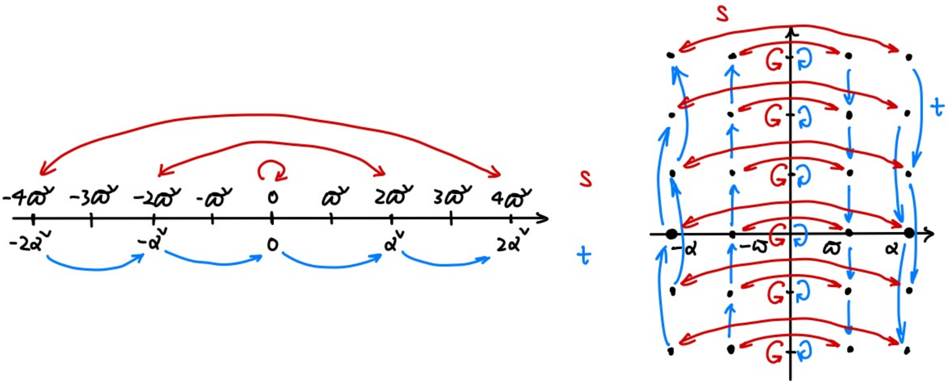
2.4. Two Actions. The affine Weyl group acts

on $Q^\vee$ **affinely**:	on $Q \oplus \mathbb{Z}\delta$ **linearly**:
$(wt_\lambda) \cdot \mu = w(\lambda + \mu)$.	$(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta$.

Here δ is a formal variable, called the **null root**. Note that the same formula defines an action on

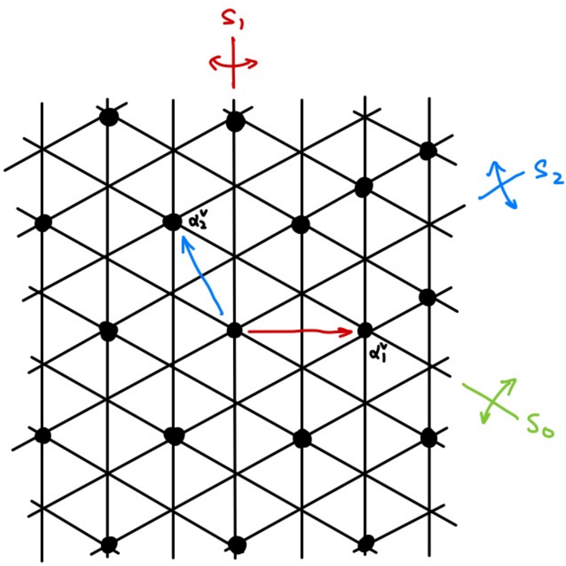
$$\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee, \qquad \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta = \mathbb{R} \otimes_{\mathbb{Z}} (Q \oplus \mathbb{Z}\delta).$$

2.5. Example. Here are the example of type A_1. We denote $\varpi^\vee = \frac{1}{2}\alpha^\vee$ and $\varpi = \frac{1}{2}\alpha$.



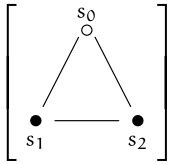
2.6. Exercise. Find the action of $s_0 = ts$ in the above example.

2.7. Example. Let us consider A_2. Let $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. Consider $s_0 = t_{\theta^\vee} s_1 s_2 s_1$. The following figure shows the action of W_a on $Q^\vee$.



With more efforts, we can see

$$W_a = \text{the Coxeter group of}$$



2.8. Roots. We define the set of **real affine roots**

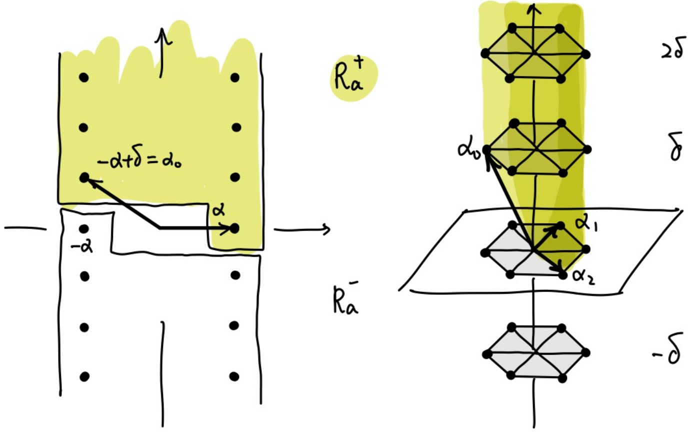
$$R_a = \{\alpha + k\delta : \alpha \in R, k \in \mathbb{Z}\} \subset Q \oplus \mathbb{Z}\delta.$$

We define the set of **positive real roots**

$$R_a^+ = \{\alpha + k\delta : k > 0 \text{ or } (k = 0 \text{ and } \alpha \in R^+)\} \subset R_a.$$

We similarly define the set of **negative roots** $R_a^- = -R_a^+$.

2.9. Examples. Here is the illustration of affine root systems of type A_1 and A_2



2.10. Reflections. For each root $\alpha + k\delta \in R_a$, we define the **reflection**

$$r_{\alpha+k\delta} = r_\alpha t_{k\alpha^\vee} \in W_a.$$

The action of $r_{\alpha+k\delta}$ on $Q \oplus \mathbb{Z}\delta$ is given by a linear reflection

$$r_{\alpha+k\delta}(\beta + n\delta) = \beta + n\delta - \langle \alpha^\vee, \beta \rangle (\alpha + k\delta).$$

The action of $r_{\alpha+k\delta}$ on $Q^\vee$ is given by the affine reflection along the hyperplane

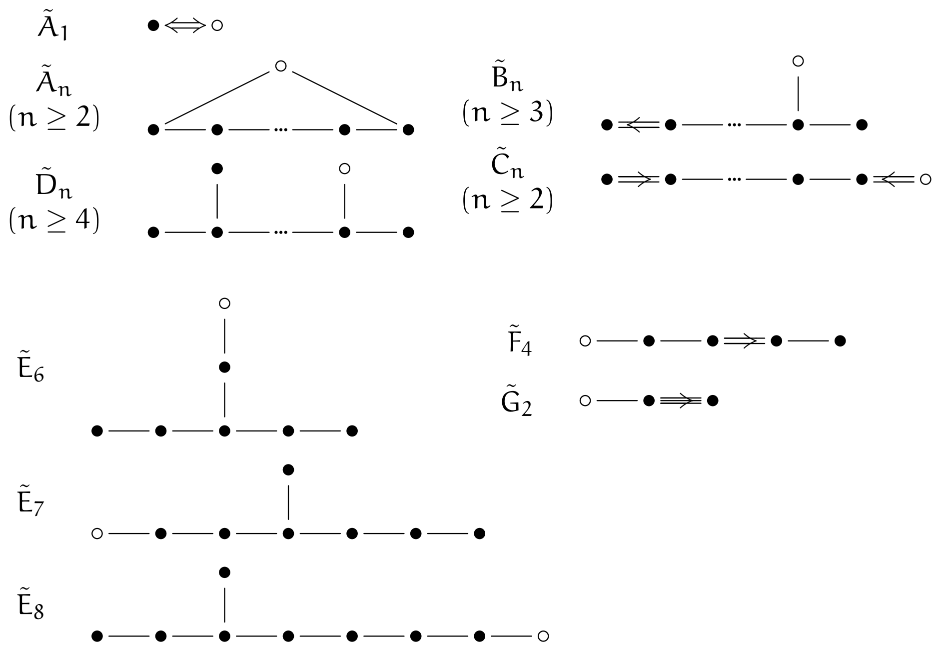
$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

2.11. Simple roots. Let $\theta \in R^+$ be the unique highest root. We denote

$$\alpha_0 = -\theta + \delta \in R_a^+, \qquad s_0 = r_{\alpha_0} = t_{\theta^\vee} r_\theta \in W_a, \qquad I_a = I \cup \{0\}.$$

Realization B.

2.12. Affine Dynkin diagram. The following are untwisted affine Dynkin diagrams



The twisted affine Dynkin diagrams are their dual.

2.13. Theorem. The affine Weyl group W_a constructed above is a Coxeter group with Coxeter generator $\{s_i : i \in I_a\}$.

2.14. Example. Let $n \geq 2$. For type A_{n-1}, the Weyl group is $\mathfrak{S}_n$ and the coroot lattice

$$Q^\vee = \{(a_1, \cdots, a_n) : a_1 + \cdots + a_n = 0\} \subset \mathbb{Z}^n.$$

The affine Weyl group admits the following realization

$$\tilde{\mathfrak{S}}_n^0 = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{c} f(i+n) = f(i) + n \\ \sum_{i=1}^n (f(i) - i) = 0. \end{array} \right\}.$$

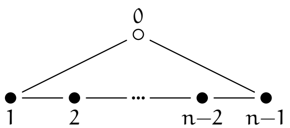
An element in $\tilde{\mathfrak{S}}_n$ is determined by its values at $1 \leq i \leq n$. The identification is given by

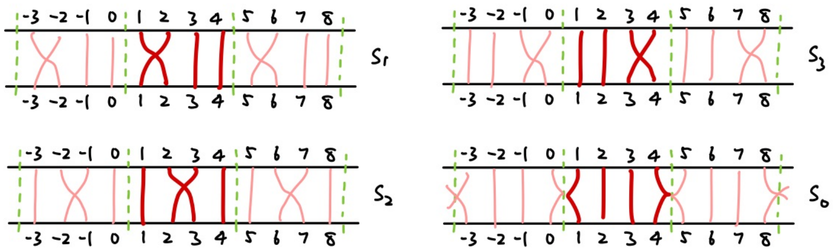
$$wt_\lambda(i) = w(i) + \lambda_i n \quad (1 \leq i \leq n).$$

Denote s_i for $i \in \mathbb{Z}/n\mathbb{Z}$ by

$$s_i = \begin{array}{c} \text{the affine permutation exchanging } j \text{ and } j+1 \\ \text{when } i \equiv j \bmod n \text{ with other numbers fixed} \end{array} \in \tilde{S}_n^0.$$

This equips the Coxeter group structure over $\tilde{S}_n^0$, where the Coxeter diagram is $(n \geq 3)$





2.15. Example. Let $n \geq 2$. For type C_n, the Weyl group is $\mathfrak{BC}_n$, and the coroot lattice is

$$\begin{aligned} Q^\vee &= \mathbb{Z}e_1 \oplus \mathbb{Z}(e_2 - e_1) \oplus \cdots \mathbb{Z}(e_n - e_{n-1}) \\ &= \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n. \end{aligned}$$

We can realize the affine Weyl group as

$$\tilde{\mathfrak{C}}_n = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{c} f(-i) = -f(i) \\ f(2n+2+i) = f(i) \end{array} \right\}.$$

Note that for any $a \in \mathbb{Z}(n+1) = \{\cdots, -(n+1), 0, n+1, 2n+2, \cdots\}$, we have

$$f(a+i) + f(a-i) = 2a.$$

An element of $\tilde{\mathfrak{C}}_n$ is determined by its value at $1 \leq i \leq n$. The identification is give by

$$wt_\lambda = w(i) - \lambda_i(2n+2).$$

The Coxeter generators are

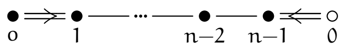
$$s_o = \left[\begin{array}{c} \text{the permutation exchanging } a{+}1 \text{ and } a{-}1 \text{ for} \\ a \in \mathbb{Z}(2n{+}2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{BC}}_n$$

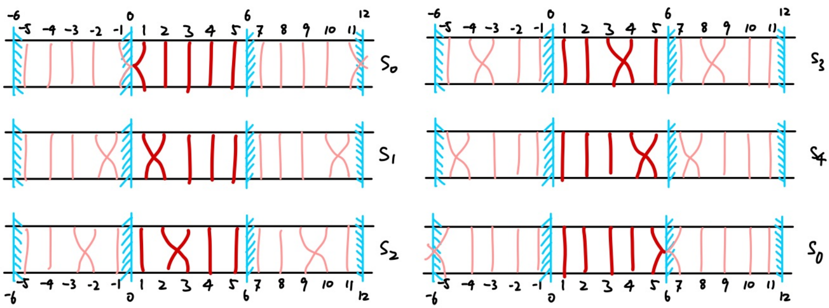
$$s_0 = \left[\begin{array}{c} \text{the permutation exchanging } a{+}1 \text{ and } a{-}1 \text{ for} \\ a \in (n{+}1) + \mathbb{Z}(2n{+}2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{BC}}_n$$

and $1 \leq i \leq n-1$,

$$s_i = \left[\begin{array}{c} \text{the permutation exchanging } a \pm i \text{ and } a \pm (i+1) \\ (a \in \mathbb{Z}(2n{+}2)) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{BC}}_n.$$

The Dynkin diagram is

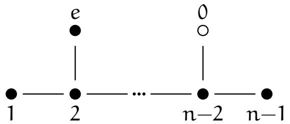




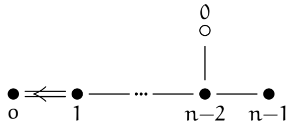
2.16. Example. We will not go into details of affine type B/D. But we mention that

$$\tilde{\mathfrak{D}}_n \subset \tilde{\mathfrak{B}}_n \subset \tilde{\mathfrak{C}}_n$$

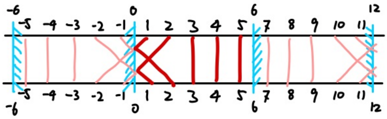
with generators described by

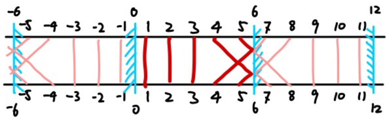


$$s_e^D = s_o s_1 s_o$$
$$s_0^D = s_0 s_{n-1} s_0$$



$$s_0^B = s_0 s_{n-1} s_0$$





Alcoves.

2.17. Alcove. For each root $\alpha + k\delta \in R_a$, we defined a **hyperplane**

$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

We define **alcoves** by

$$\{\text{alcoves}\} = \text{connected components of} \quad \mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha,k} H_{\alpha,k}.$$

Let us consider the **fundamental alcove**, i.e. the unique alcove A_0 with

$$A_0 \subset C_0, \qquad 0 \in \text{closure of } A_0.$$

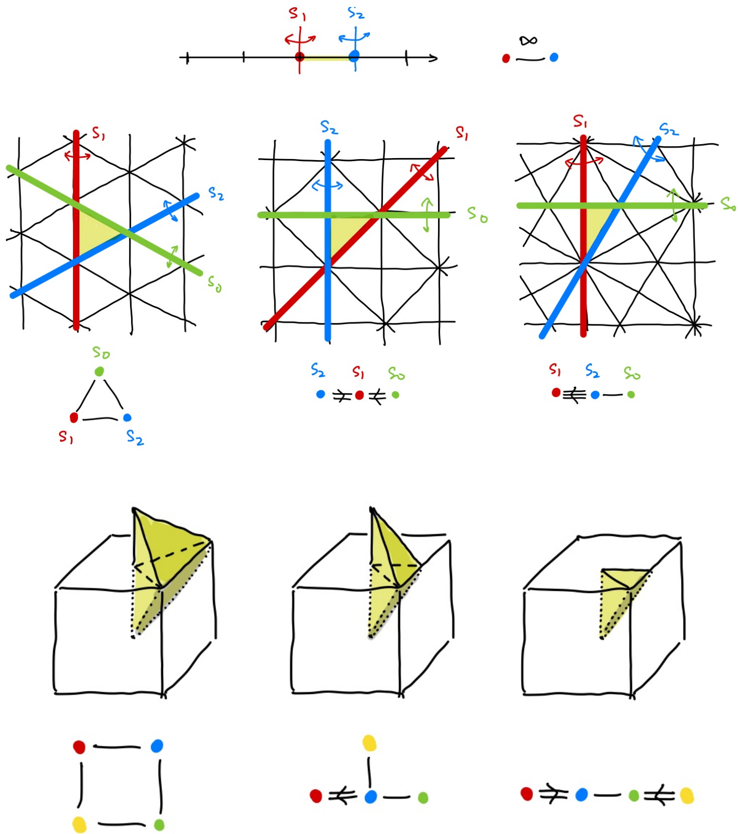
It can be described as

$$\begin{aligned} A_0 &= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha > 0\} \\ &= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha_i \rangle \text{ for } i \in I \text{ and } \langle x, \theta \rangle < 1\} \\ &= \text{bounded convex set cut by } H_{\alpha_i} \text{ for } i \in I_a \\ &= \text{interior of } \operatorname{Conv}\left(\{0\} \cup \left\{ \frac{1}{\langle \varpi_i^\vee, \theta \rangle} \varpi_i^\vee : i \in I \right\}\right). \end{aligned}$$

Here θ is the highest root. Note that

$$\langle \varpi_i^\vee, \theta \rangle = \text{coefficient of } \alpha_i \text{ in } \theta$$

Here we collect some example in small dimensions



2.18. Theorem. We have a bijection

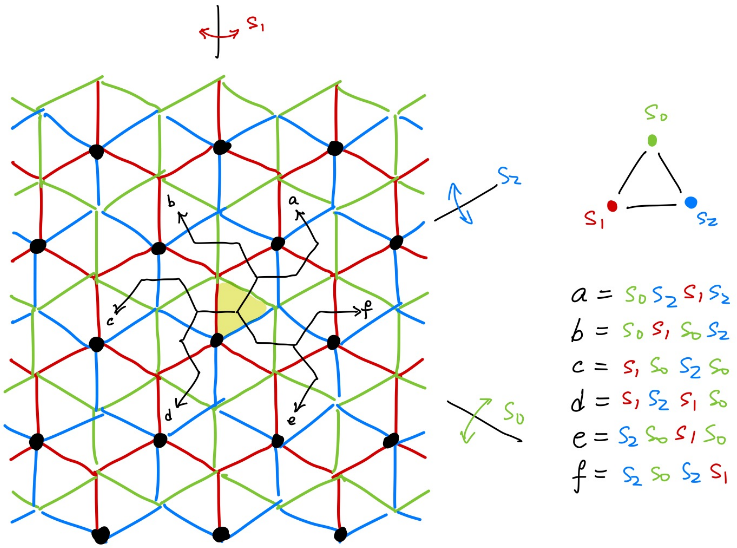
$$W_a \longrightarrow \{\text{alcoves}\}, \qquad wt_\lambda \longmapsto wt_\lambda(A_0).$$

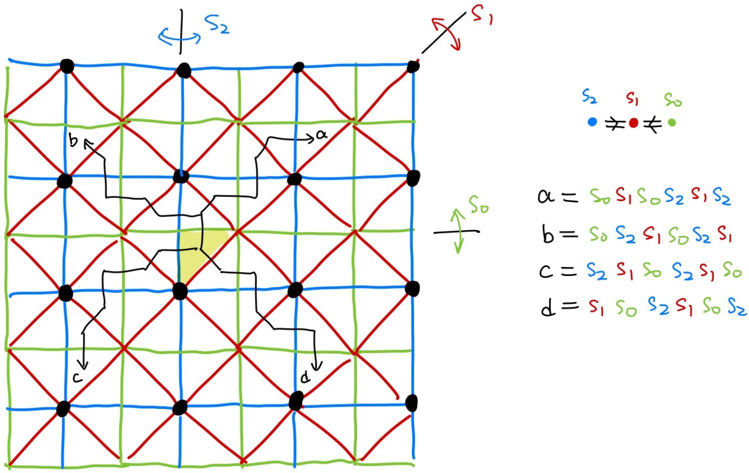
Under this bijection,

$$\text{the alcove of } s_i w t_\lambda = \begin{array}{c}\text{reflection of the alcove}\\ \text{of } w \text{ with respect to } H_{\alpha_i}\end{array}$$

$$\text{the alcove of } w t_\lambda s_i = \begin{array}{c}\text{the alcove sharing the wall}\\ wH_{\alpha_i} \text{ with the alcove of } w\end{array}$$

2.19. Example. The following shows how “alcove move” corresponds to a reduced word





3. Length formula

Iwahori–Matsumoto Formula.

3.1. Inversion. For any $wt_\lambda \in W_a$, we define set of **inversions**

$$\mathrm{Inv}(wt_\lambda) = \{\alpha + k\delta \in R_a^+ : wt_\lambda(\alpha + k\delta) \in R_a^-\}.$$

Then the length function is given by

$$\begin{aligned}\ell(wt_\lambda) &= \begin{array}{c}\text{minimal length of writing } wt_\lambda \text{ as} \\ \text{a product of simple reflections}\end{array} \\ &= \#\{\text{hyperplanes separating } A_0 \text{ and } wt_\lambda A_0\} \\ &= \#\,\mathrm{Inv}(wt_\lambda)\end{aligned}$$

There is a bijection between hyperplanes and $\mathrm{Inv}((wt_\lambda)^{-1})$.

3.2. Left inversions. Let us denote the set of **left inversions**

$$\begin{aligned}\mathrm{LInv}(wt_\lambda) = \mathrm{Inv}((wt_\lambda)^{-1}) &= \{-wt_\lambda(\alpha + k\delta) : \alpha + k\delta \in \mathrm{Inv}(wt_\lambda)\} \\ &= R_a^+ \setminus wt_\lambda R_a^+.\end{aligned}$$

There is a bijection between hyperplanes and left inversions.

3.3. Example. Let us consider A_1. The fundamental alcove A_0 is the interval $(0, \varpi)$ and

$$sA_0 = (-\varpi, 0), \qquad s_0A_0 = (\varpi, 2\varpi), \qquad tA_0 = (2\varpi, 3\varpi).$$

So we have

$$\ell(s) = \ell(s_0) = 1, \qquad \ell(t) = 2.$$

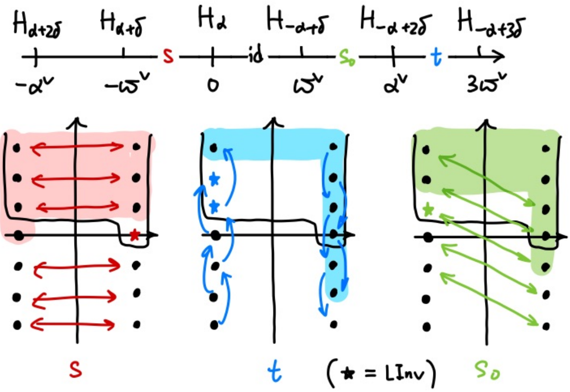
Alternatively, it is not hard to compute

$$\mathrm{Inv}(s) = \{\alpha\}, \qquad \mathrm{Inv}(s_0) = \{-\alpha + \delta\}, \qquad \mathrm{Inv}(t) = \{\alpha, \alpha + \delta\}.$$

This confirms the computation of the lengths.

$$\begin{array}{ccccccc}\cdots & H_{\alpha+2\delta} & H_{\alpha+\delta} & H_\alpha & H_{-\alpha+\delta} & H_{-\alpha+2\delta} & \cdots \\ \cdots & -2\varpi^\vee & -\varpi^\vee & 0 & \varpi^\vee & 2\varpi^\vee & \cdots\end{array}$$

Here is the diagram



3.4. Theorem. We have

$$\ell(wt_\lambda) = \sum_{\alpha>0} \left| \langle \alpha, \lambda \rangle + \delta_{w\alpha<0} \right|.$$

Here $\delta_p = 1$ if a statement p is true and equals 0 otherwise.

Proof. Fix a positive root $\alpha \in R^+$. We want to compute the contribution of

$$\pm\alpha + k\delta \in \mathrm{Inv}(wt_\lambda)$$

Note that

$$wt_\lambda(\pm\alpha + k\delta) = \pm w\alpha + (k \mp \langle \lambda, \alpha \rangle)\delta.$$

For this vector in R_a^-, we summarize four cases in the following table

	$w\alpha > 0$	$w\alpha < 0$
$\pm = +$ i.e. $k \geq 0$	$k - \langle \lambda, \alpha \rangle < 0$ i.e. $0 \leq k < \langle \lambda, \alpha \rangle$	$k - \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 \leq k \leq \langle \lambda, \alpha \rangle$
$\pm = -$ i.e. $k > 0$	$k + \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 < k \leq -\langle \lambda, \alpha \rangle$	$k + \langle \lambda, \alpha \rangle > 0$ i.e. $0 < k < -\langle \lambda, \alpha \rangle$
Total #	$\lvert\langle \lambda, \alpha \rangle\rvert$	$\lvert\langle \lambda, \alpha \rangle + 1\rvert$

This completes the proof. □

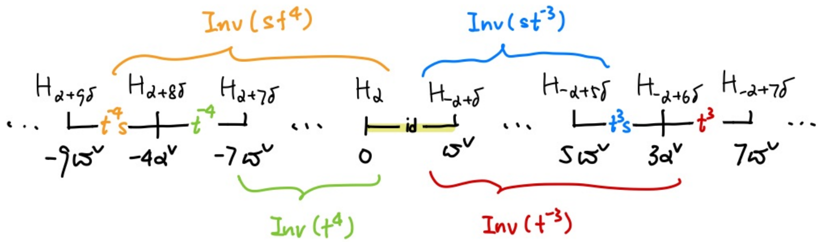
3.5. Corollary. Let us record the set of inversions for future references. For $\alpha \in R^+$, we denote

$$\mathrm{Inv}_\alpha(wt_\lambda) = \{\pm\alpha + k\delta \in \mathrm{Inv}(wt_\lambda)\}$$

the contribution of the affine positive roots as in the proof. Then the above table shows

$$\mathrm{Inv}_\alpha(wt_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \le k < \langle\lambda, \alpha\rangle + \delta_{w\alpha<0}\}, & \langle\lambda, \alpha\rangle \ge 0, \\ \{-\alpha + k\delta : 0 < k \le -\langle\lambda, \alpha\rangle - \delta_{w\alpha<0}\}, & \langle\lambda, \alpha\rangle < 0. \end{cases}$$

3.6. Example. Consider the case A_1. Recall the hyperplanes are in bijection with left inversions.



3.7. Exercise. Note that $(wt_\lambda)^{-1} = t_{-\lambda}w^{-1} = w^{-1}t_{-w\lambda}$. Check that

$$\ell(wt_\lambda) = \ell((wt_\lambda)^{-1}).$$

3.8. Example. In type $\tilde{A}_{n-1}$, we realized the affine Weyl group as $\tilde{\mathfrak{S}}_n^0$. For $f \in \tilde{\mathfrak{S}}_0^n$, we can compute the length

$$\ell(f) = \#\left\{(i,j) : \begin{array}{c} 1 \le i \le n \\ i < j, f(i) > f(j) \end{array}\right\}.$$

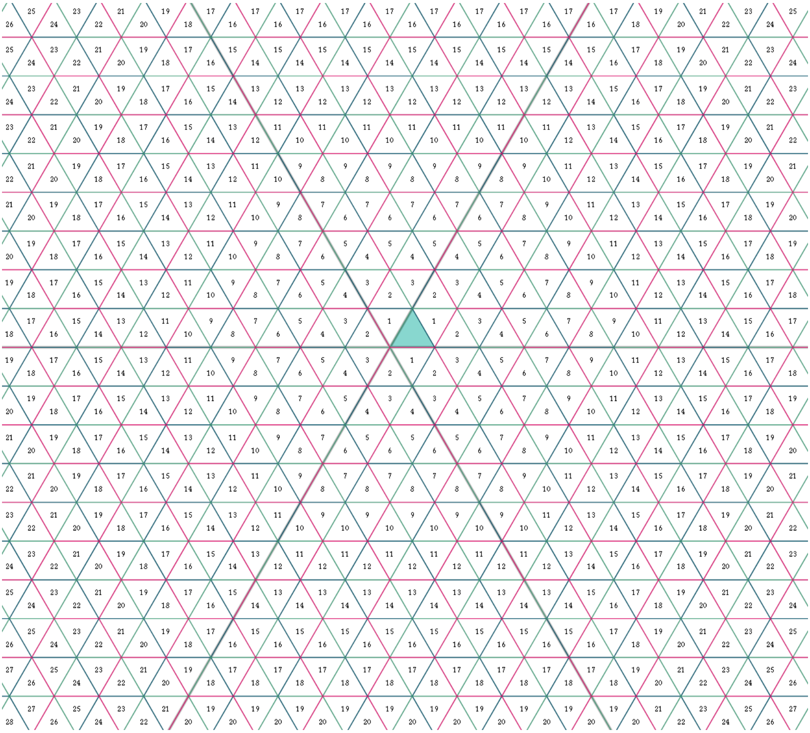
In terms of Iwahori–Matsumoto formula 3.4,

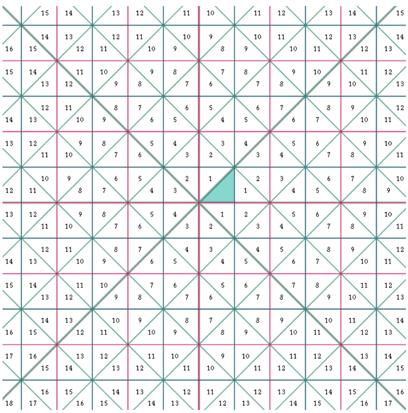
$$\ell(wt_\lambda) = \sum_{i<j} \left|\lambda_i - \lambda_j + \delta_{w(i)>w(j)}\right|.$$

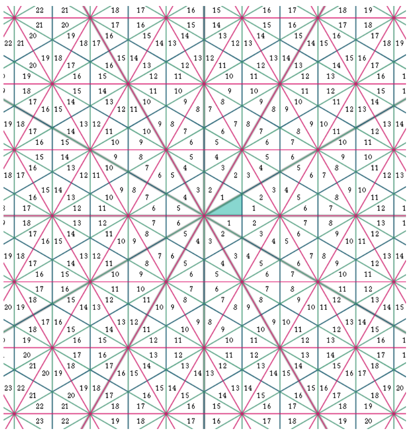
Actually, $(i, j + nk)$ corresponds to $e_i - e_j + k\delta$.

3.9. Rank 2 cases. You can visualize alcoves in rank 2 here

https://www.jgibson.id.au/lievis/affine_weyl/







Examples.

3.10. In this paragraph, we will use a lot of facts about parabolic subgroups, which is summarized at the appendix of this section.

— Length of translations.

3.11. Cartan vector. Let us denote

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i\in I}\varpi_i \in \mathfrak{h}_{\mathbb{R}}^*.$$

It satisfies

$$\rho - w\rho = \sum_{\alpha\in \mathrm{Inv}(w^{-1})}\alpha.$$

3.12. Dominant case. Let $\lambda \in Q^\vee$ be dominant. Then

$$\ell(t_\lambda) = \sum_{\alpha>0}\left|\langle\alpha,\lambda\rangle + \delta_{\alpha<0}\right| = \sum_{\alpha>0}\langle\alpha,\lambda\rangle = 2\langle\rho,\lambda\rangle.$$

3.13. General case. For general $\lambda \in Q^\vee$, we can always find $w \in W$ such that

$$w\lambda_0 = \lambda, \qquad \lambda_0 \text{ is dominant.}$$

Then

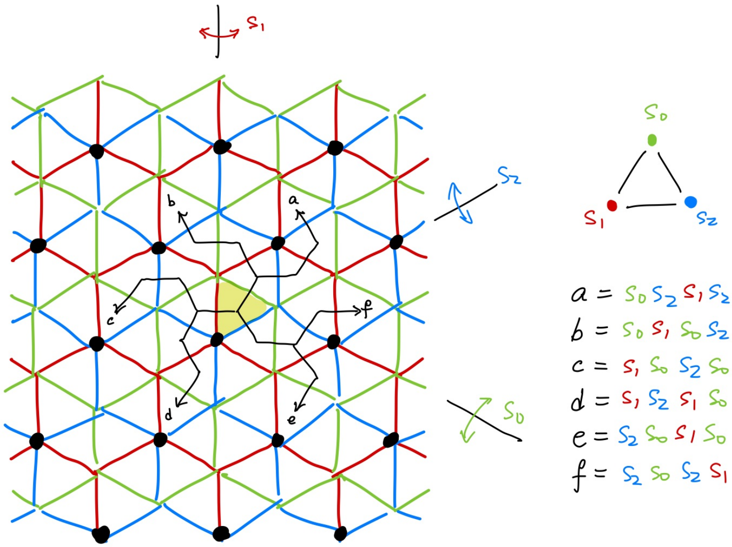
$$\begin{aligned}\ell(t_\lambda) &= \sum_{\alpha>0}\left|\langle\alpha,\lambda\rangle\right| = \sum_{\alpha>0}\left|\langle\alpha,w\lambda_0\rangle\right| \\ &= \sum_{\alpha'>0}\left|\langle w^{-1}\alpha,\lambda_0\rangle\right| = \sum_{\alpha'>0}\left|\langle\alpha',\lambda_0\rangle\right| \\ &= \sum_{\alpha'>0}\langle\alpha',\lambda_0\rangle = 2\langle\rho,\lambda_0\rangle = \ell(t_{\lambda_0}).\end{aligned}$$

Here $\alpha' = \pm w^{-1}\alpha > 0$.

3.14. Example. Consider the case A_2. Recall that $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. Then $\rho = \theta$. So

$$\ell(t_{\theta^\vee}) = \ell(t_{w\theta^\vee}) = 2\langle \rho, \theta \rangle = 4.$$

This can be seen from the first diagram of Example 2.19.



3.15. Inversion set. It would be useful to compute the set of inversions. We have

$$\mathrm{Inv}_\alpha(t_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \le k < \langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle > 0, \\ \varnothing, & \langle \lambda, \alpha \rangle = 0, \\ \{-\alpha + k\delta : 0 < k \le -\langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle < 0. \end{cases}$$

— Minimal representatives.

3.16. Formulation. We have a bijection

$$Q^\vee \xrightarrow{1:1} W_a/W, \qquad \lambda \longmapsto t_\lambda W.$$

We will describe the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda, \qquad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

3.17. Example. Let us consider type A_1. The set of minimal representative is

λ	0	$\alpha^\vee$	$-\alpha^\vee$	$2\alpha^\vee$	$-2\alpha^\vee$	$\cdots$
t_λ	id	s_0s	ss_0	$(s_0s)^2$	$(ss_0)^2$	$\cdots$
u_λ	id	s_0	ss_0	s_0ss_0	ss_0ss_0	$\cdots$
v_λ	id	s	id	s	id	$\cdots$

Equivalently, we want to find $v \in W$ such that $u^{-1} = vt_{-\lambda}$ has minimal length

$$\ell(u) = \ell(u^{-1}) = \sum_{\alpha>0} \left| - \langle \alpha, \lambda \rangle + \delta_{v\alpha<0} \right|.$$

To minimize $\ell(u)$, we wish that each summand is minimal, i.e.

$$\begin{aligned} \langle \alpha, \lambda \rangle \leq 0 \quad &\Longrightarrow \quad v\alpha > 0, \\ \langle \alpha, \lambda \rangle > 0 \quad &\Longrightarrow \quad v\alpha < 0. \end{aligned}$$

We will see, this is achievable.

3.18. Antidominant case. Let $\lambda \in Q^\vee$ be antidominant, i.e. $-\lambda$ is dominant. To minimize $\ell(u^{-1})$, it suffices to take $v = \mathrm{id}$.

3.19. General case. Let us pick $w \in W$ such that

$$\lambda = w\lambda_0, \qquad \lambda \text{ is anti-dominant.}$$

Such w's form a coset of W/W_P for W_P the stabilizer of λ_0. Let us pick the minimal one, i.e. $w \in W^P$. Then

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \langle w^{-1}\alpha, \lambda_0 \rangle < 0 \Longrightarrow w^{-1}\alpha > 0 \\ \langle \alpha, \lambda \rangle &= \langle w^{-1}\alpha, \lambda_0 \rangle = 0 \Longrightarrow w^{-1}\alpha \in R_P \overset{w \in W^P}{\Longrightarrow} w^{-1}\alpha \in R_P^+, \\ \langle \alpha, \lambda \rangle &= \langle w^{-1}\alpha, \lambda_0 \rangle > 0 \Longrightarrow w^{-1}\alpha < 0. \end{aligned}$$

It suffices to take $v = w^{-1}$.

3.20. Dominant case. Let $\lambda \in Q^\vee$ be dominant. Let $W_P = w_0 W_\lambda w_0$ be the stabilizer of $w_0\lambda$. By above computation, $v_\lambda = (w_0^P)^{-1}$ for $w_0^P = \max(W^P)$ the maximal element of W^P. Actually,

$$v_\lambda = \max(W^\lambda).$$

This is because $w_0^P = \max(W^P) = w_0 \cdot w_{0,P}$, so

$$v_\lambda = w^{-1} = w_{0,P} \cdot w_0 = w_0 \cdot w_{0,\lambda} = \max(W^\lambda).$$

3.21. Summary v1. In the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda,$$

the element v_λ is the minimal element $v \in W$ such that $v\lambda$ is anti-dominant.

3.22. Summary v2. Let $\lambda \in Q^\vee$ be anti-dominant. Denote W_P the stabilizer of $-\lambda$. Then for $w \in W^P$, the expression

$$t_{w\lambda} = (wt_\lambda) \cdot w^{-1}$$

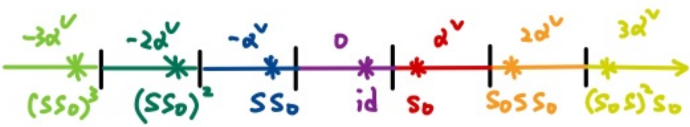
gives the parabolic decomposition. In particular,

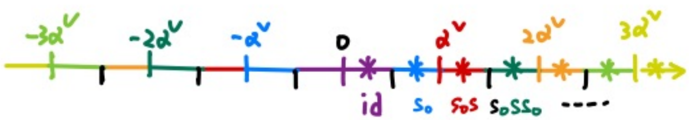
$$wt_\lambda = \min(wt_\lambda W) \iff \lambda \text{ is anti-dominant and } w \in W^\lambda.$$

3.23. Example. Consider type A_1. We have

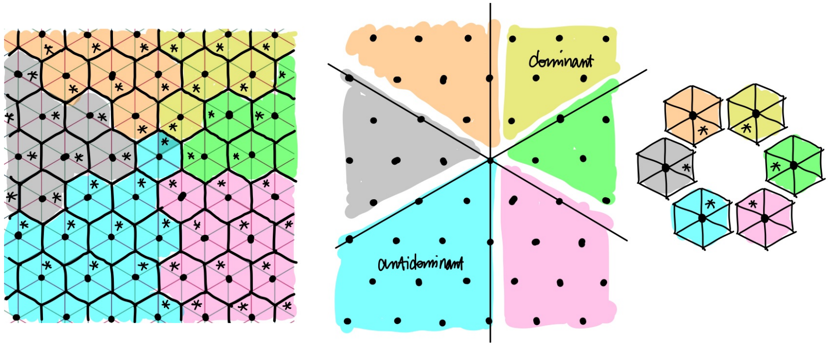
λ	0	$\alpha^\vee$	$-\alpha^\vee$	$2\alpha^\vee$	$-2\alpha^\vee$	$\cdots$
$\min(t_\lambda W)$	id	s_0	ss_0	s_0ss_0	ss_0ss_0	$\cdots$

λ	0	$-\alpha^\vee$	$\alpha^\vee$	$-2\alpha^\vee$	$2\alpha^\vee$	$\cdots$
$\min(Wt_\lambda)$	id	s_0	s_0s	s_0ss_0	s_0ss_0s	$\cdots$

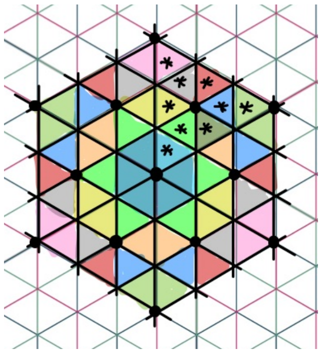




3.24. Example. The case A_2. We mark the minimal element in the cosets. Each right coset corresponds to



Each left coset corresponds to a W-orbit



3.25. Exercise. Prove that

$$wt_\lambda = \min(Wt_\lambda) \iff wt_\lambda A_0 \subset C_0.$$

This gives a bijection.

3.26. Example. For $\theta^\vee \in Q^\vee$, recall that

$$s_0 = t_{\theta^\vee} r_\theta.$$

We actually have

$$u_{\theta^\vee} = s_0, \qquad v_{\theta^\vee} = r_\theta \in W.$$

This implies

$$2\langle\rho,\theta^\vee\rangle=\ell(r_\theta)+1.$$

3.27. Example. Let us consider the case of A_{n-1}. For any $f\in\tilde{\mathfrak{S}}_n^0$, it is clear that in the decomposition

$$f=uv,\qquad u=\min(fW)\text{ and }v\in\mathfrak{S}_n$$

we have

$$\begin{aligned}u(1)&=\min(f(1),\dots,f(n)),\\ \cdots&=\cdots\\ u(n)&=\max(f(1),\dots,f(n)).\end{aligned}$$

— Double cosets.

3.28. Double cosets. We have a bijection

$$Q^\vee_{\mathrm{dom}}\xrightarrow{1:1}W\backslash W_a/W,\qquad t_\lambda\longmapsto Wt_\lambda W.$$

Recall that $wt_\lambda w^{-1}=t_{w\lambda}$. We actually have

$$Wt_\lambda W=\bigcup_{w\in W}t_{w\lambda}W.$$

Similar to one-side case, there is also a unique minimal element in each double coset.

3.29. Summary v3. Let $\lambda\in Q^\vee$ be anti-dominant. Denote W_P the stabilizer of $-\lambda$. By Summary 3.22 above,

$$W^P\times W\xrightarrow{1:1}Wt_\lambda W,\qquad (w,u)\longmapsto wt_\lambda u,$$

with

$$\ell(wt_\lambda u)=-\ell(w)+\ell(t_\lambda)+\ell(u).$$

By taking inverse, we have a dominant version.

3.30. Summary v4. Let $\lambda \in Q^\vee$ be dominant, with stabilizer W_P. We have a bijection

$$W \times W^P \longrightarrow Wt_\lambda W, \qquad (u, w) \longmapsto ut_\lambda w^{-1}$$

with

$$\ell(ut_\lambda w^{-1}) = \ell(u) + \ell(t_\lambda) - \ell(w).$$

As a result,

$$\min(Wt_\lambda W) = \min(t_\lambda W).$$

Appendix: Parabolic subgroups.

3.31. Defintion. Let I_P be a subset of I. Then we denote the **parabolic subgroup**

$$W_P = \big(\text{subgroup generated by } s_i \text{ with } i \in I_P\big) \subset W$$

and $R_P \subset R$ the root system of W_P.

3.32. Minimal representative. For any $w \in W$, there is a minimial element, called the **minimal representative**, in the right coset wW_P under the Bruhat order. Let us denote the set of **minimal representative**

$$W^P = \{\min(wW_P) : w \in W\}.$$

We have a length-additive bijection

$$W^P \times W_P \longrightarrow W, \qquad (u, v) \longmapsto uv.$$

Note that

$$w \in W_P \iff \operatorname{Inv}(w) \subset R_P^+$$
$$w \in W^P \iff \operatorname{Inv}(w) \subset R^+ \setminus R_P^+.$$

3.33. Parabolic Bruhat order. For two cosets uW_P, wW_P, we define

$$uW_P \le wW_P \iff uv \le wv' \text{for some } v, v' \in W_P.$$

The the bijection

$$W^P \longrightarrow W/W_P, \qquad w \longmapsto wW_P$$

is an isomorphism of posets.

3.34. Stabilizer. For a dominant $\lambda \in \mathfrak{h}_{\mathbb{R}}$, the stabilizer

$$W_\lambda = W_P = \{w \in W : w\lambda = \lambda\}$$

is a parabolic subgroup with

$$I_P = \{i \in I : \langle x, \alpha_i \rangle = 0\}.$$

We denote

$$W^\lambda = W^P.$$

Then we have a bijection

$$W^\lambda \longrightarrow W\lambda, \qquad w \longmapsto w\lambda.$$

4. Extended affine Weyl groups

Definition.

4.1. Weight lattice. Recall the definition of $\varpi_i^\vee$ for $i \in I$. Let us denote the **(co)weight lattice**

$$P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i \subset \mathfrak{h}_{\mathbb{R}}^*, \qquad P^\vee = \bigoplus_{i \in I} \mathbb{Z}\varpi_i^\vee \subset \mathfrak{h}_{\mathbb{R}}.$$

From the axiom of root system, we have

$$Q \subseteq P, \qquad Q^\vee \subseteq P^\vee.$$

In general, they are not equal.

4.2. Definition. The **extended affine Weyl group** is

$$W_e = W \ltimes P^\vee.$$

For $\lambda \in P^\vee$, we define $t_\lambda \in W_a$ the corresponding element.

4.3. Two Actions. The extended affine Weyl group acts

on $P^\vee$ **affinely**:	on $Q \oplus \mathbb{Z}\delta$ **linearly**:
$(wt_\lambda) \cdot \mu = w(\lambda + \mu)$.	$(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta$.

It is not hard to see R_a is stable under W_e, so the **set of inversions** also makes sense. We define the length

$$\ell(wt_\lambda) = \#\,\mathrm{Inv}(wt_\lambda).$$

It is also computed by Iwahori–Matsumoto formula 3.4.

4.4. Remark. However, W_e is not a Coxeter group in general. Actually, there would be many elements in W_e of length 0. The main purpose of this section is to study them.

4.5. The group Ω. Let us denote

$$\Omega = \{\pi \in W_e : \ell(\pi) = 0\} = \{\pi \in W_e : \pi A_0 = A_0\}.$$

Note that the norm vector of facets of $\overline{A_0}$ are simple roots. So we have

$$\begin{aligned}\Omega \hookrightarrow \mathrm{Aut}(A_0) &= \mathrm{Aut}(\text{affine Coxeter diagram}) \\ &= \mathrm{Aut}(\text{affine Dynkin diagram}).\end{aligned}$$

The last equality follows from the classification, i.e. any automorphism of affine Coxeter group preserving length. Thus Ω acts on W_a, and

$$W_e = \Omega \ltimes W_a.$$

4.6. Fundamental group. In particular the composition is an isomorphism

$$\Omega \subset W_e \twoheadrightarrow W_e/W_a = P^\vee/Q^\vee.$$

The group $P^\vee/Q^\vee$ is known to be the fundamental group of the adjoint algebraic group. Here is the table

A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n	$\mathbb{Z}/2\mathbb{Z}$
C_n	$\mathbb{Z}/2\mathbb{Z}$
D_n	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (n even) $\mathbb{Z}/4\mathbb{Z}$ (n odd)
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8, F_4, G_2	trivial

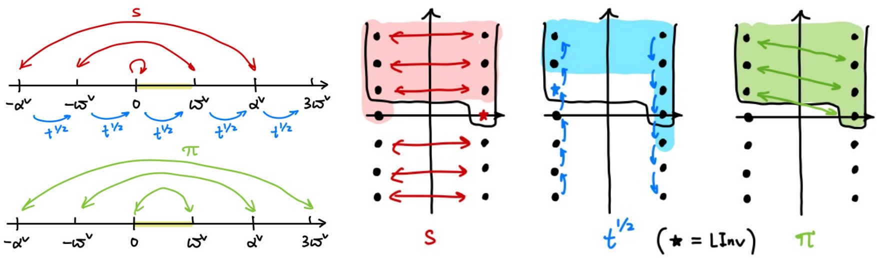
4.7. Example. Consider type A_1.

$$P^\vee = \mathbb{Z}\varpi^\vee \subset Q^\vee = \mathbb{Z}\alpha^\vee.$$

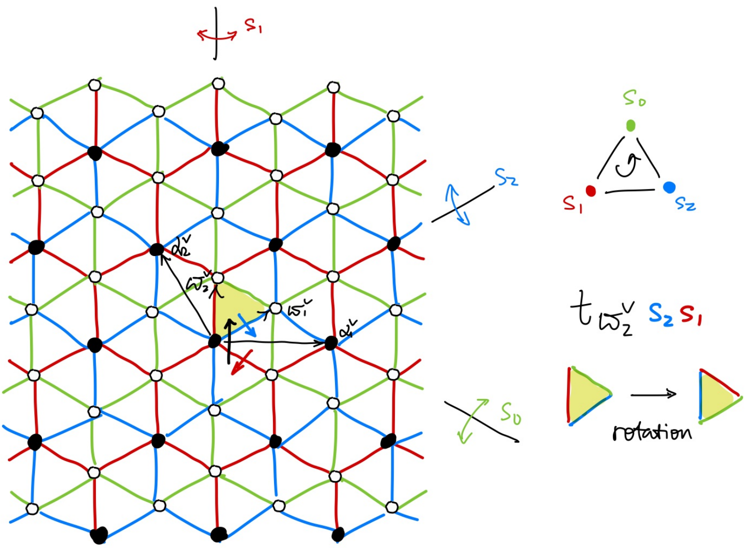
The index is 2. Let $t^{1/2} = t_{\omega^\vee} \in W_e$. We see

$$\pi := t^{1/2} s \in \Omega.$$

It acts on $P^\vee$ by reflection with respect to $\frac{\varpi^\vee}{2}$. It acts on $Q \oplus \mathbb{Z}\delta$ by interchanging α_1 and α_0.

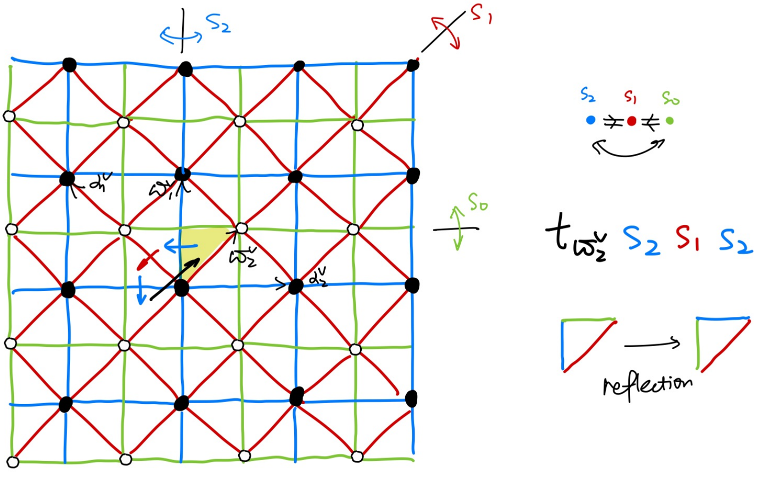


4.8. Example. Let us consider type A_2.



4.9. Exercise. Prove that $Q^\vee$ has index 3 in $P^\vee$.

4.10. Example. Let us consider type B_2.



4.11. Example. Consider type A_{n-1}. Let us first give some remark on the geometric representation. The geometric representation $\mathfrak{h}_{\mathbb{R}}^*$ can be chosen to be one of two isomorphic spaces (the subspace/quotient space realization)

$$\{(a_1, \ldots, a_n) : a_1 + \cdots + a_n = 0\} \subset \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n/\mathbb{R}(1, \ldots, 1), \qquad \{(a_1, \ldots, a_n) : a_1 + \cdots + a_n = 0\} \xrightarrow{\cong} \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$$

Then we can realize

$$\begin{array}{ccc} Q^\vee & = & \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_1 + \cdots + a_n = 0\} \\ \downarrow & & \downarrow \\ \mathbb{Z}^n & & \\ \downarrow & & \\ P^\vee & = & \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1). \end{array}$$

We thus have

$$W_a = \mathfrak{S}_n \ltimes Q^\vee \quad \subset \quad \tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n \quad \twoheadrightarrow \quad \mathfrak{S}_n \ltimes P^\vee = W_e.$$

Here

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n. \right\}.$$

For any $\lambda \in \mathbb{Z}^n$, the corresponding translation $t_\lambda \in \tilde{\mathfrak{S}}_n$ by

$$t_\lambda(i) = i + \lambda_i n, \qquad 1 \le i \le n-1.$$

Then the extended Weyl group

$$W_e = \tilde{\mathfrak{S}}_n / \langle t_{(1,\dots,1)} \rangle.$$

Actually, all theory of extended Weyl groups can be lifted to $\tilde{\mathfrak{S}}_n$. So $\tilde{\mathfrak{S}}_n$ is also called the **extended Weyl group** of type A.

Denote $\pi \in \tilde{\mathfrak{S}}_n$ by

$$\pi(i) = i + 1.$$

Note that $\pi \notin \tilde{\mathfrak{S}}_n^0$ and $\pi^n = t_{(1,\dots,1)}$. For $i \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\pi s_i \pi^{-1} = s_{i+1}.$$

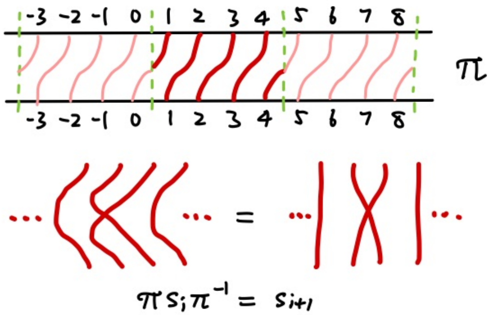
We have

$$\tilde{\mathfrak{S}}_n = \left\langle \begin{array}{c} s_0, s_1, \dots, s_{n-1} \\ \pi \end{array} : \begin{array}{c} s_i^2 = \mathrm{id}, \text{ braid relations} \\ \pi s_i \pi^{-1} = s_{i+1} \end{array} \right\rangle$$

This shows

$$\tilde{\mathfrak{S}}_n = \pi^{\mathbb{Z}} \ltimes \tilde{\mathfrak{S}}_n^0, \qquad W_e = \pi^{\mathbb{Z}} / \langle \pi^n \rangle \ltimes \tilde{\mathfrak{S}}_n^0.$$

The diagram notation



Cominuscule node.

4.12. Cominuscule node. We say a node $k \in I$ is cominuscule if

$$\langle \omega_k^\vee, \theta \rangle = 1.$$

Equivalently, for any positive roots $\alpha > 0$,

$$\langle \omega_k^\vee, \alpha \rangle \in \{0, 1\}.$$

Let us denote W_P the stabilizer of $\varpi_k^\vee$.

4.13. Elements in Ω. For any cominuscule $k \in I$, by 3.20 or 3.30, the parabolic decomposition is given by

$$t_{\varpi_k^\vee} = \pi_k \cdot w_0^P, \qquad w_0^P = \max(W^P) \text{ and } \pi_k = \min(t_{\varpi_k^\vee} W).$$

We have

$$\begin{aligned}\ell(\pi_k) &= \ell(t_{\varpi_k^\vee}(w_0^P)^{-1}) = \ell(w_0^P t_{-\varpi_k^\vee}) \\ &= \sum_{\alpha>0} \left| -\langle \alpha, \varpi_k^\vee \rangle + \delta_{w_0^P \alpha < 0} \right|.\end{aligned}$$

Note that

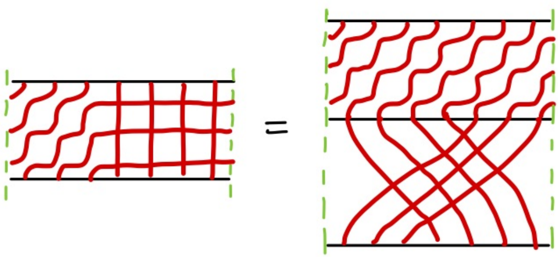
$$\begin{aligned} \alpha \in R_P^+ &\Longrightarrow \langle \alpha, \varpi_k^\vee \rangle = 0, \qquad w_0^P \alpha > 0 \\ \alpha \in R^+ \setminus R_P^+ &\Longrightarrow \langle \alpha, \varpi_k^\vee \rangle = 1, \qquad w_0^P \alpha < 0. \end{aligned}$$

Each term is zero. Thus

$$\pi_k \in \Omega.$$

4.14. Example. In type A_{n-1}, each node is cominuscule. We have $\omega_k^\vee = e_1 + \cdots + e_k$ in the quotient space realization. The following diagram reads

$$t_{\varpi_k^\vee} = \pi^k w_0^P.$$



4.15. Theorem. Denote $\pi_0 = \mathrm{id}$, and call 0 cominuscule. We have

$$\Omega = \{\pi_k : k \in I_a \text{ is cominuscule}\}.$$

4.16. Description of the automorphism. Note that for any $\pi \in \Omega$, we have

$$\pi\alpha_i = \alpha_{\pi(i)}.$$

So if

$$w_0^P \alpha_i = \alpha_j \mod \theta$$

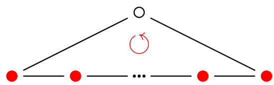
we must have

$$\pi_k \alpha_j = t_{\varpi_k^\vee}(\alpha_i + (\cdots)\delta) = \alpha_i.$$

That is, $\pi(j) = i$. In particular, since $w_0^P \alpha_k < 0$ we must have $\pi(0) = k$.

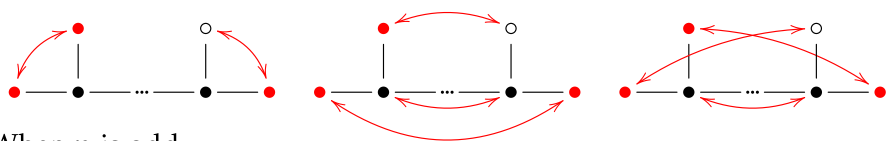
4.17. Type A. For type $\tilde{A}_{n-1}$, every node is cominuscule. The automorphism

$$\pi_k(i) = i + k \bmod n,$$

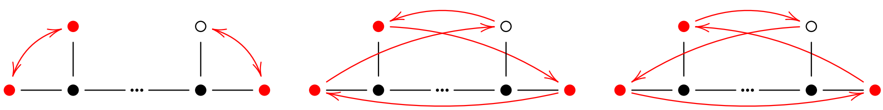


4.18. Type B and type C. For type $\tilde{B}_n$ and $\tilde{C}_n$, there is one cominuscule node

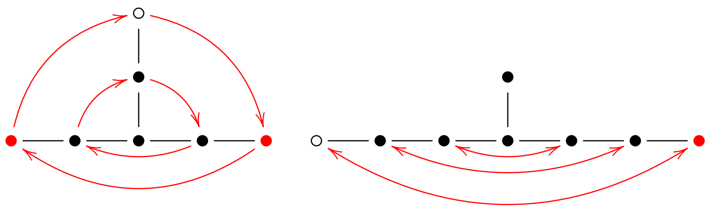
4.19. Type D. For type $\tilde{D}_n$, there are three cominuscule nodes. When n is even,



When n is odd,



4.20. Type E. For type $\tilde{E}_6$ and $\tilde{E}_7$, there are 2 and 1 cominuscule node respectively.



4.21. Corollary. A node $k \in I$ is comiuscule if and only if k is in the orbit of affine node under automorphism of affine Dynkin diagram. Moreover,

$$\mathrm{Aut}(\text{Finite Dynkin diagram}) \ltimes \Omega = \mathrm{Aut}(\text{Affine Dynkin diagram}).$$

Bruhat order.

4.22. Extending Bruhat order. We can define Bruhat order over W_e, and extend it to W_a by the disjoint union of ordering over

$$W_e = \bigcup_{\pi \in \Omega} \pi W_a.$$

Note that for any $\pi \in \Omega$,

$$ut_\mu \le wt_\lambda \iff \pi(ut_\mu)\pi^{-1} \le \pi(wt_\lambda)\pi^{-1}.$$

So it gives the same order if we use the left cosets.

4.23. Bruhat order. Let us describe the Bruhat order over

$$W_e/W \overset{1:1}{\longleftrightarrow} P^\vee.$$

We first mention that the above map is an isomorphism of W_e-sets. We denote the Bruhat order

$$\lambda \le \mu \iff t_\lambda W \le t_\mu W.$$

Note that a general fact of parabolic Bruhat order tells

$$\begin{aligned} t_\lambda W \le t_\mu W :&\iff u_\lambda \le u_\mu \\ &\iff \exists x \in t_\lambda W \text{ and } y \in t_\mu W \text{ such that } x \le y. \end{aligned}$$

Here $u_\lambda = \min(t_\lambda W)$ the minimal representative.

The Bruhat order is generated by

$$\lambda < \mu \text{ when } \mu = r_{\hat{\alpha}}\lambda \text{ for some } \hat{\alpha} \in R_a^+.$$

Note that

$$\mu < r_{\hat{\alpha}}\mu \iff \hat{\alpha} \in \mathrm{LInv}(t_\mu).$$

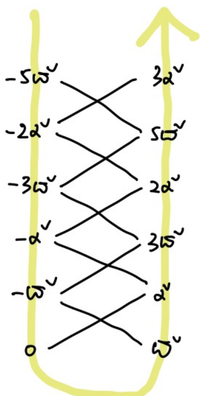
As we computed in **3.15** the inversion set of t_λ, it is not hard to conclude

- When $\langle \lambda, \alpha \rangle < 0$, $\alpha \in \mathrm{LInv}(t_\lambda)$, i.e. $r_\alpha t_\lambda < t_\lambda$. We have
$$r_\alpha \lambda < \lambda.$$
- When $\langle \lambda, \alpha \rangle > 0$, $-\alpha + \delta \in \mathrm{LInv}(t_\lambda)$, i.e. $r_{-\alpha+\delta} t_\lambda < t_\lambda$. Recall that $r_{-\alpha+\delta} = t_{\alpha^\vee} r_\alpha$, we have
$$r_\alpha \lambda + \alpha^\vee < \lambda.$$

This can also be seen from the alcove. As a result, the Bruhat order is generated by

$$\lambda < \lambda + \alpha < \alpha - \alpha < \alpha + 2\alpha < \cdots \quad \langle \lambda, \alpha \rangle = 0$$
$$\lambda < \lambda - \alpha < \alpha + \alpha < \alpha - 2\alpha < \cdots \quad \langle \lambda, \alpha \rangle = 1$$

4.24. Example. Consider type A_1.



4.25. Exercise. For $\alpha \in R^+$, denote

$$\ell_\alpha(wt_\lambda) = \#\,\mathrm{Inv}_\alpha(wt_\lambda).$$

Prove that

$\langle \lambda + k\alpha^\vee, \alpha\rangle$	$\cdots$	-8	-6	-4	-2	0	2	4	6	8	$\cdots$
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$	$\cdots$	8	6	4	2	0	1	3	5	7	$\cdots$

$\langle \lambda + k\alpha^\vee, \alpha\rangle$	$\cdots$	-7	-5	-3	-1	1	3	5	7	9	$\cdots$
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$	$\cdots$	7	5	3	1	0	2	4	6	8	$\cdots$

5. Semi-infinity

Semi-infinite length.

5.1. Length. Recall for $x \in W_e$, we defined

$$\mathrm{Inv}(x) = \{\alpha + k\delta \in R_a^+ : x(\alpha + k\delta) \in R_a^-\}.$$

Then the length function is given by

$$\begin{aligned}\ell(x) &= \#\{\text{hyperplanes separating } A_0 \text{ and } x^{-1}A_0\} \\ &= \#\,\mathrm{Inv}(x)\end{aligned}$$

There is a bijection between hyperplanes and inversions.

5.2. Semi-infinite length. For $x \in W_e$, we define

$$\ell^{\%}(x) = \ell(xt_\mu) - \ell(t_\mu)$$

for μ sufficiently dominant. Here, sufficiently dominant means,

$$\langle \mu, \alpha_i \rangle \gg 0 \qquad \text{for each } i \in I.$$

In particular, $\ell^{\%}(\pi x) = \ell^{\%}(x)$ for $\pi \in \Omega$. Note that unlike the usual length, $\ell^{\%}(x)$ might be negative and $\ell^{\%}(x) \neq \ell^{\%}(x^{-1})$ in general.

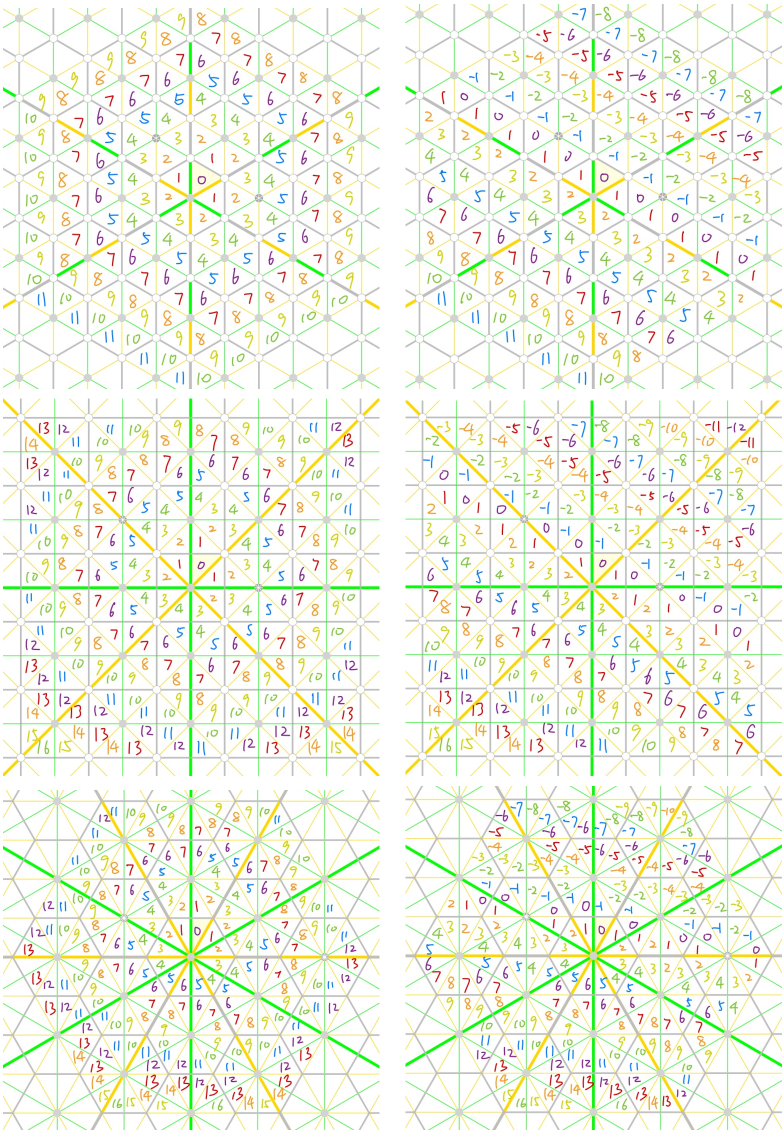
5.3. Computation. If we write $x = wt_\lambda$, then Iwahori–Matsumoto formula 3.4 implies

$$\begin{aligned}\ell^{\%}(wt_\lambda) &= \sum_{\alpha>0} \big|\langle \alpha, \lambda + \mu\rangle + \delta_{w\alpha<0}\big| - \big|\langle \mu, \alpha\rangle\big| \\ &= \sum_{\alpha>0} \big(\langle \alpha, \lambda\rangle + \delta_{w\alpha<0}\big) = \ell(w) + 2\langle \rho, \lambda\rangle.\end{aligned}$$

5.4. Example. Let us consider A_1 case. We have

x	$\cdots$	s_0ss_0	ss_0	s_0	id	s	s_0s	ss_0s	$(ss_0)^2$	$\cdots$
	$\cdots$	st^{-2}	t^{-2}	st^{-1}	id	s	t	st	t^2	$\cdots$
$\ell^{\%}(x)$	$\cdots$	-3	-2	-1	0	1	2	3	4	$\cdots$

5.5. Example. Compare length and semi-infinite length for W_a.



$\ell(x)$ on $x^{-1}A_0$ $\qquad$ $\ell^{\%}(x)$ on $x^{-1}A_0$

5.6. Example. Let us compute

$$\ell^{\%}(r_{\alpha+k\delta}) = \ell^{\%}(r_\alpha t_{k\alpha^\vee}) = \ell(r_\alpha) + 2k\langle\rho, \alpha^\vee\rangle.$$

We proved in 3.26 that

$$2\langle\rho, \theta^\vee\rangle = \ell(r_\theta) + 1.$$

So

$$\ell^{\%}(s_0) = \ell(r_\theta) - 2\langle\rho, \theta^\vee\rangle = -1.$$

5.7. A trick. Let $\alpha + k\delta \in R_a$. Notet that

$$\alpha + (k + \langle\mu, \alpha\rangle)\delta$$

We have

$$t_{-\mu}(\alpha + k\delta) \in R_a^{\pm} \text{ for sufficiently dominant } \mu \iff \alpha \in R^{\pm}.$$

5.8. Semi-infinite Inversion. Note that for $\alpha + k\delta \in R_a^+$,

$$\begin{aligned} &\alpha + k\delta \in \mathrm{LInv}(xt_\mu) \text{ for sufficiently dominant } \mu \\ \iff\ & t_{-\mu}(x^{-1}(\alpha + k\delta)) \in R_a^- \text{ for sufficiently dominant } \mu \\ \iff\ & x^{-1}(\alpha + k\delta) \bmod \delta \in R^-. \end{aligned}$$

Let us denote

$$R_{\%}^{\pm} = \{\alpha + k\delta : \alpha \in R^{\pm} \text{ and } k \in \mathbb{Z}\}.$$

We denote

$$\mathrm{LInv}_{\%}(x) = \{\alpha + k\delta \in R_a^+ : x^{-1}(\alpha + k\delta) \in R_{\%}^-\} \subset R_a^+.$$

We denote

$$\mathrm{Inv}_{\%}(x) = \{a + k\delta \in R_{\%}^+ : x(a + k\delta) \in R_a^-\} \subset R_{\%}^+.$$

5.9. Computation. We have

$$wt_\lambda(\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$$

So $\alpha + k\delta \in \mathrm{Inv}_{\%}(x)$ if and only if

$$k - \langle \lambda, \alpha \rangle < \delta_{w\alpha<0}.$$

We see

$$\mathrm{Inv}_{\%}(wt_\lambda) = \{\alpha + k\delta : \alpha \in R^+ \text{ and } k < \langle \lambda, \alpha \rangle + \delta_{w\alpha<0}\}.$$

Compare with 3.5.

5.10. Theorem. We have

$$\ell^{\%}(x) = \#\big(\mathrm{Inv}_{\%}(wt_\lambda) \setminus \mathrm{Inv}_{\%}(\mathrm{id})\big) - \#\big(\mathrm{Inv}_{\%}(\mathrm{id}) \setminus \mathrm{Inv}_{\%}(wt_\lambda)\big).$$

We can write it as

$$\ell^{\%}(x) = \sum_{\alpha+k\delta \in \mathrm{Inv}(x)} \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

5.11. Half-space. For $\alpha + k\delta \in R_a$, we defined

$$H_{\alpha+k\delta} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\}.$$

We define the **half-space**

$$H^{>0}_{\alpha+k\delta} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k > 0\}.$$

Similarly, we define $H^{<0}_{\alpha+k\delta}$ etc. One can check

$$wt_\lambda \cdot H^{>0}_{\alpha+k\delta} = H^{>0}_{wt_\lambda(\alpha+k\delta)}.$$

5.12. Alcove. Let $x \in W_e$. Let us justify the bijection

$$\mathrm{Inv}(x) \xleftrightarrow{1:1} \#\{\text{hyperplanes separating } x^{-1}A_0 \text{ and } A_0\}.$$

The key observation is

$$\alpha + k\delta \in R_a^+ \iff A_0 \subset H^{>0}_{\alpha+k\delta}.$$

As a result, for $\alpha + k\delta \in R_a^+$,

$$\begin{aligned} &\text{the hyperplane } H_{\alpha+k\delta} \text{ separates } A_0 \text{ and } x^{-1}A_0 \\ \iff\ & x^{-1}A_0 \subset H_{\alpha+k\delta}^{<0} \iff A_0 \subset H_{x(\alpha+k\delta)}^{<0} \iff x(\alpha+k\delta) \in R_a^- \\ \iff\ & \alpha + k\delta \in \mathrm{Inv}(x). \end{aligned}$$

The semi-infinite analogy is

$$\begin{aligned} \alpha + k\delta \in R_{\%}^+ &\iff A_0 + \mu \subset H_{\alpha+k\delta}^{>0} \text{ for sufficiently dominant } \mu \\ &\iff C_0 \cap H_{\alpha+k\delta}^{>0} \neq \varnothing. \end{aligned}$$

Here C_0 is the fundamental chamber. As a result, for $\alpha > 0$,

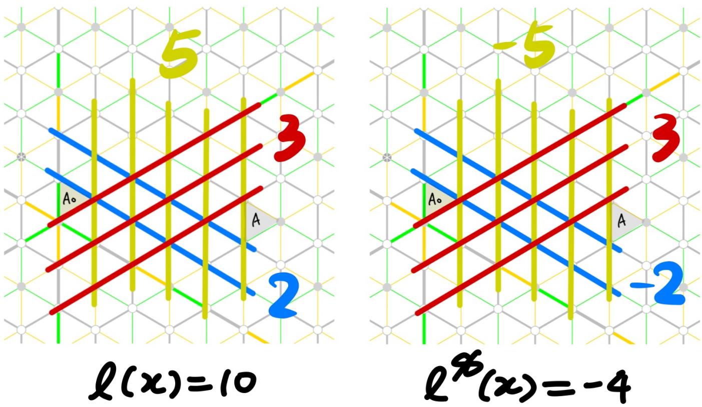
$$\begin{aligned} &\text{the hyperplane } H_{\alpha+k\delta} \text{ separates } A_0 + \mu \text{ and } x^{-1}A_0 \text{ for } \mu \text{ sufficiently dom} \\ \iff\ & x^{-1}A_0 \subset H_{\alpha+k\delta}^{<0} \iff A_0 \subset H_{x(\alpha+k\delta)}^{<0} \iff x(\alpha+k\delta) \in R_a^- \\ \iff\ & \alpha + k\delta \in \mathrm{Inv}_{\%}(x). \end{aligned}$$

As a result,

$$\ell^{\%}(x) = \sum_H \begin{cases} 1, & A_0 \subset H + C_0, \\ -1, & A_0 \subset H - C_0, \end{cases}$$

with the sum over hyperplanes H separating $x^{-1}A_0$ and A_0.

5.13. Example. Consider the case A_2.



5.14. Exercise. For $x \in W_e$, prove that
$$-\ell(x) \le \ell^{\frac{\infty}{2}}(x) \le \ell(x).$$

Semi-infinite Bruhat order.

5.15. Bruhat order. Recall the Bruhat order can be equivalently described by

- the order generated by
$$x < xr_{\hat{\alpha}} \quad \text{when} \quad \ell(xr_{\hat{\alpha}}) = \ell(x) + 1.$$
- the order generated by
$$x < xr_{\hat{\alpha}} \quad \text{when} \quad \ell(xr_{\hat{\alpha}}) > \ell(x).$$
That is, $\hat{\alpha} \in R_a^+$ and $x\hat{\alpha} \in R_a^+$.
- $x \le y$ if there is a subword of x in a reduced word of y.
- $x \le y$ if there is a subword of x in any reduced word of y.

5.16. Semi-infinite Bruhat order. For $x, y \in W_e$, we define the **semi-inifnite Bruhat order**
$$x \le_{\frac{\infty}{2}} y \iff xt_\mu \le yt_\mu \text{ for } \mu \in Q^\vee \text{ sufficiently dominant.}$$
The well-definedness follows from the description below. Note that unlike Bruhat order, $x \le_{\frac{\infty}{2}} y$ does not implies $x^{-1} \le_{\frac{\infty}{2}} y^{-1}$.

5.17. Description. The semi-inifnite Bruhat order can be equivalently described by

- the order generated by
$$x <_{\frac{\infty}{2}} xr_{\hat{\alpha}} \quad \text{when} \quad \ell^{\frac{\infty}{2}}(xr_{\hat{\alpha}}) = \ell^{\frac{\infty}{2}}(x) + 1.$$
- the order generated by
$$x <_{\frac{\infty}{2}} xr_{\hat{\alpha}} \quad \text{when} \quad \ell^{\frac{\infty}{2}}(xr_{\hat{\alpha}}) > \ell^{\frac{\infty}{2}}(x).$$
That is, $\hat{\alpha} \in R_{\frac{\infty}{2}}^+$ and $x\hat{\alpha} \in R_a^+$.
- $x \le y$ if there is a subword of xt_μ in a reduced word of yt_μ for sufficiently dominant μ.
- $x \le y$ if there is a subword of xt_μ in a reduced word of yt_μ for sufficiently dominant μ.

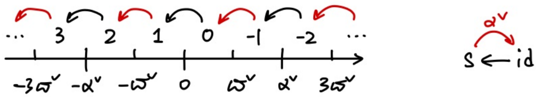
5.18. Exercise. Prove that

$$x \leq_{\%} y \iff xw_0 \geq_{\%} yw_0$$

where $w_0 = \max(W)$ the longest element in finite Weyl group.

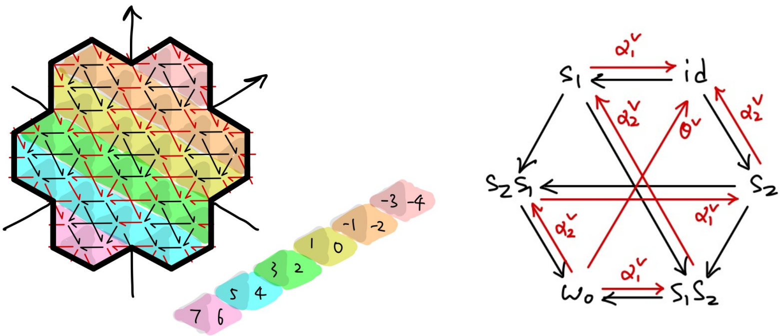
5.19. Remark. This order is also known as the **quantum Bruhat order**.

5.20. Example. Consider the case A_1.



As usual, we mark the semi-infinite length $\ell^{\%}(x)$ on $x^{-1}A_0$.

5.21. Example. Consider the case A_2.



Note that, we use $x^{-1}A_0$ to represent x, so

$$\text{left multiplication by } s_i = \text{wall-crossing}$$

5.22. Lemma. We have

$$\ell(r_\alpha) \leq 2\langle \rho, \alpha^\vee \rangle - 1.$$

Proof. Generally,

$$\rho = w\rho + \sum_{\beta \in \mathrm{LInv}(w)} \beta.$$

Substituting $w = r_\alpha$, we get

$$\langle \rho, \alpha^\vee \rangle \alpha = \sum_{\beta \in \mathrm{LInv}(r_\alpha)} \beta.$$

Note that $\beta \in \mathrm{Inv}(r_\alpha)$ implies

$$\beta - \langle \alpha^\vee, \beta \rangle \alpha < 0.$$

We must have $\langle \alpha^\vee, \beta \rangle \geq 1$. Note that $\alpha \in \mathrm{Inv}(r_\alpha)$, with $\langle \alpha^\vee, \alpha \rangle = 2$. Thus we get

$$2\langle \rho, \alpha^\vee \rangle = \sum_{\beta \in \mathrm{LInv}(r_\alpha)} \langle \beta, \alpha^\vee \rangle \geq \ell(r_\alpha) + 1.$$

This proves the inequality. □

5.23. Corollary. From the proof, for $\alpha \in R^+$, it is easy to see the inequality achieves

$$\ell(r_\alpha) = 2\langle \rho, \alpha^\vee \rangle - 1$$

exactly when

- α is long;
- the coefficient of each long simple root of α is 0.

In particular, it is always true for simply-laced types.

5.24. Computation. Let us give a more precise description of

$$x <_{\%} x r_{\hat{\alpha}} \quad \text{when} \quad \ell^{\%}(x r_{\hat{\alpha}}) = \ell^{\%}(x) + 1.$$

Firstly, the order the translation invariant, i.e.

$$x \leq_{\%} y \iff x t_\mu \leq_{\%} y t_\mu, \qquad \forall \mu \in P^\vee.$$

Let us assume $x = w \in W$, $\hat{\alpha} = \alpha + k\delta$ for $\alpha > 0$. Recall $r_{\alpha+k\delta} = r_\alpha t_{k\alpha^\vee}$. We have

$$\ell^{\%}(x r_{\alpha+k\delta}) - \ell^{\%}(x) = \ell(w r_\alpha) + 2k\langle \rho, \alpha^\vee \rangle - \ell(w) = 1.$$

So

$$2k\langle \rho, \alpha^\vee \rangle - 1 = \ell(w) - \ell(w r_\alpha).$$

We have
$$-2\langle\rho,\alpha^\vee\rangle+1\le -\ell(r_\alpha)\le 2k\langle\rho,\alpha^\vee\rangle-1\le \ell(r_\alpha)\le 2\langle\rho,\alpha^\vee\rangle-1.$$
Thus $-1<k\le 1$. When $k=0$, this is a cover relation in the finite Bruhat order
$$w<_{\%} wr_\alpha \qquad \text{when } \ell(xr_\alpha)=\ell(x)+1.$$
When $k=1$, the equality must be achieved, i.e.
$$w<_{\%} wr_\alpha t_{k\alpha^\vee} \qquad \text{when } \ell(xr_\alpha)=\ell(x)-\ell(r_\alpha) \text{ for } \alpha \text{ in } 5.23.$$

5.25. Theorem. The semi-infinite Bruhat order is generated by
$$\begin{aligned} wt_\lambda &<_{\%} wr_\alpha t_\alpha & \ell(wr_\alpha)&=\ell(w)+1 \\ wt_\lambda &<_{\%} wr_\alpha t_{\alpha+\alpha^\vee} & \ell(wr_\alpha)&=\ell(w)-\ell(r_\alpha) \text{ for } \alpha \text{ in } 5.23. \end{aligned}$$

Grassmannian elements.

5.26. Minimal representative. Let $x=wt_\lambda$. Recall that in 3.22, we get
$$x=\min(xW) \iff \lambda \text{ is anti-dominant and } w\in W^\lambda.$$
This is also true for extended Weyl group. A general facts of Weyl groups tells
$$x=\min(xW) \iff \mathrm{Inv}(x)\cap R^+=\varnothing.$$
From the computation of 3.5, we see that $x\in\min(xW)$ if and only if
$$\mathrm{Inv}(x)\subset R^-_{\%}.$$

5.27. Proposition. By 5.10, we have
$$\ell(x)=-\ell^{\%}(x) \iff x=\min(xW).$$

5.28. Example. Recall that $x=\min(xW)$ if and only if $x^{-1}A_0\subset C_0$. The above examples give examples of this theorem.

5.29. Lemma. When $x=\min(xW)$, any anti-dominant $\lambda\in P^\vee$, we have
$$xt_\mu=\min(xt_\lambda W), \qquad \ell(x)+\ell(t_\lambda)=\ell(xt_\lambda).$$
This is obvious from above description.

5.30. Theorem. When $x = \min(xW)$, for any $y \in W_e$

$$y \le x \implies y \ge_{\%} x,$$
$$y \le_{\%} x \implies y \ge x.$$

Proof. We have

$$\begin{aligned} y \le x &\implies yt_\lambda \le xt_\lambda && \text{for sufficiently anti-dominant } \lambda \\ &\implies yt_\lambda w_0 \le xt_\lambda w_0 && \text{for sufficiently anti-dominant } \lambda \\ &\implies yw_0 t_\mu \le xw_0 t_\mu && \text{for sufficiently dominant } \mu = w_0\lambda \\ &\implies yw_0 \le_{\%} xw_0 \\ &\implies y \ge_{\%} x && \text{(by 5.18).} \end{aligned}$$

$$\begin{aligned} y \le_{\%} x &\implies yw_0 \ge_{\%} xw_0 && \text{(by 5.18)} \\ &\implies yw_0 t_\mu \ge xw_0 t_\mu && \text{for sufficiently dominant } \mu \\ &\implies yt_\lambda w_0 \ge xt_\lambda w_0 && \text{for sufficiently anti-dominant } \lambda = w_0\mu \\ &\implies y \ge x && \text{(Lifting property below).} \end{aligned}$$

We are done. □

5.31. Lifting property. When $\ell(uv) = \ell(u) + \ell(v)$, we have

$$uv \le wv \implies u \le w.$$

Proof. It suffices to show when $v = s_i$. When $ws_i < w$, then $u \le us_i \le ws_i \le w$ it is obvious. When $ws_i > w$, then

$$(\text{a reduced word of } w) \oplus s_i$$

is a reduced word of ws_i. Since $us_i \le ws_i$, we can find a subword of us_i inside. If the last s_i is chosen, then drop it, we get $u \le w$. If the last s_i is not chosen, then $us_i \le w$, we also have $u \le w$. □

5.32. Corollary. For $x = \min(xW)$ and $y = \min(yW)$,

$$x \le y \iff x \ge_{\%} y.$$

5.33. Exercise. Prove that

$$xw_0 \leq_{\%} yw_0 \iff xt_\lambda \leq yt_\lambda \text{ for } \lambda \in Q^\vee \text{ sufficiently anti-dominant.}$$

When λ is sufficiently anti-dominant, $xt_\mu = \min(xt_\mu W)$ by above. So it is equivalent to say $xt_\lambda w_0 \leq yt_\lambda w_0$.

6. Combinatorics in type A

Remind.

6.1. Two presentations.

Recall that

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n \right\}.$$

Any $\lambda \in \mathbb{Z}^n$ defines a translation $t_\lambda(i) \in \tilde{\mathfrak{S}}_n$ by

$$t_\lambda(i) = i + \lambda_i n \qquad 1 \leq i \leq n-1.$$

This gives the first presentation

$$\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n.$$

Denote s_i for $i \in \mathbb{Z}/n\mathbb{Z}$ by

$$s_i = \begin{array}{c} \text{the affine permutation exchanging } j \text{ and } j+1 \\ \text{when } i \equiv j \bmod n \text{ with other numbers fixed} \end{array} \in \tilde{S}_n^0.$$

They generate the subgroup

$$\tilde{\mathfrak{S}}_n^0 = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{c} f(i+n) = f(i) + n \\ \frac{1}{n}\sum_{i=1}^n (f(i) - i) = 0 \end{array} \right\}.$$

Recall the element

$$\pi \in \tilde{\mathfrak{S}}_n, \qquad \text{given by } \pi(i) = i+1.$$

We have the second presentation

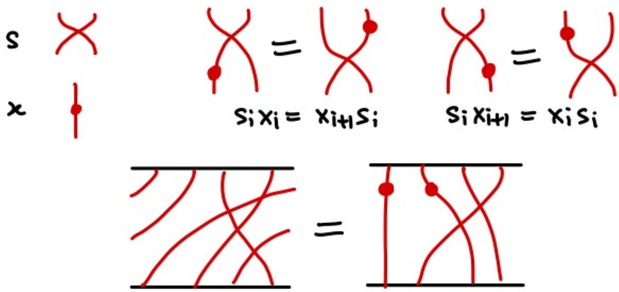
$$\tilde{\mathfrak{S}}_n = \pi^{\mathbb{Z}} \ltimes \tilde{\mathfrak{S}}_n^0.$$

6.2. Dot notation.

If we denote $x_i = t_{e_i}$, we have very explicit formula

$$x_i = (s_{i-1} \cdots s_1)\pi(s_{n-1} \cdots s_i).$$

There is another diagram notation for $\tilde{\mathfrak{S}}_n$.



6.3. Exercise. For any $f \in \tilde{\mathfrak{S}}_n$, prove the **average**

$$\mathrm{av}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(i) - i) \in \mathbb{Z}, \qquad \mathrm{av}(fg) = \mathrm{av}(f) + \mathrm{av}(g).$$

This proves $\mathrm{av} : \tilde{\mathfrak{S}}_n \to \mathbb{Z}$ defines a group homomorphism. Actually $\ker \mathrm{av} = \tilde{\mathfrak{S}}_n^0$ the affine Weyl group.

6.4. Length function. For $f \in \tilde{\mathfrak{S}}_n$, the length

$$\ell(f) = \#\left\{(i,j) : \begin{matrix} 1 \le i \le n-1 \\ i < j,\ f(i) > f(j) \end{matrix}\right\}.$$

Assume $f = wt_\lambda$, then

$$\ell(f) = \sum_{i<j} |\lambda_i - \lambda_j + \delta_{w(j)>w(i)}|.$$

In particular,

$$\ell(s_i) = 1, \qquad \ell(\pi) = 0, \qquad \ell(x_i) = n - 1$$

Actions of $\tilde{\mathfrak{S}}_n$.

6.5. Exercise. Prove that $P^\vee \cong W_e/W$ is an isomorphism of W_e-sets.

6.6. Remark. By the very definition, as a subgroup of $\mathfrak{S}_{\mathbb{Z}}$, the group $\tilde{\mathfrak{S}}_n$ acts on any objects indexed by $\mathbb{Z}$. Precisely, for any set X,

$$X^{\mathbb{Z}} = \{(\cdots, a_{-1}, a_0, a_1, \cdots), a_i \in X\},$$

the group $\tilde{\mathfrak{S}}_n$ acts by

$$f(\cdots, a_{-1}, a_0, a_1, \cdots) = (\cdots, a_{f^{-1}(-1)}, a_{f^{-1}(0)}, a_{f^{-1}(1)}).$$

That is, a_i is moves to the $f(i)$-th position, so the j-th entry is supposed to be $a_{f^{-1}(j)}$.

6.7. Action on $\mathbb{Z}^n$. The group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ linearly by

$$w(a_1, \ldots, a_n) = (a_{w^{-1}(1)}, \ldots, a_{w^{-1}(n)}).$$

We can extend this action non-linearly to $\tilde{\mathfrak{S}}_n$ by

$$wt_\lambda(a_1, \ldots, a_n) = (a_1 + \lambda_1, \ldots, a_n + \lambda_n).$$

This induces an isomorphism of $\tilde{\mathfrak{S}}_n$-set $\mathbb{Z}^n \overset{1:1}{\leftrightarrow} \tilde{\mathfrak{S}}_n/\mathfrak{S}_n$. In particular,

- since $s_0 = t_1 t_n^{-1} s_{1n}$,

  $$s_0(a_1, a_2 \ldots, a_{n-1}, a_n) = (a_n + 1, a_2, \cdots, a_{n-1}, a_1 - 1),$$

- since $\pi = t_1 s_1 \cdots s_{n-1}$,

  $$\pi(a_1, a_2, \ldots, a_{n-1}, a_n) = (a_n + 1, a_1, \cdots, a_{n-2}, a_{n-1}).$$

6.8. Example. Take $n = 3$. for simplicity, we denote $-m = \bar{m}$.

$$000 \xrightarrow{\pi} 100 \xrightarrow{s_1} 010 \xrightarrow{s_0} 11\bar{1} \xrightarrow{s_2} 1\bar{1}1 \xrightarrow{s_0} 2\bar{1}0 \xrightarrow{s_2} 20\bar{1}$$

6.9. Idenfication A. Actually, we can extend any n-tuple $(a_1, \ldots, a_n)$

$$\text{to} \quad (a_i)_{i \in \mathbb{Z}} \quad \text{by} \quad a_{kn+i} = a_i - k.$$

That is, we can embedding

$$\mathbb{Z}^n \xrightarrow{1:1} \left\{ (a_i)_{i \in \mathbb{Z}} : \begin{array}{c} a_i \in \mathbb{Z} \\ a_{i+n} = a_i - 1 \end{array} \right\} \subset \mathbb{Z}^{\mathbb{Z}}.$$

Then this can be includes in Remark 6.6 above. For example, ($\bar{k} = -k$)

$$\begin{array}{ccc}
20\bar{1} & \longmapsto & \cdots \mid 310 \mid 20\bar{1} \mid 1\bar{1}\bar{2} \mid \cdots \\
2\bar{1}0 & \longmapsto & \cdots \mid 301 \mid 2\bar{1}0 \mid 1\bar{2}\bar{1} \mid \cdots \\
1\bar{1}1 & \longmapsto & \cdots \mid 212 \mid 1\bar{1}1 \mid 0\bar{2}0 \mid \cdots \\
11\bar{1} & \longmapsto & \cdots \mid 221 \mid 11\bar{1} \mid 00\bar{2} \mid \cdots \\
010 & \longmapsto & \cdots \mid 121 \mid 010 \mid \bar{1}0\bar{1} \mid \cdots \\
100 & \longmapsto & \cdots \mid 211 \mid 100 \mid 0\bar{1}\bar{1} \mid \cdots \\
000 & \longmapsto & \cdots \mid 111 \mid 000 \mid \bar{1}\bar{1}\bar{1} \mid \cdots
\end{array}$$

Compare with the example above.

6.10. Identification B. For any n-tuple $(a_1, \ldots, a_n)$, we can associate a subset

$$A = t_a \mathbb{Z}_{<0} = \{i + (a_i - k)n : 1 \le i \le n,\ k < 0\} \subset \mathbb{Z}$$

Equivalently, we split $\mathbb{Z}$ into n copies of $\mathbb{Z}$ by

$$\mathbb{Z} \xrightarrow{1:1} (i + n\mathbb{Z}), \qquad d \longmapsto i + nd.$$

Then A is the union of the image of lower ideal $\{j < a_i\}$. This defines a bijection

$$\mathbb{Z}^n \xrightarrow{1:1} \left\{ A \subset \mathbb{Z} : \begin{array}{c} i \in A \Rightarrow i - n \in A \\ i \ll 0 \Rightarrow i \in A \\ i \gg 0 \Rightarrow i \in A \end{array} \right\} \subset 2^{\mathbb{Z}}.$$

Then this can be includes in Remark 6.6 above. For example

$$\begin{array}{l}
20\bar{1} \longmapsto \{\cdots, 4, \bar{1}, \bar{3}\} \\
2\bar{1}0 \longmapsto \{\cdots, 4, \bar{4}, 0\} \\
1\bar{1}1 \longmapsto \{\cdots, 1, \bar{4}, 3\} \\
11\bar{1} \longmapsto \{\cdots, 1, 2, \bar{3}\} \\
010 \longmapsto \{\cdots, \bar{2}, 2, 0\} \\
100 \longmapsto \{\cdots, 1, \bar{1}, 0\} \\
000 \longmapsto \{\cdots, \bar{2}, \bar{1}, 0\}
\end{array}
\qquad
\begin{array}{cccccccc}
\cdots & \bar{2} & \bar{1} & 0 & 1 & 2 & 3 & \cdots \\
\hline
\cdots & \bar{8} & \bar{5} & \bar{2} & 1 & 4 & 7 & \cdots \\
\cdots & \bar{7} & \bar{4} & \bar{1} & 2 & 5 & 8 & \cdots \\
\cdots & \bar{6} & \bar{3} & 0 & 3 & 6 & 9 & \cdots
\end{array}$$

6.11. Maya diagram. We represent any subset of $\mathbb{Z}$ by a $\mathbb{Z}$-tuple of $\{\oplus, \ominus\}$. That is, the a-th position is $\oplus$ if and only if $a \in A$. For example,

$$\begin{aligned}
20\bar{1} &\longmapsto \cdots \oplus\oplus\oplus\ominus|\oplus\ominus\ominus\oplus \cdots \\
2\bar{1}0 &\longmapsto \cdots \oplus\oplus\ominus\oplus|\oplus\ominus\ominus\oplus \cdots \\
1\bar{1}1 &\longmapsto \cdots \oplus\oplus\ominus\oplus|\oplus\ominus\oplus\ominus \cdots \\
11\bar{1} &\longmapsto \cdots \oplus\oplus\oplus\ominus|\oplus\oplus\ominus\ominus \cdots \\
010 &\longmapsto \cdots \oplus\oplus\oplus\oplus|\ominus\oplus\ominus\ominus \cdots \\
100 &\longmapsto \cdots \oplus\oplus\oplus\oplus|\oplus\ominus\ominus\ominus \cdots \\
000 &\longmapsto \cdots \oplus\oplus\oplus\oplus|\ominus\ominus\ominus\ominus \cdots
\end{aligned}$$

6.12. Partitions. The set

$$\left\{ A \subset \mathbb{Z} : \begin{array}{c} i \ll 0 \Rightarrow i \in A \\ i \gg 0 \Rightarrow i \in A \end{array} \right\} \subset 2^{\mathbb{Z}}$$

can be identified with the set of partitions with charges.

$$\{(\lambda, m) : \lambda \text{ is a partition}, m \in \mathbb{Z}\}.$$

We say (λ, m) is of center charge if $m = 0$. We identify usual partitions by a partition of center charge.

The identification is given by

$$(\lambda, m) \longmapsto \{m + 1 + \lambda_i - i : i = 1, 2, 3, \cdots\}.$$

For example,

$$(\varnothing, m) \longmapsto (\cdots, \overset{m-1}{\oplus}, \overset{m}{\oplus}, \overset{m+1}{\ominus}, \overset{m+2}{\ominus}, \cdots).$$

$$(\square, m) \longmapsto (\cdots, \overset{m-1}{\oplus}, \overset{m}{\ominus}, \overset{m+1}{\oplus}, \overset{m+2}{\ominus}, \cdots).$$

6.13. Residue. For a partition with charge (λ, m) and a box (i, j) in λ, we define

$$\operatorname{res}(\square) = j - i + m \in \mathbb{Z}$$

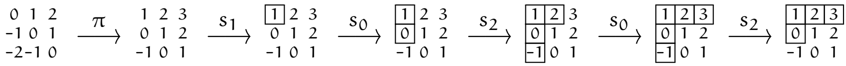
Then the action of $\tilde{\mathfrak{S}}_n$ translated to operators on partitions. By

$$\pi(\lambda, m) = (\lambda, m + 1), \qquad s_i(\lambda, m) = (s_i\lambda, m)$$

where

$$s_i\lambda = \lambda \cup \left\{ \square : \begin{array}{c} \square \text{ is addable} \\ \operatorname{res}(\square) \equiv i \bmod n \end{array} \right\} \setminus \left\{ \square : \begin{array}{c} \square \text{ is removable} \\ \operatorname{res}(\square) \equiv i \bmod n \end{array} \right\}.$$

6.14. Example. Take $n = 3$. We label the residues on the boxes.



6.15. Identification C. Under the above discussion, we can identify

$$\mathbb{Z}^n \xrightarrow{1:1} \tilde{\mathfrak{S}}_n \text{ orbit of } (\varnothing, 0) = \{(\lambda, m) : \lambda \text{ is } n\text{-core}\}.$$

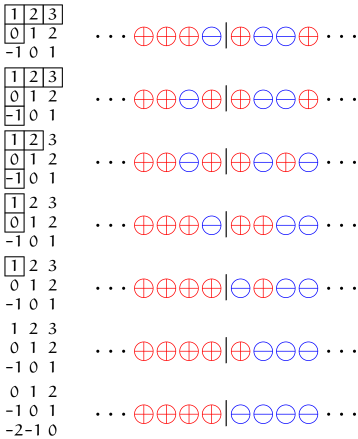
Moreover, if we restrict to

$$Q^\vee = \{(a_i)_{i\in\mathbb{Z}} : a_1 + \cdots + a_n = 0\} \subset \mathbb{Z}^n,$$

it gives

$$Q^\vee \xrightarrow{1:1} \tilde{\mathfrak{S}}_n^0 \text{ orbit of } (\varnothing, 0) = \{\lambda : \lambda \text{ is } n\text{-core}\}.$$

For example,



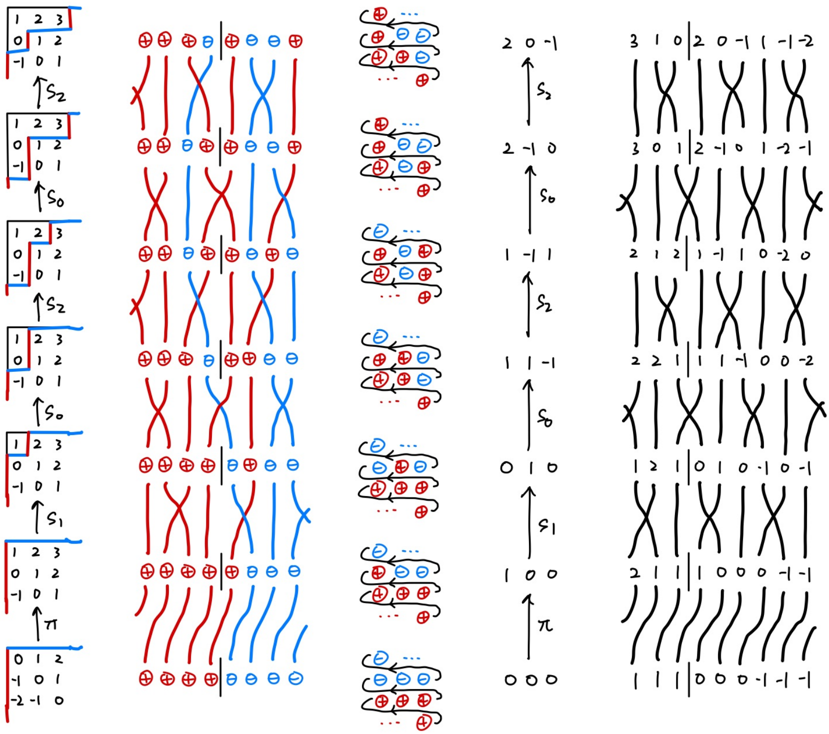
6.16. n-core partition. A partition λ is an n-core partition if there exists no $\mu \subseteq \lambda$ such that λ/μ is a ribbon of length n. Removing a ribbon of length n corresponds to the exchange

$$(\cdots \overbrace{\ominus \cdots \oplus}^{\text{distance} = n} \cdots) \longmapsto (\cdots \overbrace{\oplus \cdots \ominus}^{\text{distance} = n} \cdots).$$

As a result, if we cannot exchange, then the corresponding subset satisfies

$$i \in A \Longrightarrow i - n \in A.$$

6.17. Summary. Here is the summary of all three identification of $\mathbb{Z}^n$.



Minimal representatives.

6.18. Description. For any $f \in \tilde{\mathfrak{S}}_n^0$, it is clear that in the decomposition

$$f = uv, \qquad u = \min(fW) \text{ and } v \in \mathfrak{S}_n$$

we have

$$\begin{aligned} u(1) &= \min(f(1), \dots, f(n)), \\ \cdots &= \cdots \\ u(n) &= \max(f(1), \dots, f(n)). \end{aligned}$$

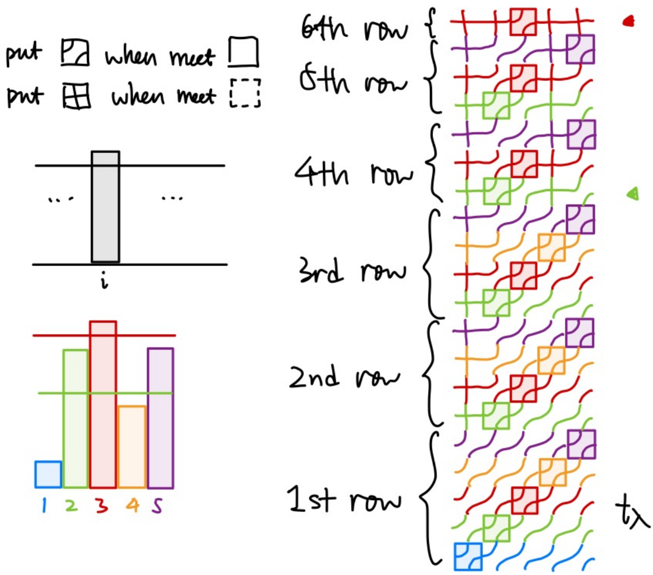
and

$$\begin{aligned} v(i) &= \text{the position of } f(i) \text{ in } \{f(1), \ldots, f(n)\} \\ &= 1 + \#\{j : f(j) < f(i)\}. \end{aligned}$$

Now, let us give a combinatorial description of

$$t_\lambda = u_\lambda v_\lambda, \qquad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

6.19. Description of t_λ. There is a combinatorial way of constructing t_λ as follows.



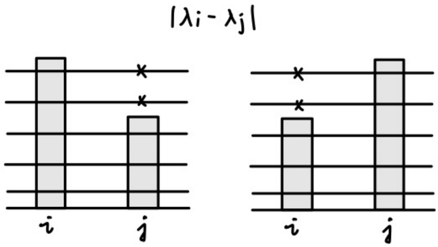
Actually, each row is a reduced word of t_μ with $\mu \in \{0,1\}^n$. For example, the above example is

$$t_{(1,5,6,3,5)} = t_{(0,0,1,0,0)} t^2_{(0,1,1,0,1)} t^2_{(0,1,1,1,1)} t_{(1,1,1,1,1)}.$$

Note that

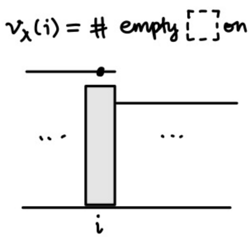
$$\ell(t_\lambda) = \sum_{i<j} |\lambda_i - \lambda_j|.$$

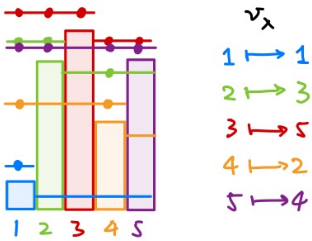
This is compatible:



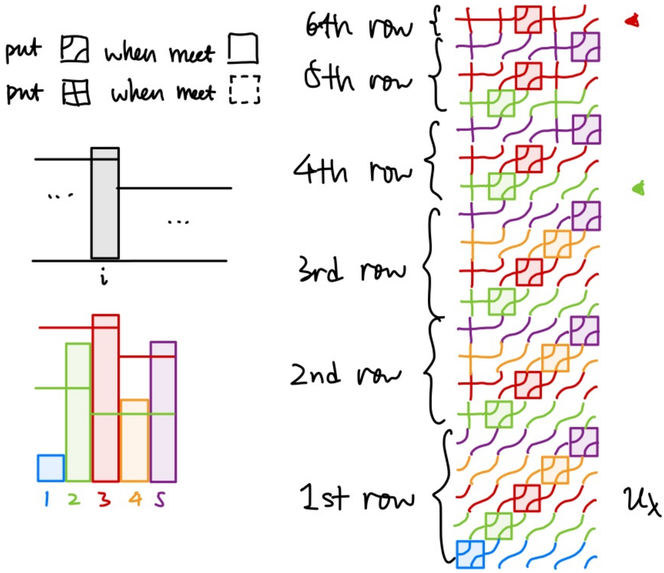
6.20. Description of ν_λ. When $f = t_\lambda$, then

$$\nu_\lambda(i) = 1 + \#\{j < i : \lambda_i \leq \lambda_j\} + \#\{j > i : \lambda_i < \lambda_j\}.$$





6.21. Description of u_λ. There is a combinatorial way of constructing u_λ as follows.



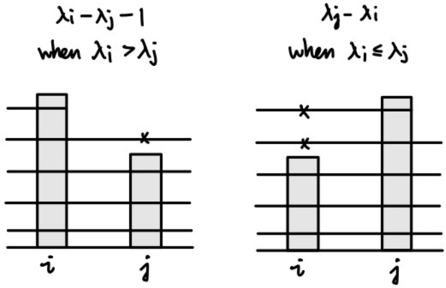
6.22. Compatible with length. Recall that the minimal representative minimizes each summand of

$$\ell(u_\lambda) = \ell(u_\lambda^{-1}) = \ell(\nu_\lambda t_{-\lambda}) = \sum_{i<j} |-\lambda_i + \lambda_j + \delta_{\nu_\lambda(j)>\nu_\lambda(i)}|.$$

That is,

$$\ell(u_\lambda) = \sum_{i<j} \begin{cases} \lambda_j - \lambda_i, & \lambda_i \leq \lambda_j, \\ \lambda_i - \lambda_j - 1, & \lambda_i > \lambda_j. \end{cases}$$

This is also compatible:

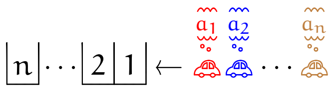


6.23. Remark. Let us identify compositions by a subset of $\mathbb{Z}^n$. Then each box (i, j) corresponds the minimal affine permutation by "creating this box", i.e. change the i-th component j-1 $\mapsto$ j. Using our identification A, the $(i$−$n)$-th component is j, we can just move it to the i-th component. When moving, we need $n - 1$ simple reflections, but once j meets another j, we do not need to exchange them, so it saves a simple reflection.

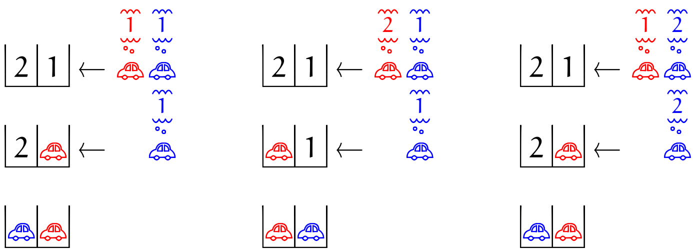
7. Funny bijections

Stable affine permutations.

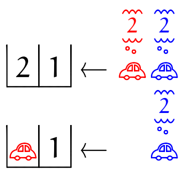
7.1. Stable affine permutations. The i-th car prefers the space a_i. If a_i is occupied, then the i-th takes the next available space. We call $(a_1, \cdots, a_n)$ a **parking function** (of length n) if all cars can park.



7.2. Example. For example, when $n = 2$, all parking functions $11, 21, 12$ are



While 22 is not:

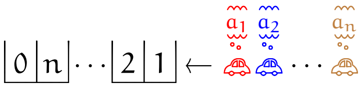


7.3. Example. For example, when $n = 3$,

111	112	121	211	113	131	311	122
212	221	123	132	213	231	312	321

7.4. Theorem. There are exactly $(n+1)^{n-1}$ parking functions.

Proof. Consider the following parking procedure:



on a circular road (i.e. the next space of $\boxed{0}$ is $\boxed{1}$). Then we see all cars can park for any preferences. It is a parking function if $\boxed{0}$ is left empty. The rotation symmetry tells that there are exactly $\frac{(n+1)^n}{n+1}$ many parking functions. $\square$

7.5. Equivalent condition. We see $(a_1, \ldots, a_n)$ is a parking function if and only if

- at most 1 car prefers position n;
- at most 2 cars prefer position $\geq n-1$;
- at most 3 cars prefer position $\geq n-2$;
- etc.

That is,

$$\{i : a_i \geq n-k+1\} \leq k.$$

In particular, does not depend on the order of cars.

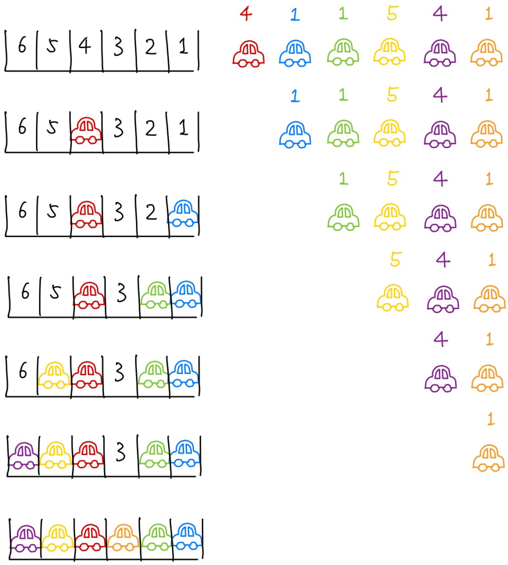
7.6. Dyck path. Dyck path (of length n) is a lattice path from $(0,0)$ to (n,n) below the diagonal. It is well-known that the number of Dyck path is Catalan number

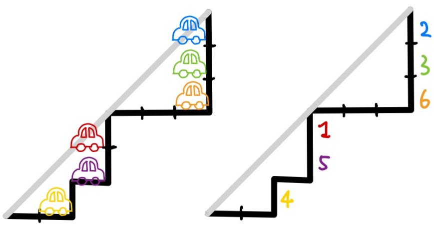
$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

7.7. Labelled Dyck path. A labeling of Dyck path is a labeling on vertical steps by $[n]$ which is increasing for consecutive vertical steps. It is not hard to find a bijection

$$\{\text{parking functions}\} \overset{1:1}{\longleftrightarrow} \{\text{labelled Dyck paths}\}.$$

That is, the labels on the vertical steps over $x = i$ gives a_{n+1-i}.





7.8. Stable affine permutation. We say $f \in \tilde{\mathfrak{S}}_n^0$ is stable if

$$f(i+n+1) = f(i+1) + n > f(i).$$

7.9. Enumeration. Note that every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ can be translated into $Q^\vee$

$$(x_1, \ldots, x_n) - \frac{x_1 + \cdots + x_n}{n}(1, \ldots, 1).$$

It would be convenient to work with

$$\mathfrak{h}^* = \{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{\oplus n}/\mathbb{R}(1, \ldots, 1).$$

The fundamental alcove is

$$A_0 = \{(x_1, \dots, x_n) \in \mathfrak{h}^* : x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - 1\}.$$

Its centroid is

$$\mathbf{c} = -\frac{1}{n}(1, 2, \cdots, n) \bmod (1, \dots, 1) \in A_0.$$

Explicit computation shows for $f \in \tilde{\mathfrak{S}}_n$,

$$f\mathbf{c} = -\frac{1}{n}(f(1), f(2), \cdots, f(n)) \bmod (1, \dots, 1) \in A_0.$$

So f is stable if and only if

$$\begin{aligned} f\mathbf{c} &\in \left\{ -\frac{1}{n}(x_1, \dots, x_n) : \begin{array}{c} x_{i+1} + n \geq x_i \\ x_1 + 2n \geq x_n \end{array} \right\} \\ &= \left\{ (x_1, \dots, x_n) : \begin{array}{c} x_i \geq x_{i+1} - 1 \\ x_n \geq x_1 - 2 \end{array} \right\} \\ &= \left\{ (x_1, \dots, x_n) : x_1 \geq x_2\text{-}1 \geq x_3\text{-}2 \geq \cdots > x_n\text{-}(n\text{-}1) \geq x_1\text{-}(n\text{+}1) \right\} \end{aligned}$$

The set is a union of alcoves, since it is bounded by hyperplanes. Moreover, its volume is the the volume of

$$(n+1)A_0 = \{(x_1, \dots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - (n+1)\}$$

(by a change of variable). This concludes

$$\#\{\text{stable permutations}\} = (n+1)^{n-1}.$$

7.10. Table of numbers. Let us fill all integers into $[n] \times \mathbb{Z}$ such that

- 0 is filled in $(0, 0)$ position;
- locally

$$\boxed{a}\boxed{b} \text{ then } b = a + n \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \text{ then } b = a + n\text{+}1.$$

For example, when $n = 3$,

⋯	-9	-6	-3	0	3	6	⋯
⋯	-5	-2	1	4	7	10	⋯
⋯	-1	2	5	8	11	14	⋯

.

7.11. A Dyck path. Let f be stable. Consider the set

$$\Delta = \{i \in \mathbb{Z} : f(i) > 0\}.$$

By periodicity and stability,

$$i \in \Delta \implies i + n \in \Delta, \quad i + n{+}1 \in \Delta.$$

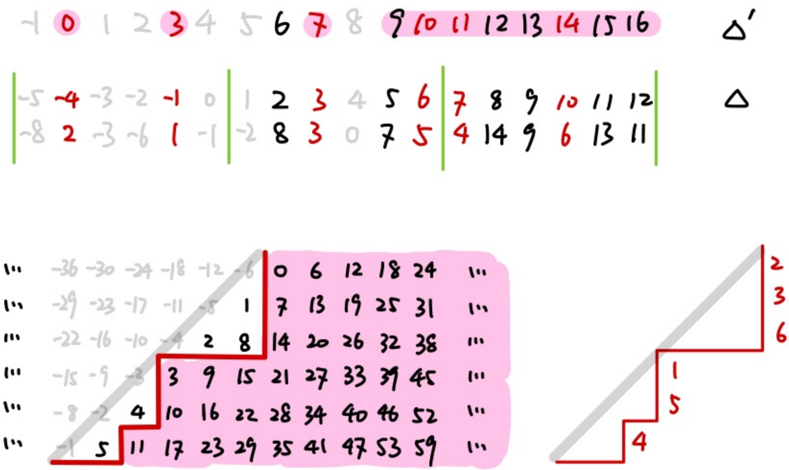
Let us shift Δ such that it has minimum 0:

$$\Delta' = \Delta - \min(\Delta).$$

Then coloring elements of Δ in the table above, we will get a Dyck path. We label $f(i + \min \Delta)$ on the leftmost colored box $\boxed{i}$ in each row. Note that the labels are exactly from $[n]$. This defines a bijection

$$\{\text{stable permutations}\} \overset{1:1}{\longleftrightarrow} \{\text{labelled Dyck paths}\}$$

7.12. Example.



7.13. References.

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Bounded affine permutations.

7.14. Setup. We call a permutation $w \in \mathfrak{S}_n$ an **k-Grassmannian permutation** if

$$w(1) < \cdots < w(k), \qquad w(k+1) < \cdots < w(n).$$

The set of k-Grassmannian permutation is in bijection with k-subset of $[n]$. In terms of Weyl group, we have

$$w = \min(wW_P), \qquad W_P = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n,$$

i.e. the set of k-Grassmannian permutations is W^P.

7.15. Description of Bruhat order in type A. For two k-subsets A, B of $[n]$, we define **Bruhat order**

$$A < B \iff \begin{cases} \min(A) < \min(B) \\ \cdots \\ \max(A) < \max(B) \end{cases}$$

Then obviously

$$A \leq B \iff [n] \setminus A \geq [n] \setminus B$$

7.16. Theorem. For two permutations $u, w \in \mathfrak{S}_n$,

$$u \leq w \iff u[k] \leq w[k] \text{ for all } 1 \leq k \leq n-1,$$

where $w[k] = \{w(1), \ldots, w(k)\}$.

7.17. Theorem. When w is Grassmannian,

$$u \leq w \iff \begin{cases} u(1) \leq w(1) \\ \cdots \\ u(k) \leq w(k) \end{cases} \text{and} \begin{cases} u(k{+}1) > w(k{+}1) \\ \cdots \\ u(n) > w(n) \end{cases}$$

Proof. Firstly

$$u[1] \leq w[1] \iff u(1) \leq w(1)$$

Since $w(1) < w(2)$,

$$u[2] \leq w[2] \overset{u(1)\leq w(1)}{\iff} u(2) \leq w(2).$$

Keep using this argument, we conclude the first set of condition is equivalent to

$$u[1] \leq w[1], \dots, u[k] \leq w[k].$$

Here is the diagram:

$$\begin{array}{ccccc} u(1) & u(2) & \cdots & u(k) \\ | \wedge & | \wedge & \cdots & | \wedge \\ w(1) < & w(2) < & \cdots < & w(k) \end{array}$$

Similarly, the the second set is equivalent to

$$u[k] \leq w[k], \dots, u[n-1] \leq w[n-1].$$

One need to notice that $[n] \setminus u[i] = u\{i+1, \dots, n\}$, and use the similar argument. □

7.18. Affine bounded permutation. We say $f \in \tilde{\mathfrak{S}}_n$ is a **bounded affine permutation** if

$$i \leq f(i) \leq i + n.$$

We say it is **k-affine permutation** if

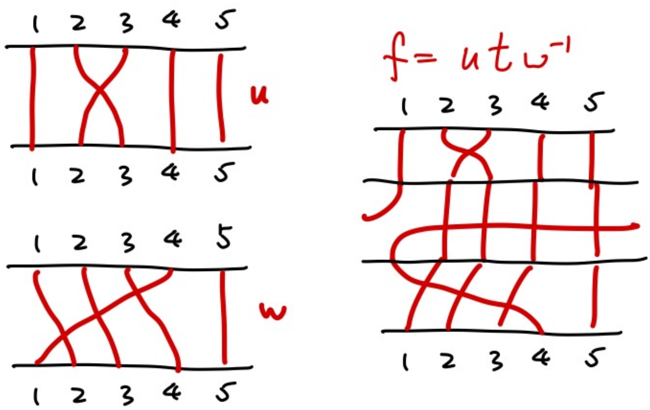
$$\mathrm{av}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k.$$

7.19. The map. Let us denote

$$\omega_k^\vee = e_1 + \cdots + e_k = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k}) \in \mathbb{Z}^n.$$

This is a lifting of the k-th fundamental coweight. We denote $t = t_{\varpi_k^\vee}$ for simplicity. For $(u, w) \in \mathfrak{S}_n \times \mathfrak{S}_n$, we define

$$f_{u,w} = utw^{-1} \in \tilde{\mathfrak{S}}_n.$$



7.20. Bijection. The map $(u, w) \mapsto f_{u,w}$ restricts to a bijection

$$\left\{ (u, w) : \begin{array}{c} u \leq w, \quad w \text{ is} \\ \text{k-Grassmannian} \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{c} \text{k-affine bounded} \\ \text{permutations} \end{array} \right\}.$$

Proof. The injectivity is not hard. Note that $vt = tv$ for any $v \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. We can decompose

$$\mathfrak{S}_n t \mathfrak{S}_n = \bigcup_{\lambda \in \mathfrak{S}_n \cdot \varpi_k^\vee} \mathfrak{S}_n t_\lambda = \bigcup_{w \text{ is k-Grassmannian}} \mathfrak{S}_n t w^{-1},$$

i.e. any element in WtW can be uniquely written as utw^{-1} where $u, w \in \mathfrak{S}_n$ and w is k-Grassmannian.

Let us check this is well-defined. Firstly, since $\mathrm{av}(t) = k$, so $\mathrm{av}(f_{u,w}) = k$. For $1 \leq i \leq n$, we have

$$f_{u,w}(i) = utw^{-1}(i) = \begin{cases} u(w^{-1}(i)) + n, & 1 \leq w^{-1}(i) \leq k, \\ u(w^{-1}(i)), & k+1 \leq w^{-1}(i) \leq n. \end{cases}$$

In the first case, write $a = w^{-1}(i)$, we have

$$f_{u,w}(i) = u(a) + n \begin{cases} \leq w(a) + n = i + n, \\ \geq 1 + n \geq i. \end{cases}$$

In the second case, write $b = w^{-1}(i)$, we have

$$f_{u,w}(i) = u(b) \begin{cases} \leq n \leq i + n, \\ \geq w(b) = i. \end{cases}$$

This proves the boundedness.

Lastly, let us check the surjectivity. Let f be a k-bounded affine permutation. Consider

$$A = \{i \in [n] : f(i) \geq n\}.$$

This must be an k-subset. The reason is the following: the bounded condition implies

$$\text{either } f(i) \in [n] \text{ or } f(i) - n \in [n].$$

The conditon $\mathrm{av}(f) = k$ implies there are exactly k many i satisifies $f(i) - n \in [n]$.

Let w be the k-Grassmannian permutation such that

$$\{w(1), \cdots, w(k)\} = A.$$

Then we define $u = fwt^{-1}$, so it rests to show $u \in \mathfrak{S}_n$ and $u \leq w$. Explicitly,

$$u(i) = \begin{cases} f(w(i)) - n, & 1 \leq i \leq k, \\ f(w(i)), & k{+}1 \leq i \leq n. \end{cases}$$

In other case the value is in $[n]$, so $u \in \mathfrak{S}_n$. In the first case,

$$u(i) = f(w(i)) - n \leq w(i) + n - n = w(i)$$

In the second case

$$u(i) = f(w(i)) \geq w(i).$$

Thus $u \leq w$. □

7.21. Length formula. We proved what when w is k-Grassmannian,

$$\ell(f_{u,w}) = \ell(u) + \ell(t) - \ell(w).$$

In our case, $\ell(t) = k(n-k)$, i.e.

$$\ell(f_{u,w}) = \ell(u) + k(n-k) - \ell(w).$$

In particular, for length to be maximal, it can only be

$$f_{w,w} = t_\lambda, \qquad \lambda = w\varpi_k^\vee.$$

7.22. Explicitly. Recall $\pi \in \tilde{S}_n$ the permutation of length 0:

$$\pi(i) = i + 1.$$

Let w_0^P be the maximal k-Grassmannian permutation, i.e.

$$w_0^P(1) = n - k + 1, \dots, w_0^P(k) = n,$$
$$w_0^P(k+1) = 1, \dots, w_0^P(n) = n - k.$$

We can write $t = \pi^k w_0^P$. Then

$$f_{u,w} = u \cdot \pi^k \cdot (w_0^P \cdot w^{-1})$$

is length additive:

$$\ell(f_{u,w}) = \ell(u) + 0 + \ell(w_0^P \cdot w^{-1}).$$

Note that

$$w \mapsto w_0^P w^{-1}$$

defines a order-reversed bijection between k-Grassmannian permutations and $(n-k)$-Grassmannian permutations.

7.23. Inversions. But it is still useful to compute the inversions. There are three types of inversions $(i < j)$. Denote $a = w^{-1}(i)$ and $b = w^{-1}(j)$.

- $a \le k < b \le n$. In this case $f(i) > n \ge f(j)$. They form the set
$$\left\{ (a, b) : \begin{matrix} 1 \le a \le k < b \le n \\ w(a) < w(b) \end{matrix} \right\}$$
- $a, b \le k$ or $k < a, b$. In this case we must have $a < b$ and $u(a) > u(b)$. They form the set
$$\left\{ (a, b) : \begin{matrix} 1 \le a, b \le k \text{ or } k < a, b \le n \\ u(a) > u(b) \end{matrix} \right\}$$
- $a \le k \le n + k < b$. In this case we must have $u(b) > u(a+n)$. They form the set
$$\left\{ (a, b+n) : \begin{matrix} 1 \le a \le k < b \le n \\ u(a) > u(b) \end{matrix} \right\}.$$

The contribution of the first type is $k(n-k) - \ell(w)$. The rest contributes $\ell(u)$.

7.24. Theorem. The set

$$\left\{ \begin{array}{c} \text{k-affine bounded} \\ \text{permutations} \end{array} \right\} = \left\{ f \in \mathfrak{S}_n : \begin{array}{c} f \le t_\lambda \text{ for some} \\ \lambda \in \mathfrak{S}_n \varpi_k^\vee. \end{array} \right\}$$

Proof. It suffices to show the left-hand-side is a lower ideal. Let f be an affine bounded permutation. We can pick a reduced word of f to be

$$(\text{a reduced word for } u)\pi^k(\text{a reduced word for } w_0^P w^{-1}).$$

Let f' be the affine permutation obtained by deleting a simple reflection and such that $\ell(f') = \ell(f) - 1$. We need to show f' is still an affine bounded permutation.

- If we delete from u. Then $f' = f_{u',w} = u'tw^{-1}$ for some $u' \le u \le w$.
- If we delete from $w_0^P w^{-1}$. Then we get $f' = f_{u,w'}$, where

  $$w' = ws_{ab} > w, \qquad a \le k < b, \qquad \ell(ws_{ab}) = \ell(w) + 1.$$

  This implies $w' = ws_{ab}$ is also k-Grassmannian. We have $u \le w \le w'$.

In both case, f' is a bounded affine permutation. □

7.25. References.

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