Positivity of Double Schubert Polynomials

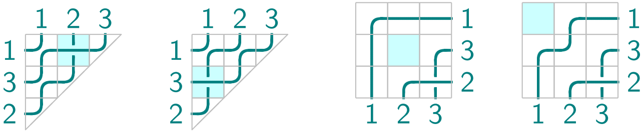
arXiv:2506.09421

Double Schubert polynomials

Double Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_\infty\}$ can be defined by

$$\boxed{\mathfrak{S}_{n\cdots 21} = \prod_{i+j\leq n}(x_i - y_j)} \qquad \boxed{\mathfrak{S}_w = \frac{\mathfrak{S}_{ws_i} - \mathfrak{S}_{ws_i}|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}} \quad \text{if } w(i) < w(i+1)}$$

Double Schubert polynomials can be computed via different combinatorial models including pipe dreams and bumpless pipe dreams. For example,



$$\mathfrak{S}_{132} = (x_1 - y_2) + (x_2 - y_1) = (x_2 - y_2) + (x_1 - y_1).$$

Monomial positivity of double Schubert polynomials

By [BB] and [LLS], we have

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\pi \in \mathsf{PD}(w)} \prod_{(i,j)=\boxplus} (x_i - y_j) = \sum_{\pi \in \mathsf{BPD}(w)} \prod_{(i,j)=\square} (x_i - y_j)$$

This implies the following monomial positivity.

Positivity

$\mathfrak{S}_w(\mathbf{x},\mathbf{y}) \in \mathbb{N}[x_i - y_j]_{i,j\geq 1}$.

Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. Experiment. Math., 2(4):257–269, 1993.

Thomas Lam, Seung Jin Lee, and Mark Shimozono. Back stable Schubert calculus. Compos. Math., 157(5):883–962, 2021.

Graham positivity of structure constants

Let us consider the structure constants

$$\mathfrak{S}_u(\mathbf{x};\mathbf{t}) \cdot \mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} c_{u,v}^{w}(\mathbf{t}) \cdot \mathfrak{S}_w(\mathbf{x};\mathbf{t}).$$

Finding a combinatorial model for these coefficients is the central open problem in (equivariant) Schubert calculus.

Graham [Gr] proved the following positivity

Positivity (Graham)

$c_{u,v}^{w}(\mathbf{t}) \in \mathbb{N}[t_i - t_j]_{i<j}$.

William Graham. Positivity in equivariant Schubert calculus. Duke Math. J., 109(3):599–614, 2001.

Positivity in Billey formulas

Billey [Bi] proved a combinatorial formula for

$$[\overline{B^- uB/B}]_T|_w = \mathfrak{S}_u(w\mathbf{y}; \mathbf{y}).$$

For a fixed reduced word $w = s_{i_1} \dots s_{i_\ell}$,

$$[\overline{B^- uB/B}]_T|_w = \sum_J \prod_{j \in J} s_{i_1} \dots s_{i_{j-1}}(\alpha_j)$$

with the sum over all reduced subword J of w for u.

Positivity (Billey)

$[\overline{B^- uB/B}]_T|_w \in \mathbb{N}[\alpha]_{\alpha \in \mathsf{Inv}(w)}$.

Sara C. Billey. Kostant polynomials and the cohomology ring for G/B. Duke Math. J., 96(1):205–224, 1999.

Positivity in triple Schubert calculus

Consider the structure constant in triple Schubert calculus [KT]

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y}) \cdot \mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} c_{u,v}^{w}(\mathbf{y},\mathbf{t}) \cdot \mathfrak{S}_w(\mathbf{x};\mathbf{t}).$$

Our main theorem is the following.

Positivity (Gao and X.; conjectured by Samuel [Sa])

$c_{u,v}^{w}(\mathbf{y},\mathbf{t}) \in \mathbb{N}[t_i - y_j]_{i,j \geq 1}$.

A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. 119 (2003), 221–260.

Matthew J. Samuel. A Molev-Sagan type formula for double Schubert polynomials. J. Pure Appl. Algebra, 228(7):Paper No. 107636, 39, 2024.

Examples of triple Schubert (1)

Let us give some examples of triple Schubert calculus.

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y})\cdot\mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w\in S_\infty} c_{u,v}^{w}(\mathbf{y},\mathbf{t})\cdot\mathfrak{S}_w(\mathbf{x};\mathbf{t}).$$

The main reference for this part is [FGX] and [Sa].

(1) When $v = w = \mathrm{id}$, by setting $\mathbf{x} = \mathbf{t}$, we get

$$c_{u,v}^{w}(\mathbf{y},\mathbf{t}) = \mathfrak{S}_u(\mathbf{t};\mathbf{y}).$$

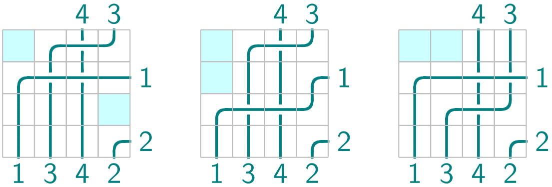
Neil J. Y. Fan, Peter L. Guo, and Rui Xiong. Bumpless pipe dreams meet puzzles. Adv. Math., 463:Paper No. 110113, 29, 2025.

Matthew J. Samuel. A Molev-Sagan type formula for double Schubert polynomials. J. Pure Appl. Algebra, 228(7):Paper No. 107636, 39, 2024.

Examples of triple Schubert (2)

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y}) \cdot \mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} c_{u,v}^{w}(\mathbf{y},\mathbf{t}) \cdot \mathfrak{S}_w(\mathbf{x};\mathbf{t}).$$

(2) When $\mathrm{maxdes}(u) \leq \mathrm{mindes}(v)$, there is a combinatorial formula in term of "pipe puzzles" [FGX]; see also [Sa].



It includes the cases considered in [MS].

A.I. Molev and B.E. Sagan, A Littlewood–Richardson rule for factorial Schur functions, Trans. Amer. Math. Soc. 351 (1999), 4429–4443.

Examples of triple Schubert (3)

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y})\cdot\mathfrak{S}_v(\mathbf{x};\mathbf{t})=\sum_{w\in S_\infty} c_{u,v}^w(\mathbf{y},\mathbf{t})\cdot\mathfrak{S}_w(\mathbf{x};\mathbf{t}).$$

(3) When $u=s_{k-r+1}\cdots s_k$, i.e.

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y})=s_{1^r}(x_1,\ldots,x_k;y_1,\ldots,y_n).$$

is a factorial Schur polynomial of a column shape. Then we have triple Pieri formula that

$$c_{u,v}^w(\mathbf{y},\mathbf{t})=0 \text{ or } s_{1^{r-\ell}}(t_{p(1)},t_{p(2)},\ldots;y_1,\ldots,y_n),$$

for some $p\in S_n$; see [Sa, FGX].

Neil JY Fan, Peter L Guo, and Rui Xiong. Pieri and Murnaghan–Nakayama type rules for Chern classes of Schubert cells. arXiv preprint arXiv:2211.06802, 2022.

Examples of triple Schubert (4)

(4) As explained in [FGX] and [Sa], we have

$$\partial_{w/v}\mathfrak{S}_u(\mathbf{x}) = c^w_{u,v}(\mathbf{0}, \mathbf{x})$$

where $\partial_{w/v}$ is the *skew divided difference operator* [Ma].

In particular, we proved Kirillov's conjecture [Ki].

Theorem (Gao and X.; conjectured by [Ki])

$\partial_{w/v}\mathfrak{S}_u(\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$.

I. G. Macdonald. Schubert polynomials. In Surveys in combinatorics, 1991 (Guildford, 1991), volume 166 of London Math. Soc. Lecture Note Ser., pages 73–99. Cambridge Univ. Press, Cambridge, 1991.

Anatol N. Kirillov. Skew divided difference operators and Schubert polynomials. SIGMA Symmetry Integrability Geom. Methods Appl., 3:Paper 072, 14, 2007.

Graham positivity Theorem

Theorem (Graham positivity [Gr])

Let B act on a non-singular variety X. If Y is a T-invariant effective cycle in X, then there exist B-invariant effective cycles $Z_1, \ldots, Z_m$ such that

$$[Y]_T \in \sum_{i=1}^{m} \mathbb{N}[\alpha]_{\alpha \in \Delta^+} \cdot [Z_i]_T$$

in $H_T^*(X)$, where Δ is the set of T-weights of $\mathrm{Lie}\, U$.

When $X = G/B$ and $Y = \overline{BwB/B} \cap \overline{B^- uB/B}$, this implies Graham's positivity for structure constants. Note that Δ^+ is the set of positive roots in a root system.

A refined version

Let $w \in W$ be a Weyl group element. Define the following closed subgroups of G

$$N(w) = N \cap wNw^{-1}, \qquad B(w) = T \ltimes N(w).$$

Our refined version of Graham positivity theorem is

Theorem (Gao, X.)

Let B act on a non-singular variety X. If Y is a $B(w)$-invariant effective cycle in X, then there exist B-invariant effective cycles $Z_1, \ldots, Z_m$ such that

$$[Y]_T \in \sum_{i=1}^{m} \mathbb{N}[\alpha]_{\alpha \in \mathsf{Inv}(w)} \cdot [Z_i]_T$$

in $H_T^*(X)$.

Positivity via degeneration

Let us remind the proof in [Gr]; see also [AF, 19.4.4]. Assume we are given

$$\begin{array}{ccc} & & T \\ & & \cap \\ B' & \subset & B \\ \cup & & \cup \\ U' & \underset{\chi}{\subset} & U \end{array}$$

- $B = T \ltimes U$ is a solvable group with maximal torus T and unipotent radical U;
- $U' \subset U$ is a normal subgroup of B with $\dim U/U' = 1$, and $B' = T \ltimes U'$;
- the T-weight of Lie $U/$ Lie U' is $\chi \in X^*(T)$.

David Anderson and William Fulton. Equivariant cohomology in algebraic geometry, volume 210 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2024.

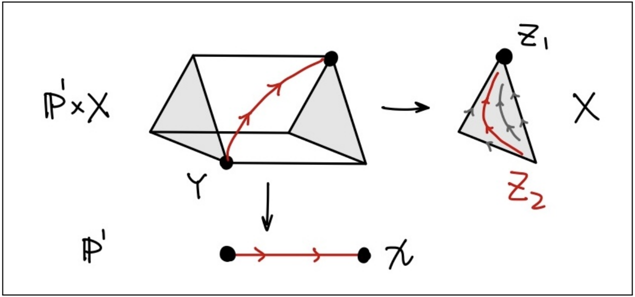
Positivity via degeneration (cont')

Lemma ([Gr]; see also [AF, 19.4.4])

Let B act on a non-singular variety X. If Y is a B'-invariant effective cycle in X, then there exist B-invariant effective cycles Z_1, Z_2 such that

$$[Y]_T = [Z_1]_T + \chi[Z_2]_T$$

in $H_T^*(X)$.



Heights of roots

For a unipotent group, we can always find a normal series. So Graham's positivity theorem follows from induction. We need an explicit choice of normal series. There are two ways.

For a positive root α in a root system Δ^+, its *height* is

$$\mathrm{ht}(\alpha) = \sum_{i \in I} c_i > 0, \qquad \text{if } \alpha = \sum_{i \in I} c_i \alpha_i.$$

Let us enumerate the positive roots $\beta_1, \dots, \beta_{\ell(w_0)}$ such that $\mathrm{ht}(\beta_1), \dots, \mathrm{ht}(\beta_{\ell(w_0)})$ is decreasing. Then we can construct a normal series

$$\{1\} = N_0 \subset N_1 \subset \cdots \subset N_{\ell(w_0)} = N$$

such that $\mathrm{Lie}\, N_i = \mathsf{span}(E_{\beta_1}, \dots, E_{\beta_i}) \subset \mathrm{Lie}\, G$. Note that each N_i is a normal subgroup of B.

Reflection order

Recall a *reflection order* of a root system Δ is a total order $\leq$ on the positive roots Δ^+ such that

$$\text{if } \alpha < \beta \text{ such that } \alpha + \beta \in \Delta^+, \text{ then } \alpha < \alpha + \beta < \beta.$$

Assume $\Delta^+ = \{\beta_1 < \cdots < \beta_{\ell(w_0)}\}$. We can construct a normal series

$$\{1\} = N_0 \subset N_1 \subset \cdots \subset N_{\ell(w_0)} = N$$

such that $\mathrm{Lie}\, N_i = \mathrm{span}(E_{\beta_1}, \ldots, E_{\beta_i}) \subset \mathrm{Lie}\, G$. Note that N_i is *NOT* necessarily a normal subgroup of N, since normal subgroup is not a transitive relation.

Actually, we only need the inequality $<$ to show N_{i-1} is a normal subgroup of N_i.

Example ($G = GL_4$)

Using heights of positive roots, we can construct a series like

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&*\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&*&*\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&*&*\\&1&&*\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&&*\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&*&*\\&&1&*\\&&&1\end{bmatrix}$$

Recall that a T-equivariant normal subgroup of $N \subset GL_n$ corresponds to a Dyck path. The normal series is given by a "flag" of Dyck paths.

On the other hand, the reflection order

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4.$$

gives the following series

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&*&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&*&*&*\\&1&*&*\\&&1&*\\&&&1\end{bmatrix}.$$

The proof of the refined version

Theorem (Gao, X.)

Let B act on a non-singular variety X. If Y is a $B(w)$-invariant effective cycle in X, then there exist B-invariant effective cycles $Z_1, \ldots, Z_m$ such that

$$[Y]_T \in \sum_{i=1}^{m} \mathbb{N}[\alpha]_{\alpha \in \mathsf{Inv}(w)} \cdot [Z_i]_T$$

in $H_T^*(X)$.

For a Weyl group element $w \in W$, we can always find some reflection order such that

$$\text{positive roots not in } \mathsf{Inv}(w) < \text{roots in } \mathsf{Inv}(w).$$

Moreover, the group $N(w) = N \cap wNw^{-1}$ is exactly the corresponding subgroup at this "$<$".

Positivity in Billey's formula (sketch)

The T-fixed points of G/B are wB/B for $w \in W$. The key observation is

$$\boxed{\text{the fixed point } wB/B \text{ is } B(w)\text{-invariant}}.$$

Applying our theorem to $[wB/B]_T$, we have

$$[wB/B]_T \in \sum_{u \in W} \mathbb{N}[\alpha]_{\alpha \in \mathsf{Inv}(w)} \cdot [\overline{BuB/B}]_T.$$

By applying the Poincaré pairing with $[\overline{B^- uB/B}]_T$, we get

$$[\overline{B^- uB/B}]_T|_w \in \mathbb{N}[\alpha]_{\alpha \in \mathsf{Inv}(w)}.$$

This proves the positivity in Billey's formula.

Positivity in Triple Schubert calculus (sketch)

The trick is, we need to work with $G = GL_{2n}$, where we rename the equivariant parameters $y_1 = t_{n+1}, \ldots, y_n = t_{2n}$ in $H_T^*(\mathrm{pt})$. Then we can translate the left-hand side of

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y}) \cdot \mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} c_{u,v}^{w}(\mathbf{y},\mathbf{t}) \cdot \mathfrak{S}_w(\mathbf{x};\mathbf{t})$$

as a generically transversal intersection which is invariant under the following subgroup $\left[\begin{array}{c|c} \begin{matrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{matrix} & \\ \hline 0 & \begin{matrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \end{matrix} \end{array}\right]$. This group is a transposition of $B(\tau)$ for some $\tau \in S_{2n}$, and

$$\mathrm{Inv}(\tau) = \{-t_i + y_j : 1 \leq i, j \leq n\}.$$

The positivity follows from (a transposed version of) our theorem.

Open problems

- Similar question could be asked for double Grothendieck polynomials

$$\mathfrak{G}_u(x;y)\cdot\mathfrak{G}_v(x;t)=\sum_{w\in S_\infty}\tilde{c}_{u,v}^{w}(y,t)\cdot\mathfrak{G}_w(x;t).$$

We conjecture that $\tilde{c}_{u,v}^{w}(y,t)\in\mathbb{N}[\beta][t_i\ominus y_j]_{i,j\geq 1}$.

- Similar question for quantum double Schubert polynomials

$$\mathfrak{S}_u^q(x;y)\cdot\mathfrak{S}_v^q(x;t)=\sum_{w\in S_\infty}\sum_{d\geq 0}c_{u,v}^{w,d}(y,t)\cdot q^d\mathfrak{S}_w^q(x;t).$$

We conjecture that $c_{u,v}^{w,d}(y,t)\in\mathbb{N}[t_i-y_j]_{i,j\geq 1}$.

- Similar question for quantum Grothendieck polynomials.


5 Ways to Live a
More Positive Life
1. Tell yourself positive things
2. Chat with positive people
3. Do Schubert calculus
4. Do double Schubert calculus
5. Do triple Schubert calculus
Kimi AI 生成