Toric Varieties and their Applications

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`date: April 24, 2024`

0 Plan

This is a rough plan for the Reading group "**Toric Varieties and its Applications**". The final purpose is to understand the proof of **Read's conjecture** stating that the absolute value of coefficients of the **chromatic polynomial of a graph is unimodal (sinkless)**. The proof is established in a series of papers with main ideas originally from **algebraic geometry**, e.g.

- June Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc. [arXiv:1008.4749]
- June Huh, Eric Katz, Log-concavity of characteristic polynomials and the Bergman fan of matroids 2011, Mathematische Annalen. [arXiv:1104.2519]
- Karim A. Adiprasito, June Huh, Eric Katz. Hodge theory for combinatorial geometries, Annals Of Mathematics 2015. [arXiv:1511.02888]

See also the following survey

- Eric Katz, Matroid theory for algebraic geometers. [arXiv:1409.3503]

Note that June Huh was awarded the **Fields Medal** in 2022 due to the mentioned work and more work in this direction.

The main geometric object is toric variety, a sort of algebraic variety parametrized by combinatorial objects — fans. It provides many examples (and counterexamples) in algebraic geometry. The standard book of toric variety is

- W. Fulton. Introduction to Toric Varieties.

Most of the combinatorial applications (including the proof we promised) of toric variety use **Hodge theory**. For general information on combinatorial applications, we recommend

- R.P. Stanley. Combinatorial Applications of the Hard Lefschetz Theorem.
- J. Huh. Combinatorial Applications of the Hodge–Riemann Relations.

My plan is the following,

- I will give the talk for each meet except when any audience wants to share. Actually, it seems to me that a detailed note directed to this topic is not yet written.

  In other words, you are not supposed to give any talk.

- To maximize the possible achievement each time, I will first review the algebraic geometry I will use later.

  In other words, if you are combinatorics-allergic and literally lost in the discussion, at least, you would know some algebraic geometry.

- There will be an **offline** talk each week. The exact time and room will be announced later. It would probably be the **unit just before Joint Ottawa/Carleton Algebra Seminar**.

- We need to require basic knowledge of algebra geometry, for example,

  $\boxed{\textbf{Try to figure out which ring } R \textbf{ such that } \operatorname{Spec} R = \mathbb{C} \setminus \{0\}}$.

> Actually, we will mainly use the complex algebraic geometry, so not absolutely the style of Hartshorne.

If you are willing to join or have questions, please contact me: `rxion043@uOttawa.ca`.

1 Introduction and Constructioins

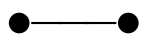
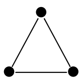
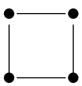
Introduction

1.1. Chromatic polynomials Let G be a **graph**. You definitely know the story of four-color theorem. Historically, Birkhoff defined the **chromatic polynomial** of a graph in an attempt to prove the four color theorem. To be exact, chromatic polynomial χ_G is the unique polynomial such that

$$\chi_G(q) = \#\{\text{vertex } q\text{-colorings of } G\}.$$

It is a good exercise to prove this number is polynomial-dependent in q.

1.2. Example Here we list some examples

Graph	chromatic polynomial	remarks
	$\chi_G(q) = q$	For G with n isolated vertices, $\chi_G(q) = q^n$.
	$\chi_G(q) = q(q-1) = q^2 - q$	More edges between two vertices do not affect the polynomial.
	$\chi_G(q) = q(q-1)(q-2)$ $= q^3 - 3q^2 + 2q$	More general, we can find the chromatic polynomial for K_n.
	$\chi_G(q)$ $= q(q-1)(q-2)^2+q(q-1)^2$ $= q^4 - 4q^3 + 6q^2 - 3q$	A good exercise is to find the chromatic polynomial of an n-gon.

1.3. Coefficients of chromatic polynomials Assume

$$\chi_G(q) = \mu_0 + \mu_1 q + \cdots + \mu_n q^n.$$

By a pure algebraic approach, one can show that the sign is alternating. **Read's conjecture** says

$$|\mu_0| \le \cdots \le |\mu_{k-1}| \le |\mu_k| \ge |\mu_{k+1}| \ge \cdots \ge |\mu_n|,$$

i.e. **unimodal**. This conjecture was first proved by Huh using algebraic geometry. At the end of semester, we will discuss a simplified proof by Huh and Katz. The vital geometric object in the proof is **toric variety**. The proof goes as follows

- We first construct a subvariety in the permutohedral variety encoding information of the graph.
- By computing the product with two **nef** line bundles, we will translate $|\mu_i|$ as an intersection number.
- Apply **Hodge index theorem** to conclude $\det \begin{bmatrix} |\mu_{i-1}| & |\mu_i| \\ |\mu_i| & |\mu_{i+1}| \end{bmatrix} < 0$. In particular, the sequence $|\mu_i|$ is unimodal.

In particular, the knowledge of line bundles and cohomology ring of toric variety is necessary. On the geometric side, positivity of algebraic geometry (e.g. properties of **nef bundles**) and Hodge theory (e.g. Hodge–Riemman relations) play important roles. We will deal with them step by step and we will also meet other applications. Today's topic is the construction of toric varieties.

Definitions and Constructions

Assumption We fix an algebrally closed field $\Bbbk$. We will identify a variety by its closed points.

1.4. Semigroup ring Let Q be a subset of $\mathbb{Z}^N$ of the form

$$Q = \mathbb{Z}_{\geq 0}\mathbf{u}_1 + \cdots + \mathbb{Z}_{\geq 0}\mathbf{u}_n \subseteq \mathbb{Z}^N.$$

Note that Q is a **monoid**. We can formulate the **semigroup ring** over a field $\Bbbk$,

$$\Bbbk[Q] := \Bbbk[x^{\mathbf{u}}]_{\mathbf{u}\in Q} \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_N^{\pm 1}],$$

which is a commutative $\Bbbk$-algebra. Note that it is finitely generated and integral.

1.5. Example Let us see some example when $N = 1$.

- When $Q = 0$, then $\Bbbk[Q] = \Bbbk$.
- When $Q = \mathbb{Z}_{\geq 0}$, then $\Bbbk[Q] = \Bbbk[x]$ the ring of polynomials.
- When $Q = \mathbb{Z}_{\leq 0}$, then $\Bbbk[Q] = \Bbbk[x^{-1}]$, isomorphic the ring of polynomials.
- When $Q = \mathbb{Z}$, then $\Bbbk[Q] = \Bbbk[x^{\pm}]$ the ring of Laurant polynomials.

- In general:
  - if $Q = d\mathbb{Z}_{\geq 0}$, for some $d \in \mathbb{Z} \setminus 0$, then $\Bbbk[Q] = \Bbbk[x^d]$, isomorphic to the ring of polynomials.
  - If $Q = d\mathbb{Z}$, for some $d \in \mathbb{Z} \setminus 0$, then $\Bbbk[Q] = \Bbbk[x^{\pm d}]$, isomorphic to the ring of Laurant polynomials.

1.6. Affine Toric Varieties are nothing but $\operatorname{Spec} \Bbbk[Q]$. Recall that

$$\begin{aligned}\operatorname{Spec} \Bbbk[Q] &= \operatorname{Hom}_{\Bbbk\text{-}\mathsf{Alg}}(\Bbbk[Q], \Bbbk) = \operatorname{Hom}_{\mathsf{Monoid}}(Q, \Bbbk) \\ &= \left\{ Q \xrightarrow{f} \Bbbk : \begin{array}{l} f(\mathbf{0}) = 1 \\ f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1) f(\mathbf{u}_2) \end{array} \right\}.\end{aligned}$$

1.7. Example For example, for any $d \in \mathbb{Z} \setminus 0$

$$\operatorname{Spec} \Bbbk[x^d] \cong \Bbbk =: \mathbb{A}^1 \qquad \operatorname{Spec} \Bbbk[x^{\pm d}] \cong \Bbbk \setminus 0 =: \mathbb{G}_m.$$

For example, compare $2\mathbb{Z}_{\geq 0}$ and $3\mathbb{Z}$

		-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6		
$\cdots$	$\cdots$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	1	$\square$	z	$\square$	z^2	$\square$	z^2	$\cdots$	$\cdots$

		-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6		
$\cdots$	$\cdots$	$\frac{1}{z^2}$	$\square$	$\square$	$\frac{1}{z}$	$\square$	$\square$	1	$\square$	$\square$	z	$\square$	$\square$	z^2	$\cdots$	$\cdots$

It seems that d does not play an role in the theory. But actually the morphisms induced by $\Bbbk[Q] \to \Bbbk[x^{\pm 1}]$ are different when d varies.

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk[x^{\pm 1}] \arrow[r] \arrow[d, equal] & \operatorname{Spec} \Bbbk[x^d] \arrow[r, "\sim"] & \operatorname{Spec} \Bbbk[t] \arrow[d, equal] \\
\Bbbk \setminus 0 \arrow[rr, "x \mapsto x^d"] & & \Bbbk.
\end{tikzcd}$$

When $d = \pm 1$, the morphism $[x \to x^d]$ is injective. We can recognize

$$\begin{aligned}\operatorname{Spec} \Bbbk[x] &= \Bbbk, \\ \operatorname{Spec} \Bbbk[x^{\pm 1}] &= \Bbbk \setminus 0, \\ \operatorname{Spec} \Bbbk[x^{-1}] &= \Bbbk \setminus 0 \sqcup \{\infty\}.\end{aligned}$$

1.8. Torus action Denote $\mathbb{G}_m = \Bbbk^\times$ the multiplication group. Let $T = \mathbb{G}_m^N$ be a torus. Note that an action of T on $\operatorname{Spec}\Bbbk[Q]$ is nearly the same thing as a $\mathbb{Z}^N$-grading on $\Bbbk[Q]$. We have a natural action $T \curvearrowright \Bbbk[Q]$ by

$$x^{\mathbf{u}} \xmapsto{\mathbf{z}\in T} (zx)^{\mathbf{u}} = z^{\mathbf{u}} x^{\mathbf{u}}.$$

Due to geometric reason, we shall view it as a **right action**. The T-action on $\operatorname{Spec}\Bbbk[Q]$ can be translated to be the following. For $\mathbf{z} \in T$ and $f \in \operatorname{Spec}\Bbbk[Q]$,

$$(\mathbf{z}\cdot f)(\mathbf{u}) = z^{\mathbf{u}} f(\mathbf{u}).$$

We call $\operatorname{Spec}\Bbbk[Q]$ an **affine toric variety**.

For example, the image of

$$\begin{array}{ccccccccccccccccc} & & {\scriptstyle -6} & {\scriptstyle -5} & {\scriptstyle -4} & {\scriptstyle -3} & {\scriptstyle -2} & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & & \\ \cdots & \cdots & \square & \square & \square & \square & \square & \square & 1 & \square & 2 & \square & 4 & \square & 8 & \cdots & \cdots \end{array}$$

under the action of $t \in T = \mathbb{G}_m$ is

$$\begin{array}{ccccccccccccccccc} & & {\scriptstyle -6} & {\scriptstyle -5} & {\scriptstyle -4} & {\scriptstyle -3} & {\scriptstyle -2} & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & & \\ \cdots & \cdots & \square & \square & \square & \square & \square & \square & 1 & \square & 2t^2 & \square & 4t^4 & \square & 8t^6 & \cdots & \cdots \end{array}$$

1.9. Example Note that

$$\operatorname{Spec}\Bbbk[\mathbb{Z}^N] = \operatorname{Hom}_{\mathsf{Mnd}}(\mathbb{Z}^N, \Bbbk) = (\Bbbk^\times)^N = T. \tag{$*$}$$

Correspondingly, for any $\mathbf{z} \in T$, it corresponds to

$$[\mathbf{u} \longmapsto z^{\mathbf{u}}] \in \operatorname{Spec}\Bbbk[\mathbb{Z}^N].$$

As a result, the identification $(*)$ is T-equivariant.

Here is an example when $N = 2$, when $(x, y) \in \mathbb{G}_m \times \mathbb{G}_m$, the corresponding point is

$$\begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \dfrac{y^2}{x} & y^2 & xy^2 & x^2y^2 & \cdots \\ \cdots & \dfrac{y}{x} & y & xy & x^2y & \cdots \\ \cdots & \dfrac{1}{x} & 1 & x & x^2 & \cdots \\ \cdots & \dfrac{1}{xy} & \dfrac{1}{y} & \dfrac{x}{y} & \dfrac{x^2}{y} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

1.10. Dual notations Let

$$\sigma = \mathbb{R}_{\geq 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{\geq 0}\mathbf{v}_n \subseteq \mathbb{R}^N.$$

We assume it is

$$\begin{array}{lll} \text{rational} & \text{each } \mathbf{v}_i \in \mathbb{Q}^N & \\ \text{pointed} & \sigma \cap (-\sigma) = 0 & \text{i.e. not line inside } \sigma \end{array}$$

We denote

$$Q_\sigma = \{\mathbf{u} \in \mathbb{Z}^N : {}^{\forall x \in \sigma,}\ \langle \mathbf{u}, x\rangle \geq 0\}.$$

We call $\operatorname{Spec} \Bbbk[Q_\sigma]$ the **affine toric variety** for σ. Later, we will always deal with them.

1.11. Remark Note that Q_σ is always finitely generated, and

$$\begin{array}{lll} \text{of full rank,} & \text{i.e.} & \mathbb{R}Q_\sigma = \mathbb{R}^N \\ \text{and saturated,} & \text{i.e.} & Q = \mathbb{R}_{\geq 0}Q \cap \mathbb{Z}^N. \end{array}$$

Actually,

$$\sigma = \{x \in \mathbb{R}^N : {}^{\forall \mathbf{u} \in Q,}\ \langle \mathbf{u}, x\rangle \geq 0\}.$$

For example, only $d = \pm 1$ is allowed for $d\mathbb{Z}_{\geq 0}$.

1.12. An open embedding Note that the inclusion $\Bbbk[Q_\sigma] \to \Bbbk[\mathbb{Z}^N]$ induces

$$T \longrightarrow \operatorname{Spec} \Bbbk[Q_\sigma] \tag{$*$}$$

is an inclusion. Explicitly, for any $\mathbf{z} \in T$,

$$[\mathbf{u} \longmapsto z^{\mathbf{u}}] \in \operatorname{Spec} \Bbbk[Q_\sigma].$$

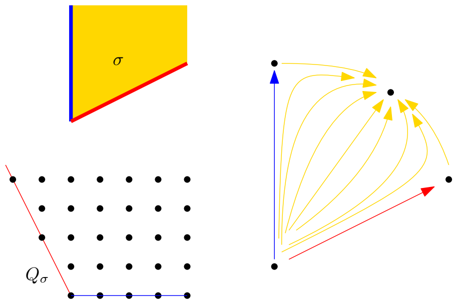
Actually, $(*)$ is an open embedding, since $\Bbbk[\mathbb{Z}^N]$ can be obtained from $\Bbbk[Q_\sigma]$ by localization. In other words, the image of T contains those $f \in \operatorname{Spec} \Bbbk[Q_\sigma] = \operatorname{Hom}_{\mathsf{Monoid}}(Q, \Bbbk)$ which can be extended globally to $\mathbb{Z}^N$.

1.13. Example Let $\sigma \subseteq \mathbb{R}^2$ be the fan

$$\sigma = \operatorname{span}_{\geq 0}(\mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2).$$

Then

$$Q_\sigma = \mathbb{Z}_{\geq 0}\mathbf{e}_1 + \mathbb{Z}_{\geq 0}(2\mathbf{e}_2 - \mathbf{e}_1) + \mathbb{Z}_{\geq 0}\mathbf{e}_2.$$



As a result,

$$\Bbbk[Q_\sigma] = \Bbbk[x, y, u] / \left\langle u^2 = xy \right\rangle .$$

Say,

$$\begin{array}{cccccc}
y^2 & \cdots & u^4 & \cdots & \cdots & \cdots \\
 & \cdots & u^3 & \cdots & \cdots & \cdots \\
 & y & u^2 & \cdots & \cdots & \cdots \\
 & & u & \cdots & \cdots & \cdots \\
 & & 1 & x & x^2 & \cdots
\end{array}$$

Thus $\operatorname{Spec} \Bbbk[Q_\sigma]$ is a quadratic cone.

1.14. Limit It turns out we can read some information about limit from σ. Now we assume $\Bbbk = \mathbb{C}$. For any vector $v \in \mathbb{R}^N$, we define

$$\exp(vt) = (e^{v_1 t}, \cdots, e^{v_N t}) \in T,$$

where $t \in \mathbb{R}$. We are going to compute

$$\lim_{t \to -\infty} \exp(vt) \cdot \mathbf{1}$$

where $[\mathbf{1} : Q_\sigma \to \mathbb{C}] \in \operatorname{Spec} \Bbbk[Q_\sigma]$ the constant map. Denote $z_t = \exp(t) \cdot \mathbf{1}$, i.e.

$$z_t(\mathbf{u}) = e^{(u_1 v_1 + \cdots + u_N v_N)t} = e^{\langle \mathbf{u}, \mathbf{v} \rangle t}.$$

Before going further, let us compute an example.

The case $v = (1,1)$**.**

$$\begin{array}{ccccc}
e^{2t} & e^{3t} & e^{4t} & e^{5t} & e^{6t} \\
 & e^{2t} & e^{3t} & e^{4t} & e^{5t} \\
 & e^{t} & e^{2t} & e^{3t} & e^{4t} \\
 & & e^{t} & e^{2t} & e^{3t} \\
 & & 1 & e^{t} & e^{2t}
\end{array}
\longrightarrow
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 \\
 & & 1 & 0 & 0
\end{array}$$

The case $v = (2,1)$**.**

$$\begin{array}{ccccc}
1 & e^{2t} & e^{4t} & e^{6t} & e^{8t} \\
 & e^{t} & e^{3t} & e^{5t} & e^{7t} \\
 & 1 & e^{2t} & e^{4t} & e^{6t} \\
 & & e^{t} & e^{3t} & e^{5t} \\
 & & 1 & e^{2t} & e^{4t}
\end{array}
\longrightarrow
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 \\
 & & 1 & 0 & 0
\end{array}$$

The case $v = (-1,1)$**.**

$$\begin{array}{ccccc}
e^{-6} & e^{-5t} & e^{-4t} & e^{-3t} & e^{-2t} \\
 & e^{-4t} & e^{-3t} & e^{-2t} & e^{-t} \\
 & e^{-3t} & e^{-2t} & e^{-t} & 1 \\
 & & e^{-t} & 1 & e^{t} \\
 & & 1 & e^{t} & e^{2t}
\end{array}
\longrightarrow
\begin{array}{ccccc}
\infty & \infty & \infty & \infty & \infty \\
 & \infty & \infty & \infty & \infty \\
 & \infty & \infty & \infty & 1 \\
 & & \infty & 1 & 0 \\
 & & 1 & 0 & 0
\end{array}$$

Now we can conclude the general result. Recall that

$$\lim_{t \to -\infty} e^{at} = \begin{cases} 0, & a > 0, \\ 1, & a = 0, \\ \infty \ (\nexists), & a < 0. \end{cases}$$

In particular, the limit z_t exists if and only if $(u_1v_1+\cdots+u_Nv_N)\geq 0$ for any $\mathbf{u}\in Q_\sigma$, i.e. $v\in\sigma$. In this case, assume the limit is z, then

$$z(\mathbf{u})=\begin{cases}1, & \langle v,\mathbf{u}\rangle=0,\\ 0, & \langle v,\mathbf{u}\rangle>0,\end{cases}=\begin{cases}1, & \mathbf{u}\perp\tau\\ 0, & \text{otherwise.}\end{cases}$$

where τ is the maximal face of σ containing v. We summarize the discussion above as follows.

1.15. Theorem For any face $\tau\subseteq\sigma$, we define $\mathbf{1}_\tau\in\operatorname{Spec}\Bbbk[Q_\sigma]$, to be

$$\mathbf{1}_\tau(\mathbf{u})=\begin{cases}1, & \mathbf{u}\perp\tau,\\ 0, & \text{otherwise.}\end{cases}$$

For example, $\mathbf{1}_0=\mathbf{1}$. This is a monoid homomorphism since τ is a face:

$$\begin{array}{c|c|c} + & \in\tau^\perp & \notin\tau^\perp\\ \hline \in\tau^\perp & \in\tau^\perp & \notin\tau^\perp\\ \hline \notin\tau^\perp & \notin\tau^\perp & \notin\tau^\perp \end{array}\quad\longleftrightarrow\quad\begin{array}{c|c|c} * & 1 & 0\\ \hline 1 & 1 & 0\\ \hline 0 & 0 & 0\end{array}$$

Then

$$\lim_{t\to-\infty}\exp(vt)\cdot\mathbf{1}=\begin{cases}\mathbf{1}_\tau, & \tau\text{ is the minimal face of }\sigma\text{ containing }v,\\ \nexists, & v\notin\sigma.\end{cases}$$

In other word, σ is the “local traffic map” at the point $\mathbf{1}$ telling the end of different direction. See the figure in Example 1.13.

1.16. Toric Variety Let $\tau\subseteq\sigma$ be a face. Then $Q_\sigma\subseteq Q_\tau$ induces

$$\operatorname{Spec}(\Bbbk[Q_\tau])\longrightarrow\operatorname{Spec}(\Bbbk[Q_\sigma]).\tag{$*$}$$

By our idenfication, this map is given by restricting any $[Q_\tau\to\Bbbk]\in\operatorname{Spec}(\Bbbk[Q_\tau])$ to Q_σ to get a new map $[Q_\tau\to\Bbbk]\in\operatorname{Spec}(\Bbbk[Q_\sigma])$. Since everything is of full rank, $(*)$ is injective. The philosophy is to glue using these morphisms.

Let Δ be a fan. That is, a collection of cones which is closed under (1) taking face and (2) taking intersection. We define **toric variety**

$$X(\Delta)=\varinjlim_{\sigma\in\Delta}\operatorname{Spec}\Bbbk[Q_\sigma].$$

Note that the smaller σ is, the bigger Q_σ is. Let us use the convention $|\Delta|$ to stand for the union of all cones in Δ.

1.17. Points The closed point can be understood as follows

- Any point of $X(\Delta)$ can be represented by a homomorphism $[Q_\sigma \xrightarrow{f} \Bbbk]$ for some $\sigma \in \Delta$.
- Two points $[Q_{\sigma_1} \xrightarrow{f_1} \Bbbk]$ and $[Q_{\sigma_2} \xrightarrow{f_1} \Bbbk]$ represents the same point if there exists a common extension $[Q_{\sigma_1 \cap \sigma_2} \to \Bbbk]$.

For example,

$$\operatorname{Spec} \Bbbk[Q_0] = \operatorname{Spec} \Bbbk[\mathbb{Z}^N] = T$$

is the set of $[\mathbb{Z}^N \xrightarrow{f} \Bbbk]$, i.e. globally defined homomorphism. Actually, $\operatorname{Spec} \Bbbk[Q_0]$ is an open (thus dense) subset of $X(\Delta)$ isomorphic to T. The reason $X(\Delta)$ is called **toric** variety.

1.18. Example Consider the following fan

$$\Delta = \{\mathbb{R}_{\geq 0}, 0, \mathbb{R}_{\leq 0}\}.$$

By the discussion, $X(\Delta) = \mathbb{P}^1$.

$$\left[\begin{tikzcd} & X(\Delta) & \\ \operatorname{Spec} \Bbbk[x] \arrow[ur, dotted] & & \operatorname{Spec} \Bbbk[x^{-1}] \arrow[ul, dotted] \\ & \operatorname{Spec} \Bbbk[x^{\pm 1}] \arrow[ul] \arrow[ur] & \end{tikzcd}\right] = \left[\begin{tikzcd} & \mathbb{P}^1 & \\ \Bbbk \arrow[ur, dotted] & & \Bbbk \setminus 0 \cup \{\infty\} \arrow[ul, dotted] \\ & \Bbbk \setminus 0 \arrow[ul] \arrow[ur] & \end{tikzcd}\right]$$

Actually, $\mathbb{P}^1$ can be understood as follows

$$\begin{tikzcd}[column sep=small] 0 & \cdots & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \arrow[u] & 1 & 1 & 1 & 1 & \cdots \\ \mathbb{P}^1 \ni z & \cdots & \frac{1}{z^4} & \frac{1}{z^3} & \frac{1}{z^2} & \frac{1}{z} & 1 \arrow[u, dash] \arrow[d] & z & z^2 & z^3 & z^4 & \cdots \\ \infty & \cdots & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \cdots \end{tikzcd}$$

1.19. Theorem We similarly define $\mathbf{1}_\sigma$ for any $\sigma \in \Delta$. Then

$$\lim_{t \to -\infty} \exp(vt) \cdot \mathbf{1} = \begin{cases} \mathbf{1}_\tau, & \begin{array}{l}\tau \text{ is the minimal} \\ \text{cone in } \Delta \text{ containing } v,\end{array} \\ \nexists, & v \notin |\Delta|. \end{cases}$$

1.20. Theorem Here we list some basic properties.

- The toric variety $X(\Delta)$ is always normal and separable (=Hausdorff).
- The toric variety $X(\Delta)$ is complete (=compact) if and only if $|\Delta| = \mathbb{R}^N$.

We refer Fulton's book for a proof.

1.21. Projective Spaces Let Δ be the fan of $\mathbb{R}^n$ with $|\Delta| = \mathbb{R}^n$ with one dimensional cones spanned by

$$\mathbf{e}_1, \ \ldots, \ \mathbf{e}_n, \ -\mathbf{1},$$

where $-\mathbf{1} = (-1, \ldots, -1) \in \mathbb{R}^n$. I claim that

$$\mathbb{P}^n = X(\Delta). \tag{$*$}$$

Recall that

$$\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus 0)/\mathbb{G}_m = \Big\{ [x_0 : \cdots : x_n] : \mathbf{x} \neq 0 \Big\}.$$

There are two ways to see this — a dirty way, and a sophisticated way.

Dirty Way We will illustrate this way by analyzing the case $n = 2$. Recall that $\mathbb{P}^2$ can be covered by

$$\{x_0 \neq 0\}, \qquad \{x_2 \neq 0\}, \qquad \{x_1 \neq 0\}.$$

Recall that we identify

$$\mathbb{A}^2 \subseteq \mathbb{P}^2, \qquad (x_1, x_2) \text{ identified as } [1 : x_1 : x_2].$$

Moreover,

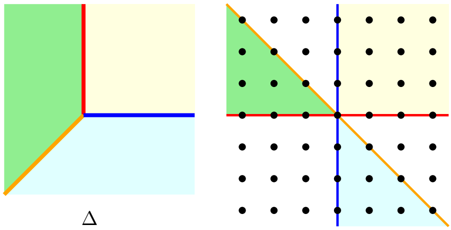
$$\{x_1 \neq 0\} \cong \mathbb{A}^2, \qquad [x_0 : x_1 : x_2] \longmapsto (\tfrac{x_0}{x_1}, \tfrac{x_2}{x_1}).$$

$$\{x_2 \neq 0\} \cong \mathbb{A}^2, \qquad [x_0 : x_1 : x_2] \longmapsto \left(\tfrac{x_0}{x_2}, \tfrac{x_1}{x_2}\right).$$

When restricting to $\mathbb{A}^2$, i.e. $x_0 = 1$, the two map correspond to

$$(x_1, x_2) \longmapsto \left(\tfrac{1}{x_1}, \tfrac{x_2}{x_1}\right), \qquad (x_1, x_2) \longmapsto \left(\tfrac{1}{x_2}, \tfrac{x_1}{x_2}\right).$$

We can find that $\mathbb{P}^2$ is glued from $\mathbb{A}^2$ by two $\mathbb{A}^2$ using above two maps respectively.



In this case, there are three cones of full dimension 2.

$$\mathrm{span}_{\geq 0}(\mathbf{e}_1, \mathbf{e}_2), \qquad \mathrm{span}_{\geq 0}(\mathbf{e}_1, -\mathbf{1}), \qquad \mathrm{span}_{\geq 0}(\mathbf{e}_2, -\mathbf{1}).$$

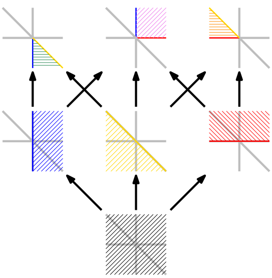
It is not hard to see the corresponding monoids are

$$\mathbb{Z}_{\geq 0}\mathbf{e}_1 + \mathbb{Z}_{\geq 0}\mathbf{e}_2, \qquad -\mathbb{Z}_{\geq 0}\mathbf{e}_2 + \mathbb{Z}_{\geq 0}(\mathbf{e}_1 - \mathbf{e}_2), \qquad -\mathbb{Z}_{\geq 0}\mathbf{e}_1 + \mathbb{Z}_{\geq 0}(\mathbf{e}_2 - \mathbf{e}_1).$$

As a result, the corresponding ring is

$$\Bbbk[x_1, x_2], \qquad \Bbbk\left[\tfrac{1}{x_2}, \tfrac{x_1}{x_2}\right], \qquad \Bbbk\left[\tfrac{1}{x_1}, \tfrac{x_2}{x_1}\right].$$

The first corresponds to $\mathbb{A}^2 \subseteq \mathbb{P}^2$. The last two corresponds to $\{x_2 \neq 0\}$ and $\{x_1 \neq 0\}$. We left to reader to check the identification.



$$\begin{tikzcd}
\{x_2 \neq 0\} & & \{x_0 \neq 0\} & & \{x_1 \neq 0\} \\
\{x_0x_2 \neq 0\} \arrow[u] \arrow[urr] & & \{x_1x_2 \neq 0\} \arrow[ull] \arrow[urr] & & \{x_0x_1 \neq 0\} \arrow[u] \arrow[ull] \\
& & \{x_0x_1x_2 \neq 0\} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Sophisticated Way Consider $\widetilde{\Delta}$ to be the fan of **proper** faces of

$$\mathbb{R}_{\geq 0}\mathbf{e}_0 + \cdots + \mathbb{R}_{\geq 0}\mathbf{e}_n \subseteq \mathbb{R}^n.$$

It is not hard to see

$$X(\widetilde{\Delta}) = \Bbbk^{n+1} \setminus 0.$$

We have a "morphism" of fan $\widetilde{\Delta} \to \Delta$, which will "induce"

$$X(\widetilde{\Delta}) \longrightarrow X(\Delta).$$

One can show that this map coincides with the quotient map (well, the problem is local)

$$\Bbbk^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n.$$

Exercises

1.22. Complement of Coordinate Planes Let $\triangle$ be a simplex over $\{1, \ldots, n\}$. That is, $\triangle$ is a family of subset of $\{1, \ldots, n\}$ such that

$$B \subseteq A \in \triangle \Longrightarrow B \in \triangle.$$

Let Δ be the collection

$$\left\{ \operatorname{span}_{\geq 0}(\mathbf{e}_a)_{a \in A} \right\}_{A \in \triangle}.$$

Note that Δ is a fan. Show that

$$X(\Delta) = \Bbbk^k \setminus \bigcup_{A \in \triangle} \begin{pmatrix} \text{coordinate plane of} \\ \{1, \ldots, n\} \setminus A. \end{pmatrix}.$$

1.23. Product of Projective Lines Figure our what is $X(\Delta)$ for Δ the fan of $\mathbb{R}^n$ with $|\Delta| = \mathbb{R}^n$ with one dimensional cones spanned by

$$\pm\mathbf{e}_1, \ \ldots, \ \pm\mathbf{e}_n.$$

Say, Δ is a direct product of n-copies of Δ_1, with Δ_1 the unique fan over $\mathbb{R}_1$ such that $|\Delta_1| = \mathbb{R}$.

1.24. A_n **singularity** Actually, any 2 dimensional affine toric variety is of the form

$$Q_\sigma = \operatorname{span}_{\geq 0}(\mathbf{e}_1, \mathbf{e}_1 + m\mathbf{e}_2) \cap \mathbb{Z}^2,$$

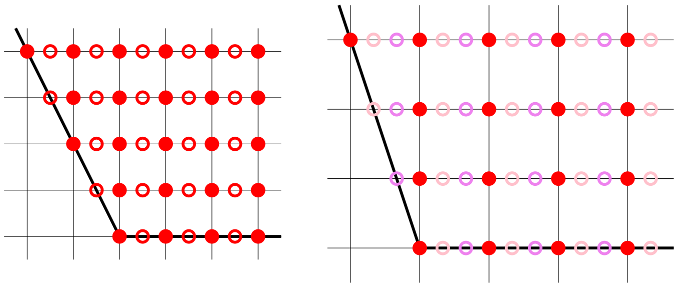
up to an isomorphism. So that

$$\Bbbk[Q_\sigma] = \Bbbk[x, y, u] / \langle xy = u^m \rangle .$$

Remark Actually,

$$\operatorname{Spec} \Bbbk[Q_\sigma] = \mathbb{C}^2 / C_m,$$

where C_m is a cyclic group of degree m in $\mathrm{SL}_2(\mathbb{C}^2)$. One way to think is



That is, we can introduce Q which isomorphic to $\mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}$ containing Q_σ. We define an action of C_m such that $Q^{C_m} = Q_\sigma$. Then it is easy to see $\Bbbk[x, y]^{C_m} = \Bbbk[Q]^{C_m} = \Bbbk[Q_\sigma]$. The argument works for general simplicial σ.

Next Time

Next time, we will discuss **line bundles/divisors**.

2 Divisors and Line Bundles

Geometry Background

2.1. Let X be a variety over $\Bbbk$.

2.2. Class Groups We denote the group of **Weil divisors** to be

$$\mathrm{Div}(X) = \bigoplus_{Y \subseteq X} \mathbb{Z} \cdot [Y], \qquad \text{with each } Y \text{ a subvariety of codimension 1.}$$

Recall subvariety means integral (i.e. reduced and irreducible). For any nonzero rational function $f \in \mathscr{K}(X) = \mathrm{Mor}(X, \mathbb{P}^1)$, we can define

$$\operatorname{div} f = \sum_U v_Y(f)[Y] \in \mathrm{Div}(X).$$

Roughly speaking

$$\operatorname{div} f = [\text{zeros}] - [\text{poles}] \qquad (\text{counting multiplicity}).$$

We define **Class group** by

$$\mathrm{Cl}(X) = \mathrm{Div}(X) / \left\langle \operatorname{div} f : f \in \mathscr{K}(X)^\times \right\rangle.$$

Formally, class group can be putted in the following exact sequence

$$\mathscr{K}(X)^\times \xrightarrow{\operatorname{div}} \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

2.3. Picard Groups We define **Picard group**

$$\mathrm{Pic}(X) = \{\text{line bundles over } X\} / \cong .$$

Recall that line bundle is nothing but vector bundle of rank 1. It forms a group under the multiplication induced by tensor product. When X is nonsingular (or more generally, local factorial), we have a natural **isomorphism**

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X).$$

To be exact, for any $D \in \mathrm{Div}(X)$, we define a line bundle

$$\mathscr{O}(D) : U \longmapsto \{f \in \mathscr{K}(X) : \operatorname{div} f + D|_U \geq 0\},$$

where we take the convention that $f = 0$ always satisfies the condition. To be exact, if locally $D|_U = \operatorname{div} f$, then $\mathscr{O}(D)$ is locally generated by f^{-1} as $\mathscr{O}_X$-module. For example, when $D = 0$, locally, f can be any element of $\mathscr{O}$, thus $\mathscr{O}(D) = \mathscr{O}_X$.

2.4. Example For projective space $\mathbb{P}^n$,

$$\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}[H]$$

where H is any hyperplane section. To be exact, for any codimension one subvariety must be zero loci of a homogeneous f of degree d. Assume $H = \{x_0 = 0\}$, then $f/x_0^d \in \mathscr{K}(\mathbb{P}^n)$ with

$$\operatorname{div} \tfrac{f}{x_0^d} = [Y] - d[H].$$

As a result, $[Y] = d[H] \in \mathrm{Cl}(\mathbb{P}^n)$. We remind reader that $f \notin \mathscr{K}(\mathbb{P}^n)$ if $d > 0$. Actually,

$$\mathscr{O}(dH) = \mathscr{O}(d).$$

2.5. Positivities In algebraic geometry, there are two sorts of positivity

	cycles	sheaves
≥ 0	effective	globally generated
$*$	intersection product	tensor product

Since we have a nice correspondence for line bundle and codimension 1 cycle, there are a lot of terminologies. Let $\mathcal{L} = \mathscr{O}(D)$ be a line bundle. We say $\mathcal{L}$ or D

(i) is **globally generated** if $\mathcal{L}$ is;

(ii) is **effective** if D is equivalent to an effective class;

(iii) is **very ample** if it induces closed embedding into projective space;

(iv) is **ample** if one of the equivalent condition holds

  (a) For any coherent sheaf $\mathcal{F}$, $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ is globally generated for $n \gg 0$;

  (b) For any closed subvariety W, $\langle D^{\dim W}, W \rangle > 0$;

(v) is **numerically effective (nef)** if $\langle D^{\dim W}, W \rangle \geq 0$ for any closed subvariety W.

We have

$$\begin{array}{ccccc} \text{very ample} & \Longrightarrow & \text{globally generated} & \Longrightarrow & \text{effective} \\ \Downarrow & & \Downarrow & & \\ \text{ample} & \Longrightarrow & \text{nef} & & \end{array}$$

2.6. Asymptotic Riemann–Roch Let $\mathscr{O}(D)$ be a line bundle. Then

$$\chi(\mathscr{O}(mD)) = \frac{\langle D^n \rangle}{n!} m^n + o(m^n), \tag{$*$}$$

where $n = \dim X$,

$$\chi(\mathscr{O}(mD)) = \sum (-1)^i \dim H^i(X, \mathscr{O}(md)),$$

and

$$\langle D^n \rangle = \deg(D \overset{n}{\cdots} D) = \begin{array}{c}\text{number of points}\\ \text{on self-intersection.}\end{array}$$

When D is nef, then the LHS of $(*)$ can be replaced by dimension of global sections

$$\dim \Gamma(\mathscr{O}(mD)) = \frac{\langle D^n \rangle}{n!} m^n + o(m^n).$$

This fact is known as **asymptotic Riemann–Roch**.

Divisors over Toric Varieties

2.7. Let Δ be a fan in $\mathbb{R}^N$. Let us denote $\Delta(i)$ to be the collection of all cones of dimension i. For example $\Delta(0) = \{0\}$.

2.8. Recall that

$$X(\Delta) = \frac{\bigcup_{\sigma \in \Delta} \left\{ Q_\sigma \xrightarrow{f} \mathbb{k} : \begin{array}{l} f(\mathbf{0}) = 1 \\ f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1) f(\mathbf{u}_2) \end{array} \right\}}{f \sim g \iff \text{there is a common extension}}.$$

Recall that for $\tau \in \Delta$, we defined

$$\mathbf{1}_\tau : Q_\tau \longrightarrow \mathbb{k}, \qquad \mathbf{u} \longmapsto \begin{cases} 1, & \mathbf{u} \in \tau^\perp, \\ 0, & \mathbf{u} \notin \tau^\perp. \end{cases}.$$

Let us denote $\mathcal{O}_\tau$ the orbit of $\mathbf{1}_\tau$. For example, $\mathcal{O}_0$ is T.

2.9. Theorem We have

$$X(\Delta) = \bigcup_{\tau \in \Delta} \mathcal{O}_\tau.$$

Moreover, the codimension of $\mathcal{O}_\tau$ is $\dim \tau$.

Proof Since we can cover toric varieties by open affine toric varieties, it suffices to show when Δ is affine. Assume Δ is the fan of all faces of σ. Then

$$X(\Delta) = \operatorname{Spec} \Bbbk[Q_\sigma] = \{Q_\sigma \xrightarrow{f} \Bbbk : f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1)f(\mathbf{u}_2)\}.$$

For any $[Q_\sigma \xrightarrow{f} \Bbbk] \in X(\Delta)$, we see

$$\operatorname{supp} f = \{\mathbf{u} \in Q_\sigma : f(\mathbf{u}) \neq 0\}$$

is a face of Q_σ, correspondent to a face τ of σ, say $Q_\sigma \cap \tau^\perp$. Abstractly, a face F of Q_σ is a sub-monoid with

$$\begin{array}{c|c|c} + & \in F & \notin F \\ \hline \in F & \in F & \notin F \\ \hline \notin F & \notin F & \notin F \end{array} \quad \xrightarrow{f} \quad \begin{array}{c|c|c} * & \neq 0 & = 0 \\ \hline \neq 0 & \neq 0 & = 0 \\ \hline = 0 & = 0 & = 0 \end{array}$$

Now,

$$\begin{aligned} \{f \in X(\Delta) : \operatorname{supp} f = \tau\} &= \left\{ \tau^\perp \cap Q_\sigma \xrightarrow{f} \Bbbk^\times : \begin{array}{l} f(\mathbf{0}) = 1 \\ f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1)f(\mathbf{u}_2) \end{array} \right\} \\ &= \left\{ \tau^\perp \cap \mathbb{Z}^N \xrightarrow{f} \Bbbk^\times : \begin{array}{l} f(\mathbf{0}) = 1 \\ f(\mathbf{u}_1 + \mathbf{u}_2) = f(\mathbf{u}_1)f(\mathbf{u}_2) \end{array} \right\} \\ &= \operatorname{Hom}_{\mathsf{Mnd}}(\tau^\perp \cap \mathbb{Z}^N, \Bbbk^\times) \cong \mathbb{G}_m^{N - \dim \tau}. \end{aligned}$$

Here we use the fact that $\tau^\perp \cap \mathbb{Z}^N$ is free of rank $N - \dim \tau$. Since $\tau^\perp \cap \mathbb{Z}^N$ is a direct summand of Q_σ, we see this is a single T-orbits (containing $\mathbf{1}_\tau$). Q.E.D.

2.10. Closure We can prove that the closure

$$\overline{\mathcal{O}}_\tau = \bigcup_{\tau \subseteq \sigma} \mathcal{O}_\sigma.$$

In particular,

$$\begin{aligned} \Delta(N) &\xleftrightarrow{1:1} \{\ T\text{-fixed points of } X(\Delta)\ \}, \\ \Delta(N-1) &\xleftrightarrow{1:1} \{\ T\text{-equivariant curves}\ \}, \\ \Delta(1) &\xleftrightarrow{1:1} \left\{ \begin{array}{c} T\text{-equivariant subvarieties} \\ \text{of codimension } 1 \end{array} \right\}, \\ \Delta(0) &\xleftrightarrow{1:1} \{\ T\ \} = \text{a single point}. \end{aligned}$$

2.11. Rational Fields To figure out $\mathrm{Cl}(X(\Delta))$, by definition

$$\mathscr{K}(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Recall that the torus T is embedded in $X(\Delta)$, say

$$T \ni \mathbf{z} = [\mathbf{u} \longmapsto z^{\mathbf{u}}] \in \operatorname{Spec} \Bbbk[Q_0].$$

As a result,

$$\mathscr{K}(X(\Delta)) = \mathscr{K}(T) = \Bbbk(x_1, \ldots, x_N).$$

However, both $\mathscr{K}(X)^\times$ and $\mathrm{Div}(X)$ are too huge to control.

2.12. Lemma For a monomial $x^{\mathbf{u}} \in \Bbbk(x_1, \ldots, x_N)$ with $\mathbf{u} \in \mathbb{Z}^N$, we have

$$\operatorname{div} x^{\mathbf{u}} = \sum_{\ell \in \Delta(1)} \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, [\overline{\mathcal{O}}_\ell],$$

where $\mathbf{v}_\ell$ is the first nonzero integer vector on the ray $\ell \in \Delta(1)$.

Proof It is clear that $\operatorname{div} x^{\mathbf{u}}$ is T-equivariant, so it suffices to take $[\overline{\mathcal{O}}_\ell]$ into consideration. The problem is local, and thus reduce to $\operatorname{Spec} \Bbbk[Q_\ell]$ — we can assume $\mathbf{v}_\ell = \mathbf{e}_1 = (1, 0, \ldots)$. Note that

$$Q_\ell \cong \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{N-1}, \qquad \operatorname{Spec} \Bbbk[Q_\ell] = \Bbbk \times (\Bbbk^\times)^{N-1}.$$

Under this identification

$$\overline{\mathcal{O}}_\ell = (\Bbbk \times (\Bbbk^\times)^{N-1}) \setminus (\Bbbk^\times \times (\Bbbk^\times)^{N-1}) = 0 \times (\Bbbk^\times)^{N-1}.$$

Note that the restriction $x^{\mathbf{u}}$ is $x_1^{u_1} x_2^{u_2} \cdots x_N^{u_N}$ whose zero loci is $\overline{\mathcal{O}}_\ell$ with multiplicity u_1 (note that $x_2^{u_2} \cdots x_N^{u_N}$ is a unit). So we can conclude the multiplicity of $\overline{\mathcal{O}}_\ell$ in $\operatorname{div} x^{\mathbf{u}}$ is $u_1 = \langle \mathbf{u}_1, \mathbf{v}_\ell \rangle$. Q.E.D.

2.13. Theorem We have

$$\mathrm{Cl}(X(\Delta)) = \bigoplus_{\ell \in \Delta(1)} \mathbb{Z} \cdot [\overline{\mathcal{O}}_\ell] \Big/ \Big\langle \sum_{\ell \in \Delta(1)} \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, [\overline{\mathcal{O}}_\ell] : \mathbf{u} \in \mathbb{Z}^N \Big\rangle.$$

In other words, we have

$$\mathbb{Z}^N \xrightarrow{(*)} \mathbb{Z}^{\Delta(1)} \longrightarrow \mathrm{Cl}(X(\Delta)) \longrightarrow 0,$$

where $(*)$ is given by $\mathbf{u} \mapsto \sum \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, \mathbf{e}_\ell$.

Proof We have excision sequence,

$$\mathbb{Z}^{\Delta(1)} \xrightarrow{(*)} \mathrm{Cl}(X(\Delta)) \longrightarrow \underbrace{\mathrm{Cl}(T)}_{=0} \longrightarrow 0.$$

The kernel of $(*)$ is generated by $\operatorname{div} f$ for $f \in \mathscr{K}(X(\Delta))$ which is invertible over T. That is,

$$f \in \Bbbk[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^\times = \bigcup_{\mathbf{u}} \Bbbk^\times x^{\mathbf{u}}.$$

The proof is now complete. Q.E.D.

2.14. Example For $\mathbb{P}^n$, we see

$$\mathbb{Z}^n \xrightarrow{(*)} \mathbb{Z}^{n+1} \longrightarrow \mathrm{Cl}(\mathbb{P}^n) \longrightarrow 0,$$

where $(*)$ sends $\mathbf{e}_i$ to $\mathbf{e}_i - \mathbf{e}_0$. As a result, in $\mathrm{Cl}(\mathbb{P}^n)$, all classes $[\overline{\mathcal{O}}_\ell]$ are equal in $\mathrm{Cl}(\mathbb{P}^n)$.

2.15. Cartier Divisor Note that when $X(\Delta)$ is smooth, $\mathrm{Cl}(X(\Delta)) = \mathrm{Pic}(X(\Delta))$. In the general case, toric variety can be singular, there is one way to describe Picard group, and one can find examples such that $\mathrm{Cl}(X(\Delta)) \neq \mathrm{Pic}(X(\Delta))$. Speaking of this, we are at the position to discuss smoothness of toric varieties.

2.16. On smoothness Since smoothness is local, let us state the equivalent condition for an affine toric variety. The affine toric variety $\mathrm{Spec}[Q_\sigma]$ is non-singular if and only if we have $\sigma = \mathrm{span}_{\geq 0}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ for some r where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a $\mathbb{Z}$-base of $\mathbb{Z}^N \subseteq \mathbb{R}^N$. Note that in this case,

$$\mathrm{Spec}[Q_\sigma] = \Bbbk^r \times (\Bbbk^\times)^{N-r}.$$

As a result, non-singular affine variety is boring.

Linear bundles over Toric Varieties

2.17. Linear bundles Let D be a Weil divisor. By definition

$$\Gamma(\mathcal{O}(D)) = \{f \in \mathscr{K}(X(\Delta)) : \operatorname{div} f + D \geq 0\}$$

Let find when a monomial $x^{\mathbf{u}} \in \Gamma(\mathcal{O}(D))$ for $\mathbf{u} \in \mathbb{Z}^N$. Assume $D = \sum_\ell c_\ell [\overline{\mathcal{O}}_\ell]$,

$$\operatorname{div} x^{\mathbf{u}} + D = \sum (\langle \mathbf{u}, \mathbf{v}_\ell \rangle + c_\ell) \cdot [\overline{\mathcal{O}}_\ell].$$

Thus we should define a convex set

$$P_D = \{\mathbf{u} \in \mathbb{R}^N : \langle \mathbf{u}, \mathbf{v}_\ell \rangle + c_\ell \geq 0\}.$$

2.18. Theorem For any Weil divisor D, we have

$$\Gamma(\mathscr{O}(D)) = \bigoplus_{\mathbf{u} \in P_D} \Bbbk \cdot x^{\mathbf{u}}.$$

In particular, $\dim \Gamma(\mathscr{O}(D))$ equals to the number of lattice points inside P_D.

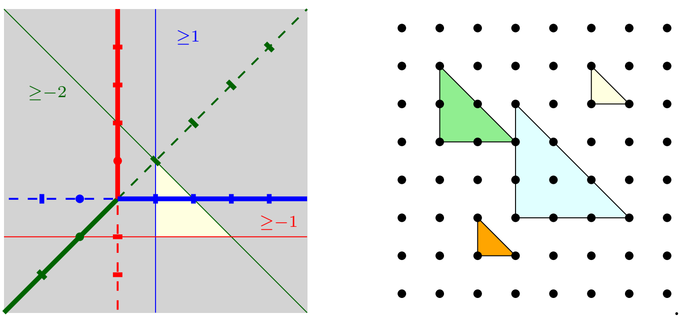
Proof Since D is T-equivariant, $\Gamma(\mathscr{O}(D))$ is a T-representation. For any $f \in \Gamma(\mathscr{O}(D))$, by definition f has no pole over T, thus

$$f \in \Bbbk[x_1^{\pm 1}, \ldots, x_N^{\pm 1}].$$

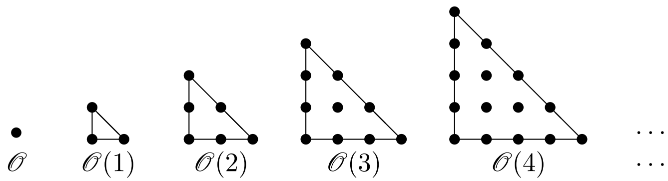
This shows $\Gamma(\mathscr{O}(D))$ decompose into weight modules, i.e. is spanned by monomials in $\Gamma(\mathscr{O}(D))$. Q.E.D.

2.19. Example Here is an example of $\mathbb{P}^2$. Consider

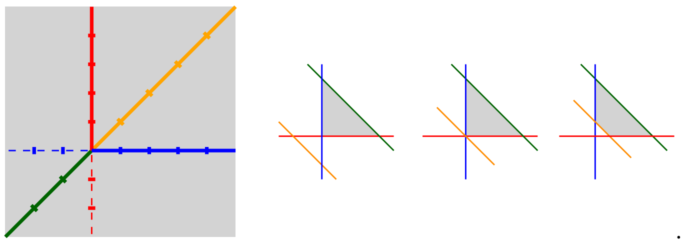
$$D = -[\overline{\mathcal{O}}_{\ell_1}] + [\overline{\mathcal{O}}_{\ell_2}] + 2[\overline{\mathcal{O}}_{\ell_2}].$$



In particular, $\dim \Gamma(\mathscr{O}(D))$ is 0 or a triangular number.



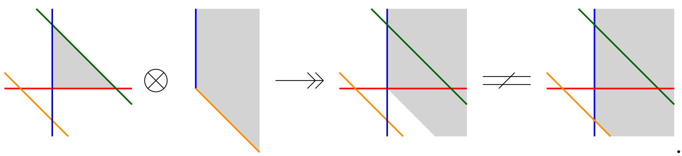
2.20. Example For every property of line bundles mentioned at the beginning of this section, there is an equivalent combinatorial description known. We will not state the exact conditions, since it is sort of technical. But it will be clear after seeing an example. Consider



The first line bundle is NOT generated by global section, actually, global sections do not generate over $X(\sigma)$ for the cone σ spanned by **orange** and **blue** rays. That is,

$$\Gamma(\mathscr{O}(D)) \otimes \mathscr{O}_U \longrightarrow \mathscr{O}(D)|_U$$

is not surjective. Look at



The second is globally generated and thus nef. The last is ample and very ample.

2.21. Remark In general,

- for toric varieties, nef is equivalent to globally generated;
- for non-singular toric varieties, ample is equivalent to very ample.

- for toric varieties, nef implies cohomology trivial i.e. $H^{\geq 1} = 0$.

We refer Fulton's book for a proof.

2.22. Corollary For any nef divisor D (in particular $P_D \neq \varnothing$), we have

$$\frac{\langle D^N \rangle}{N!} = \mathrm{Vol}(P_D).$$

Proof By the asymptotic Riemann–Roch,

$$\dim \Gamma(\mathscr{O}(mD)) = \frac{\langle D^N \rangle}{N!} m^N + o(m^N),$$

we have

$$\begin{aligned} \frac{\langle D^N \rangle}{N!} &= \lim_{m \to \infty} \frac{\dim \Gamma(\mathscr{O}(mD))}{m^N} \\ &= \lim_{m \to \infty} \frac{\#(mP_D \cap \mathbb{Z}^N)}{m^N} \\ &= \lim_{m \to \infty} \frac{\#\big(P_D \cap \frac{1}{m}\mathbb{Z}^N \big)}{\#\big([0,1]^N \cap \frac{1}{m}\mathbb{Z}^N\big)} = \mathrm{volume}(P_D). \end{aligned}$$

2.23. As a result, the

$$\left(\begin{matrix} \text{algebraic} \\ \text{cycles} \end{matrix}, \cap, \deg \right) \qquad \text{v.s.} \qquad \left(\begin{matrix} \text{coherent} \\ \text{sheaves} \end{matrix}, \otimes, \chi \right)$$

is reflected as

$$\big(\text{volume}\big) \qquad \text{v.s.} \qquad \big(\text{lattice points}\big).$$

A finer Riemann–Roch over toric surface will give Pick theorem. We will meet this later.

Exercises

2.24. Translation Assume $D_1 - D_2 = \operatorname{div} x^{\mathbf{u}}$. Show that $P_{D_2} = P_{D_1} + \mathbf{u}$.

2.25. Mixed volume Let D_1, D_2 be two nef Weil divisors over a toric surface. We known that

$$\frac{1}{2}\langle D_1, D_1\rangle = \mathrm{area}(P_{D_1}), \qquad \frac{1}{2}\langle D_2, D_2\rangle = \mathrm{area}(P_{D_2}).$$

What is $\langle D_1, D_2\rangle$? Hint: think about $\frac{1}{2}\langle D_1 + D_2, D_1 + D_2\rangle$. In particular, if we translate them over $\mathbb{P}^2$, we will give the Bézout theorem for $\mathbb{P}^2$.

Next Time

Next time, we will discuss **cohomology/Chow ring**.

3 Cohomology and Chow ring

Generalities on Chow rings

3.1. Chow Groups Let X be an algebraic variety. Define the group of algebraic cycles

$$Z^k(X) = \bigoplus_{Y \subseteq X} \mathbb{Z} \cdot [Y], \qquad \text{with } Y \text{ integral of codimension } k.$$

We define **Chow group**

$$\mathrm{CH}^k(X) = Z^k(X) / \begin{pmatrix} \text{rational} \\ \text{equivalence} \end{pmatrix},$$

where two cycles $[Y]$ and $[W]$ are rational equivalent if there exists $Y \in Z^k(X \times \mathbb{P}^1)$ such that

$$[Y] = [\text{fibre of } 0], \qquad [W] = [\text{fibre of } \infty].$$

We call $[Y]$ the fundamental class of a subvariety Y of X. In particular,

$$\mathrm{CH}^0(X) = \mathbb{Z} \cdot [X], \qquad \mathrm{CH}^1(X) = \mathrm{Cl}(X).$$

3.2. Chow Rings If X is smooth, $\mathrm{CH}^\bullet(X)$ is a graded ring under transversal intersection and will be called **Chow ring**. To be exact, let Y, W be two subvarieties,

$$[Y] \cdot [W] = \begin{cases} 0, & \dim(Y \cap W) < \text{expected dimension}, \\ [Y \pitchfork W], & Y \text{ intersects } W \text{ (generically) transversally}, \\ \text{unknown}, & \dim(Y \cap W) > \text{expected dimension}, \end{cases}$$

where $d = \dim Y + \dim Z - \dim X$ is the expected dimension.

3.3. Torus Fixed Loci Let X be a smooth complete variety acted by $\mathbb{G}_m$ algebraically. Assume

$$X \text{ can be covered by } \mathbb{G}_m\text{-invariant open affine subvarieties.}$$

Let X_0 be the fixed loci of X. For any connected component $Z \in \pi_0(X_0)$, denote

$$\mathrm{Attr}(Z) = \left\{ x \in X : \lim_{t \to 0} t \cdot x \in Z \right\},$$

where the limit $\lim_{t\to 0} tx$ means the value of 0 extending $\mathbb{G}_m \xrightarrow{t\mapsto t\cdot x} X$. Note that X_0 and $\mathrm{Attr}(Z)$ is always smooth. To be exact, for $x \in Z \subseteq X_0$, assume

$$\mathscr{T}_X(x) = \bigoplus\nolimits_{i\in\mathbb{Z}} \mathscr{T}_X^i(x), \qquad \mathscr{T}_X^i(x) = \{v \in \mathscr{T}_X(x) : t\cdot v = t^i v\}.$$

Then

$$\mathscr{T}_{X_0}(x) = \mathscr{T}_X^0(x), \qquad \mathscr{T}_{\mathrm{Attr}(Z)}(x) = \mathscr{T}_X^+(x) := \bigoplus\nolimits_{i\geq 0} \mathscr{T}_X^i(x).$$

3.4. Białynicki-Birula theorem The **Białynicki-Birula theorem** states that

$$X = \bigsqcup_{Z\in\pi_0(X_0)} \mathrm{Attr}(Z), \qquad \text{and} \qquad \begin{gathered}\mathrm{Attr}(Z) \xrightarrow{\lim_{t\to 0}} Z \\ \text{is an affine bundle.}\end{gathered}$$

In particular, if $\dim X_0 = 0$ (thus finite), X can be decomposed into strata with each of them isomorphic to affine space $\mathbb{A}^{\ell(Z)}$ for some $\ell(Z)$.

3.5. Stratification Recall a **stratification** $\mathbb{S}$ on X is a decomposition

$$X = \bigsqcup_{S\in\mathbb{S}} S, \qquad \text{with} \qquad \begin{gathered}\text{each } \overline{S} = \text{finite union} \\ \text{of many members of } \mathbb{S}.\end{gathered}$$

We call a stratification $\mathbb{S}$ **affine** if each of them $S \in \mathcal{S}$ is isomorphic to an affine space $\mathbb{A}^{\ell(S)}$ for some $\ell(S)$. In this case,

$$\mathrm{CH}^\bullet(X) = H^{2\bullet}(X) = \bigoplus\nolimits_{S\in\mathbb{S}} \mathbb{Z}\cdot[\overline{S}].$$

In particular, $H^{\mathrm{odd}}(X) = 0$.

3.6. General Tori Now, let T be a torus, and X be a smooth complete T-variety. Assume

$$X \text{ can be covered by } T\text{-invariant open affine subvarieties.}$$

For each one-parameter subgroup $\lambda \in \mathbf{1PS}(T) = \mathrm{Hom}_{\mathsf{AlgGrp}}(\mathbb{G}_m, T)$,

$$\lambda : \mathbb{G}_m \longrightarrow T$$

defines a $\mathbb{G}_m$-action on X. Then for general $\lambda \in \mathbf{1PS}(T)$, we have

$$X^T = X^{\lambda(\mathbf{G}_m)}.$$

Actually, it suffices to avoid some hyperplanes determined by weights appearing tangent bundle of X^T. In particular, if $\dim X^T = 0$ (thus finite), X can be decomposed into strata with each of them isomorphic to affine space $\mathbb{A}^{\ell(Z)}$ for some $\ell(Z)$.

3.7. Equivariant Cohomology Assume X is a T-variety, we can define **equivariant cohomology** $H_T^\bullet(X)$.

(1) We have
$$H_T^\bullet(\mathsf{pt}) = \mathrm{Sym}_{\mathbb{Z}}(\mathbf{ch}(T)) = \mathbb{Z}[t_1, \cdots, t_N].$$
To be exact, for an equivariant line bundle $\mathbf{k}_\mathbf{u}$ over a point corresponding to character $\mathbf{u} \in \mathbf{ch}(T)$, we denote
$$\mathbf{u} = c_1(\Bbbk_\mathbf{u}) \in H_T^2(\mathsf{pt}).$$
In particular $H_T^{\mathrm{odd}}(\mathsf{pt}) = 0$.

(2) For any $H_T^\bullet(X)$, we have two ring homomorphisms
$$\begin{aligned} &\text{structure morphism } H_T^\bullet(\mathsf{pt}) \longrightarrow H_T^\bullet(X) \\ &\text{forgetful morphism } H_T^\bullet(X) \longrightarrow H^\bullet(X). \end{aligned}$$
Actually, there is a spectral sequence
$$E_2^{pq} = H^p(X) \otimes H_T^q(\mathsf{pt}) \Longrightarrow H_T^{p+q}(X).$$

(3) Assume X is a complete nonsingular variety, the spectral sequence always degenerate (due to Deligne). In particular, we have
$$\begin{aligned} H_T^\bullet(X) &\cong H_T^\bullet(\mathsf{pt}) \underset{\mathbb{Z}}{\otimes} H^\bullet(X) && \text{as } \mathrm{CH}_T(\mathsf{pt}) \text{ modules} \\ H^\bullet(X) &= H_T^\bullet(X) \underset{H_T^\bullet(\mathsf{pt})}{\otimes} \mathbb{Z} = \frac{H_T^\bullet(X)}{\langle H_T^2(\mathsf{pt}) \rangle} && \text{as a ring} \end{aligned}$$

We remark that equivariant Chow ring can be understood as the cohomology theory for T-varieties with base ring $H_T^\bullet(\mathsf{pt})$.

Fundamental Classes of Toric Varieties

3.8. Let Δ be a fan such that the toric variety $X(\Delta)$ is smooth and complete.

3.9. Basis Recall that

- $X(\Delta)$ can be covered by $X(\sigma) = \mathrm{Spec}\,\Bbbk[Q_\sigma]$ for $\sigma \in \Delta$;
- $X(\Delta)^T$ is discrete and in bijection to $\Delta(N)$.

Thus, we can conclude that $\mathrm{CH}^\bullet(X(\Delta))$ is a free $\mathbb{Z}$-module.

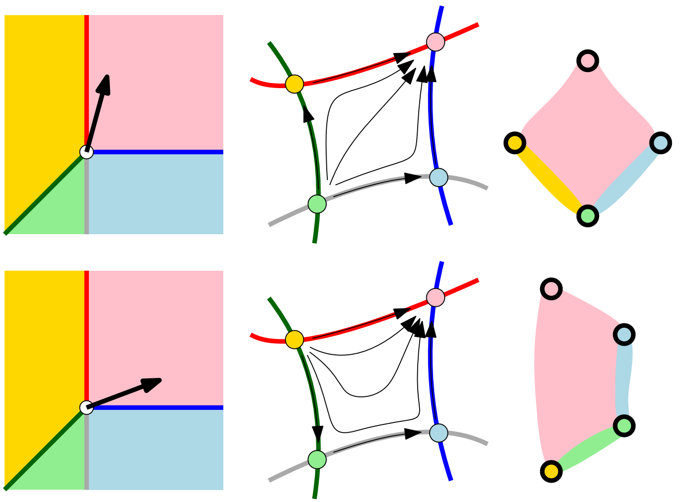
3.10. Remarks Actually, for a chosen $\lambda \in \mathbf{1PS}(T)$, we can compute the limit $\lim_{t\to 0} \lambda(t) \cdot f$ for any $f \in X(\Delta)$ following the same principle as we do for $f = \mathbf{1}_0$. To be exact, assume

$$f \in \mathcal{O}_\tau \subseteq \mathrm{Hom}_{\mathsf{Mnd}}(Q_\tau, \Bbbk).$$

Let σ be the maximal $\sigma \in \Delta$ such that $\lambda \in \sigma - \tau$. We have

$$\lim_{t\to 0} \lambda(t) \cdot f \in \mathcal{O}_\sigma \in \mathrm{Hom}_{\mathsf{Mnd}}(Q_\sigma, \Bbbk).$$

3.11. Example Here is an example,



3.12. Generators By direct computation of limit above, for generic $\lambda \in \mathbf{1PS}(T)$, and a fixed point corresponds $\sigma \in \Delta(N)$,

$$\overline{\mathrm{Attr}(\mathbf{1}_\sigma)} = \overline{\mathcal{O}}_\tau \qquad \text{for minimal } \tau \in \Delta \text{ such that } \lambda \in \sigma - \tau.$$

In particular,

$$\mathrm{CH}^k(X(\Delta)) = \sum_{\tau \in \Delta(k)} \mathbb{Z} \cdot [\overline{\mathcal{O}}_\tau].$$

3.13. Poincaré Polynomial We hope to read the Betti numbers

$$\beta^k = \mathrm{rank}\, \mathrm{CH}^k(X(\Delta))$$

directly from the fan. Let us denote the **Poincaré polynomial**

$$P_\Delta(t) = \sum \operatorname{rank} \mathrm{CH}^k(X(\Delta)) \cdot t^k.$$

By Poincaré duality, we have

$$P_\Delta(t) = t^N P_\Delta(t^{-1}).$$

Let us denote **face polynomial**

$$F_\Delta(t) = \sum_{\sigma\in\Delta} t^{\dim\sigma} = \sum \#\Delta(k) \cdot t^k.$$

Note that the orbit decomposition is a stratification. To be exact,

$$X(\Delta) = \bigsqcup_\tau \mathcal{O}_\tau, \quad \text{with} \quad \overline{\mathcal{O}_\tau} = \bigsqcup_{\sigma\supseteq\tau} \mathcal{O}_\sigma.$$

But this stratification is not affine,

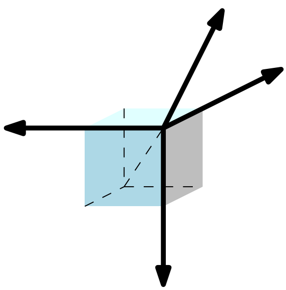
$$\mathcal{O}_\sigma \cong \mathbb{G}_m^{N-\dim\sigma}$$

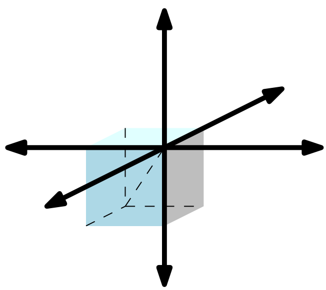
rather than an affine space. But we see that by suitable combination, it will become an affine stratification, thus

$$P_\Delta(t) = \sum_{\sigma\in\Delta} (t-1)^{N-\dim\sigma} = (t-1)^N F_\Delta\left(\tfrac{1}{t-1}\right).$$
$$P_\Delta(t) = t^N P(t^{-1}) = (1-t)^N F_\Delta\left(\tfrac{t}{1-t}\right).$$

Equivalently, $F(t) = t^N P(1+\frac{1}{t})$.

3.14. Example For Poincaré polynomials, there is one way to compute the coefficients using "difference operators". Here we give two examples of computation and left to readers to figure out the algorithm.





$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&4&&& \\ &&1&&3&&6&& \\ &1&&2&&3&&4& \\ 1&&1&&1&&1&&0 \end{array} \qquad \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&6&&& \\ &&1&&5&&12&& \\ &1&&4&&7&&8& \\ 1&&3&&3&&1&&0 \end{array}$$

Cup product over Toric Varieties

3.15. Recall Recall that

$$\mathrm{CH}^1(X(\Delta)) = \bigoplus_{\ell\in\Delta(1)} \mathbb{Z}\cdot[\overline{\mathcal{O}}_\ell] \Big/ \Big\langle \sum_{\ell\in\Delta(1)} \langle \mathbf{u}, \mathbf{v}_\ell\rangle\,[\overline{\mathcal{O}}_\ell] : \mathbf{u}\in\mathbb{Z}^N \Big\rangle,$$

where $\mathbf{v}_\ell$ is the first integer vector over the ray ℓ.

3.16. Lemma For $\ell\in\Delta(1)$ and $\sigma\in\Delta$ not containing ℓ, then

$$[\overline{\mathcal{O}}_\sigma]\cdot[\overline{\mathcal{O}}_\ell] = \begin{cases} [\overline{\mathcal{O}}_{\sigma'}], & \sigma' = \mathrm{span}_{\geq 0}(\sigma,\ell)\in\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Proof In the first case, let us choose a maximal cone $\alpha\in\Delta(N)$ containing σ'. Since we assume $X(\Delta)$ to be smooth, locally $X(\alpha)$ is nothing but $\Bbbk^N$. Note

$$\overline{\mathcal{O}_\sigma}\cap X(\alpha), \qquad \overline{\mathcal{O}_\ell}\cap X(\alpha), \qquad \overline{\mathcal{O}_{\sigma'}}\cap X(\alpha)$$

are all coordinate subspaces. Thus, it is easy to see that the intersection is transversal, thus

$$[\overline{\mathcal{O}}_\sigma]\cdot[\overline{\mathcal{O}}_\ell] = [\overline{\mathcal{O}}_{\sigma'}].$$

In the second case, $\overline{\mathcal{O}}_\sigma$ and $\overline{\mathcal{O}}_\ell$ are actually disjoint. Q.E.D.

3.17. Remark It is funny to see what is the product when σ containing ℓ_1, From the Lemma above, we see that

$$[\overline{\mathcal{O}}_\sigma] = [\overline{\mathcal{O}}_{\ell_1}]\cdots[\overline{\mathcal{O}}_{\ell_r}]$$

for $\sigma = \mathrm{span}_{\geq 0}(\ell_1,\ldots,\ell_r)$. We can use the relation for divisors to "move" $[\overline{\mathcal{O}}_{\ell_1}]$ such that it intersects $[\overline{\mathcal{O}}_\sigma]$ "transversally". Precisely, we can pick $\mathbf{u}\in\mathbb{Z}^N$ such that

$$\langle \mathbf{v}_{\ell_1}, \mathbf{u}\rangle = 1, \qquad \langle \mathbf{v}_{\ell_2}, \mathbf{u}\rangle = \cdots = \langle \mathbf{v}_{\ell_r}, \mathbf{u}\rangle = 0,$$

Then we have

$$[\overline{\mathcal{O}}_{\ell_1}] + \sum_{\ell \notin \{\ell_1, \dots, \ell_r\}} \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, [\overline{\mathcal{O}}_\ell] = 0.$$

Note that the condition of ℓ in the sum is equivalent to say ℓ is not contained in σ. Then we successfully move $[\overline{\mathcal{O}}_{\ell_1}]$ out of $[\overline{\mathcal{O}}_\sigma]$. So that

$$\begin{aligned} [\overline{\mathcal{O}}_\sigma] \cdot [\overline{\mathcal{O}}_\ell] &= - \sum_{\ell \notin \{\ell_1, \dots, \ell_r\}} \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, [\overline{\mathcal{O}}_\sigma] \cdot [\overline{\mathcal{O}}_{\ell_1}] \\ &= - \sum_{\ell \notin \{\ell_1, \dots, \ell_r\}} \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, [\overline{\mathcal{O}}_{\mathrm{span}_{\geq 0}(\sigma, \ell)}], \end{aligned}$$

where $[\overline{\mathcal{O}}_{\dots}]$ is understood as zero if not defined.

3.18. Theorem The Chow ring $\mathrm{CH}^\bullet(X(\Delta)) = H^{2\bullet}(X(\Delta))$ is generated by

$$D_\ell = [\overline{\mathcal{O}}_\ell]$$

for all $\ell \in \Delta(1)$ with the following relations

- $D_{\ell_1} \cdots D_{\ell_r} = 0$ if $\mathrm{span}_{\geq 0}(\ell_1, \dots, \ell_r) \notin \Delta$.
- $\sum \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, D_\ell$ for $\mathbf{u} \in \mathbb{Z}^N$.

Proof Let $A^\bullet$ be the ring generated by D_ℓ for $\ell \in \Delta(1)$ with above relations. It is clear that both of them are relations and we have an induced map

$$A^\bullet \longrightarrow \mathrm{CH}^\bullet(X(\Delta)).$$

This is surjective. There are two ways to show it is injective.

(1) The first method is to "move" as above remark and to show that

$$\{D_\tau : \overline{\mathcal{O}}_\tau = \overline{\mathrm{Attr}(\mathbf{1}_\sigma)} \text{ for some } \sigma \in \Delta(N)\}$$

generates $A^\bullet$, where $D_\tau = D_{\ell_1} \cdots D_{\ell_r}$ if $\tau = \mathrm{span}_{\geq 0}(\ell_1, \dots, \ell_r)$. We refer Fulton's book for details.

(2) The second method is to "lift" the result to **equivariant Chow ring/cohomology** which we will explain now.

3.19. Basis We have

$$H_T^\bullet(X(\Delta)) = \bigoplus_{\sigma\in\Delta(N)} H_T^\bullet(\mathsf{pt})\cdot[\overline{\mathrm{Attr}(\mathbf{1}_\sigma)}].$$

In particular,

$$H_T^\bullet(X(\Delta)) = \sum_{\sigma\in\Delta} H_T^\bullet(\mathsf{pt})\cdot[\overline{\mathcal{O}}_\sigma].$$

In particular, the Poincaré series is

$$\sum \operatorname{rank}\mathrm{CH}_T^k(X(\Delta))\cdot t^k = \frac{P_\Delta(t)}{(1-t)^N} = F_\Delta\left(\frac{t}{1-t}\right).$$

3.20. Product The equivariant Chow ring $\mathrm{CH}_T^\bullet(X(\Delta))$ is generated by

$$D_\ell = [\overline{\mathcal{O}}_\ell]_T$$

for all $\ell\in\Delta(1)$ with the following relations

$$D_{\ell_1}\cdots D_{\ell_r} = 0 \quad \text{if} \quad \mathrm{span}_{\geq 0}(\ell_1,\dots,\ell_r)\notin\Delta.$$

Proof Let

$$\mathcal{R}_\Delta = \mathbb{Z}[D_\ell]_{\ell\in\Delta(1)}$$

and $\mathcal{I}_\Delta$ be the ideal generated by

$$D_{\ell_1}\cdots D_{\ell_r}, \qquad \text{if } \mathrm{span}_{\geq 0}(\ell_1,\dots,\ell_r)\notin\Delta.$$

Actually this is famous — the ideal $\mathcal{I}_\Delta$ is known as **Stanley–Reisner ideal**. By the same line as above, we have an induced map

$$\mathcal{R}_\Delta/\mathcal{I}_\Delta \longrightarrow \mathrm{CH}_T^\bullet(X(\Delta)).$$

This is surjective since

$$D_\sigma := D_{\ell_1}\cdots D_{\ell_r} \longmapsto [\overline{\mathcal{O}}_{\ell_1}]\cdots[\overline{\mathcal{O}}_{\ell_r}] = [\overline{\mathcal{O}}_\sigma]_T$$

if $\sigma = \mathrm{span}_{\geq 0}(\ell_1,\dots,\ell_r)$. Thus it suffices to prove $\mathcal{R}_\Delta/\mathcal{I}$ is graded free abelian and to compute the Hilbert series of $\mathcal{R}_\Delta/\mathcal{I}_\Delta$. This is purely algebraic.

For any multi-index $\mathbf{a}\in\mathbb{Z}_{\geq 0}^{\Delta(1)}$, we denote $D^{\mathbf{a}} = \prod D_\ell^{a_\ell}$. We denote

$$\operatorname{supp}\mathbf{a} = \{\ell\in\Delta(1) : a_\ell\neq 0\}$$
$$\mathrm{span}_{\geq 0}(\mathbf{a}) = \mathrm{span}_{\geq 0}(\operatorname{supp}\mathbf{a})$$

It is clear that

$$\mathcal{I}_\Delta = \bigoplus_{\mathrm{span}_{\geq 0}(\mathbf{a}) \notin \Delta} \mathbb{Z} \cdot D^{\mathbf{a}}.$$

As a result,

$$\begin{aligned} \mathcal{R}_\Delta/\mathcal{I}_\Delta &= \bigoplus_{\mathrm{span}_{\geq 0}(\mathbf{a}) \in \Delta} \mathbb{Z} \cdot D^{\mathbf{a}} = \bigoplus_{\sigma \in \Delta} \left(\bigoplus_{\mathrm{span}_{\geq 0}(\mathbf{a}) = \sigma} \mathbb{Z} \cdot D^{\mathbf{a}} \right) \\ &= \bigoplus_{\sigma \in \Delta} \mathcal{R}_\sigma \cdot D_\sigma, \qquad \text{where } \mathcal{R}_\sigma = \mathbb{Z}[D_\ell]_{\ell \in \sigma(1)}. \end{aligned}$$

In particular, $\mathcal{R}_\Delta/\mathcal{I}_\Delta$ is graded free abelian and has Hilbert series

$$\sum \mathrm{rank}(\mathcal{R}_\Delta/\mathcal{I}_\Delta)_k \cdot t^k = \sum_{\sigma \in \Delta} \left(\frac{t}{1-t} \right)^{\dim \sigma} = F_\Delta\left(\frac{t}{1-t} \right),$$

which coincides with Poincaré polynomial of $\mathrm{CH}^\bullet_T(X(\Delta))$. Q.E.D.

3.21. Equivariant structure Moreover, the structure morphism $\mathrm{CH}^\bullet_T(\mathsf{pt}) \to \mathrm{CH}^\bullet_T(X(\Delta))$ sends

$$\mathbf{u} \longmapsto \sum \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, D_\ell,$$

for any $\mathbf{u} \in \mathbb{Z}^N = \mathbf{Ch}(T)$ viewed as $c_1(\Bbbk_{\mathbf{u}}) \in \mathrm{CH}^1_T(\mathsf{pt})$.

Proof Denote

$$D_{\mathbf{u}} = \sum \langle \mathbf{u}, \mathbf{v}_\ell \rangle \, D_\ell.$$

We see

$$\Gamma(\mathcal{O}(D_{\mathbf{u}})) = \Bbbk \cdot x^{-\mathbf{u}}.$$

There is a subtle sign problem — the weight of $x^{-\mathbf{u}}$ is $-\mathbf{u}$ under the **right action**, thus it is of weight $\mathbf{u}$ under the left action. Q.E.D.

Exercises

3.22. Localization Assume $\sigma = \mathrm{span}(\ell_1, \dots, \ell_N) \in \Delta(N)$ for $\ell_i \in \Delta(1)$. We denote $\mathbf{u}_{\sigma/\ell_i} \in \mathbb{Z}^N$ with

$$\left\langle \mathbf{u}_{\sigma/\ell_i}, \mathbf{v}_{\ell_j} \right\rangle = \delta_{ij}.$$

For general $\ell \in \Delta(1)$, show that

$$D_\ell|_\sigma = \begin{cases} \mathbf{u}_{\sigma/\ell}, & \ell \subseteq \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

where

$$\cdot|_\sigma : \mathrm{CH}_T(X(\Delta)) \longrightarrow \mathrm{CH}_T(\mathbf{1}_\sigma).$$

Hint: locally $\mathcal{O}(D_\ell)$ is trivial with T-weight $\mathbf{u}_{\sigma/\ell}$.

3.23. GKM picture For $\tau \in \Delta(N-1)$, we denote $\mathbf{u}_\tau$ the $\mathbb{Z}$-generator of $\tau^\perp \subseteq \mathbb{Z}^N$. This vector is unique up to a sign. Assume $\sigma_1, \sigma_2 \in \Delta(N)$ with $\sigma_1 \cap \sigma_2 \in \Delta(N-1)$. Show that

$$\mathbf{u}_\tau \text{ divides } D_\ell|_{\sigma_1} - D_\ell|_{\sigma_2}.$$

Actually, by **GKM theory**, the localization map

$$\mathrm{CH}_T^\bullet(X(\Delta))_\mathbb{Q} \longrightarrow \bigoplus_{\sigma\in\Delta(N)} \mathrm{CH}_T^\bullet(\mathbf{1}_\sigma)_\mathbb{Q}$$

is injective with image

$$\left\{ (z_\sigma)_\sigma : \begin{array}{c} \text{for any } \sigma_1 \cap \sigma_2 \in \Delta(N-1) \\ \mathbf{u}_{\sigma_1\cap\sigma_2} \mid z_{\sigma_1} - z_{\sigma_2} \end{array} \right\}.$$

3.24. Example Consider the case $\mathbb{P}^1$. We name two fixed point

$$0 = \mathbf{1}_{\mathbb{R}_{\geq 0}}, \qquad \infty = \mathbf{1}_{\mathbf{R}_{\leq 0}}.$$

We see

$$H_T^\bullet(\mathbb{P}^1) = \mathbb{Z}[D_0, D_\infty] / \langle D_0 D_\infty = 0 \rangle,$$

with the equivariant parameter $t = D_0 - D_\infty$. For $f \in \mathbb{Z}[D_0, D_\infty]$, we have

$$f|_0 = f(t, 0), \qquad f|_\infty = f(0, -t).$$

It is clear that

$$(f|_0, f|_\infty) = 0 \iff f \in \langle D_0 D_\infty \rangle.$$

Moreover,

$$t \mid f_\infty - f|_0$$

since $f|_\infty$ and $f|_0$ share the same constant term.

Next Time

We will come to the combiantoricial application of toric varieties after this talk. In other words, everything will be more combinatorial. Next time, we will discuss **Pick theorem**, which is the shadow of **Riemann–Roch theorem**.

4 Riemann–Roch and Pick theorem

Riemann–Roch

4.1. Let X be a non-singular variety.

4.2. Chern classes We can define **Chern classes** for any vector bundles by

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots \in \mathrm{CH}^\bullet(X)$$

such that

(i) for any divisor $D \in \mathrm{Pic}(X)$ (**normalization**)

$$c(\mathcal{O}(D)) = 1 + D;$$

(ii) for any morphism $f : X \to Y$ (**functoriality**)

$$c(f^*\mathcal{F}) = f^*c(\mathcal{F});$$

(iii) for any sub-bundle $\mathcal{G} \subseteq \mathcal{F}$ (**Whitney formula**)

$$c(\mathcal{F}) = c(\mathcal{G}) \cdot c(\mathcal{F}/\mathcal{G}).$$

Using the Whitney formula, we can define Chern classes for coherent sheaves (using **Hilbert's syzygy theorem**).

4.3. Example For a codimension one closed subvariety $D \subseteq X$, let us denote $\mathcal{O}_D$ the extension by zero out of D. We have

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

In particular, $c(\mathcal{O}) = c(\mathcal{O}_D) \cdot c(\mathcal{O}(-D))$, or

$$c(\mathcal{O}_D) = \frac{1}{1-D} = 1 + D + D^2 + \cdots \qquad \text{(finite)}.$$

4.4. Example For a vector bundle $\mathcal{F}$, with

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots,$$

Then

$$c(\mathcal{F}^\vee) = 1 - c_1(\mathcal{F}) + c_2(\mathcal{F}) - \cdots.$$

Actually, this is true for line bundles $\mathcal{O}(D)$, since $\mathcal{O}(D)^\vee = \mathcal{O}(-D)$. By Whitney formula, this is also true for vector bundle admits a filtration of line bundles. The general case follows from splitting principle, that roughly speaking,

If an identity holds for all vector bundles admitting a filtration of line bundles, then it is true for all vector bundles.

To be exact, for each $\mathcal{F}$, we can always find $f : F \to X$ such that $f^*\mathcal{F}$ admitting a filtration of line bundles, and $f^* : \mathrm{CH}(X) \to \mathrm{CH}(F)$ is injective.

4.5. K-theory Let us denote

$$K(X) = \bigoplus_{\text{coherent } \mathscr{F}} \mathbb{Z} \cdot [\mathscr{F}] \Big/ \left\langle \begin{array}{c} [\mathcal{F}] + [\mathcal{H}] = [\mathcal{G}] \quad \text{if we have a short} \\ \text{exact sequence } 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \end{array} \right\rangle.$$

It is generated by fibre bundles (we assume X to be non-singular), and forms a ring under $\otimes$.

4.6. Chern character We can define **Chern character** $\mathrm{ch} : K(X) \to \mathrm{CH}(X)_{\mathbb{Q}}$ such that

(i) for any divisor $D \in \mathrm{Pic}(X)$ (**normalization**)

$$\mathrm{ch}(\mathcal{O}(D)) = e^D = 1 + D + \frac{D^2}{2} + \cdots \in \mathrm{CH}(X;\mathbb{Q});$$

(ii) for any morphism $f : X \to Y$ (**functoriality**)

$$\mathrm{ch}(f^*\mathcal{F}) = f^* \mathrm{ch}(\mathcal{F});$$

(iii) for any sub-bundle $\mathcal{G} \subseteq \mathcal{F}$ (**additive**)

$$\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{G}) + \mathrm{ch}(\mathcal{F}/\mathcal{G}).$$

Using splitting principle, we can conclude Chern character is a ring homomorphism. Say,

$$\operatorname{ch}(\mathcal{F}\otimes\mathcal{G}) = \operatorname{ch}(\mathcal{F})\operatorname{ch}(\mathcal{G})$$

for two vector bundles $\mathcal{F}$ and $\mathcal{G}$. This follows from splitting principle:

This is true for line bundles
$\Longrightarrow$This is true for vector bundles admitting a filtration of lines bundles
$\Longrightarrow$This is true for all vector bundles.

4.7. Example Assume

$$c_1(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots$$

then

$$\operatorname{ch}(\mathcal{F}) = 1 + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2 - c_2(\mathcal{F})}{4} + \cdots .$$

Actually, this follows from

$$(1+x_1)\cdots(1+x_n) = 1 + \textstyle\sum_i x_i + \sum_{i<j} x_i x_j + \cdots$$

$$e^{x_1} + \cdots + e^{x_n} = 1 + \textstyle\sum_i x_i + \frac{1}{2}\sum_i x_i^2 + \cdots$$

$$x_1^2 + \cdots + x_n^2 = \frac{(\sum x_i)^2 - \sum_{i<j} x_i x_j}{2}.$$

4.8. Example Chern character is good enough with respect to pullback. But how about pushforward? We will only deal with the case when pushing forward to a point. To be exact, when X is complete, we have the "trace map"

$$\chi : K(X) \longrightarrow K(\mathsf{pt}) = \mathbb{Z}, \qquad \mathcal{F} \longmapsto \sum (-1)^i \overbrace{\dim H^i(X,\mathcal{F})}^{\text{finite dimensional}} .$$

$$\deg : \mathrm{CH}(X) \longrightarrow \mathrm{CH}(\mathsf{pt}) = \mathbb{Z}, \qquad [Y] \longmapsto \underbrace{\text{number of points on } Y}_{(=0 \text{ if } \dim > 0)}$$

We do not have

$$\begin{tikzcd} K(X) \arrow[r, "\chi"] \arrow[d, "\mathrm{ch}"'] \arrow[dr, phantom, "\substack{\text{NOT}\\ \text{commutative}}"] & \mathbb{Z} \arrow[d, equal] \\ \mathrm{CH}(X;\mathbb{Q}) \arrow[r, "\deg"'] & \mathbb{Q}. \end{tikzcd}$$

For example, for $\mathbb{P}^1$, denote x the class of a point,

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, mapsto] \arrow[d, mapsto] & 2 = \dim \Gamma(\mathcal{O}(1)) - 0 \arrow[d, phantom, sloped, "\neq"] \\ 1 + x \arrow[r, mapsto] & 1. \end{tikzcd}$$

4.9. Hirzebruch–Riemann–Roch theorem For this problem, there is a solution — we only need to twist Chern characters by **Todd classes**. The Todd class is defined to satisfy

(i) for any divisor $D \in \operatorname{Pic}(X)$ (**normalization**)

$$\operatorname{Td}(\mathcal{O}(D)) = \frac{D}{1 - e^{-D}} = 1 + \frac{D}{2} + \frac{D^2}{12} + \cdots \in \operatorname{CH}^{\bullet}(X; \mathbb{Q}).;$$

(ii) for any morphism $f : X \to Y$ (**functoriality**)

$$\operatorname{Td}(f^* \mathcal{F}) = f^* \operatorname{Td}(\mathcal{F});$$

(iii) for any sub-bundle $\mathcal{G} \subseteq \mathcal{F}$ (**multiplicative**)

$$\operatorname{Td}(\mathcal{F}) = \operatorname{Td}(\mathcal{G}) \operatorname{Td}(\mathcal{F}/\mathcal{G}).$$

We denote $\operatorname{Td}(X) = \operatorname{Td}(\mathscr{T}_X)$. Then **Hirzebruch–Riemann–Roch** tells that

$$\begin{tikzcd} K(X) \arrow[r, "\chi"] \arrow[d, "\operatorname{Td}(X) \cdot \operatorname{ch}(-)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[d, equal] \\ \operatorname{CH}(X; \mathbb{Q}) \arrow[r, "\deg"'] & \mathbb{Q}. \end{tikzcd}$$

4.10. Example Let still consider $\mathbb{P}^1$. In this case, $\mathscr{T}_{\mathbb{P}^1} = \mathcal{O}(2)$. and thus

$$\operatorname{Td}(\mathbb{P}^1) = \operatorname{Td}(\mathscr{T}_{\mathbb{P}^1}) = \frac{2x}{1 - e^{-2x}} = 1 + \frac{2x}{2} = 1 + x.$$

As a result,

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, mapsto] \arrow[d, mapsto] & 2 = \dim \Gamma(\mathcal{O}(1)) - 0 \arrow[d, equal] \\ (1 + x)(1 + x) = 1 + 2x + 0 \arrow[r, mapsto] & 2. \end{tikzcd}$$

4.11. Example Assume

$$c_1(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots$$

then

$$\mathrm{Td}(\mathcal{F}) = 1 + \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12} + \cdots.$$

Actually, this follows from

$$(1+x_1)\cdots(1+x_n) = 1 + \textstyle\sum_i x_i + \sum_{i<j} x_i x_j + \cdots$$

$$\frac{x_1}{1-e^{-x_1}} \cdots \frac{x_1}{1-e^{-x_1}} = 1 + \tfrac{1}{2}\textstyle\sum_i x_i + \tfrac{1}{12}\sum_i x_i^2 + \tfrac{1}{4}\sum_{i<j} x_i x_j + \cdots$$

$$\tfrac{1}{12}\textstyle\sum_i x_i^2 + \tfrac{1}{4}\sum_{i<j} x_i x_j = \tfrac{1}{12}\big((\sum x_i)^2 + \sum_{i<j} x_i x_j\big).$$

Riemann–Roch theorem on Toric Varieties

4.12. Theorem Over toric varieties, we have a short exact sequence

$$0 \longrightarrow \Omega_{X(\Delta)} \longrightarrow \mathcal{O}_{X(\Delta)} \otimes_{\mathbb{Z}} \mathbb{Z}^N \longrightarrow \bigoplus_{\ell \in \Delta(1)} \mathcal{O}_{\overline{\mathcal{O}}_\ell} \longrightarrow 0.$$

We refer Fulton's book for a proof. Actually we have the Euler sequence

$$0 \longrightarrow \mathcal{O}_{X(\Delta)}^{\oplus s} \longrightarrow \bigoplus_{\ell \in \Delta(1)} \mathcal{O}(D_\ell) \longrightarrow \mathscr{T}_{X(\Delta)} \longrightarrow 0,$$

where $s = |\Delta(1)| - N$. This follows from the quotient construction of toric variety.

4.13. Recall that for a smooth complete toric variety $X(\Delta)$,

$$H_T^\bullet(X(\Delta)) = \mathcal{R}_\Delta / \mathcal{I}_\Delta$$
$$H^\bullet(X(\Delta)) = \mathrm{CH}(X(\Delta)) = \mathcal{R}_\Delta \big/ (\mathcal{I}_\Delta + \mathcal{J}_\Delta),$$

where

$$\begin{aligned}
\mathcal{R}_\Delta &= \mathbb{Z}[D_\ell]_{\ell \in \Delta(1)}, \\
\mathcal{I}_\Delta &= \big\langle D_{\ell_1} \cdots D_{\ell_r} : \mathrm{span}_{\geq 0}(\ell_1, \ldots, \ell_r) \notin \Delta \big\rangle, \\
\mathcal{J}_\Delta &= \big\langle \textstyle\sum \langle \mathbf{u}, \mathbf{v}_\ell \rangle D_\ell : \mathbf{u} \in \mathbb{Z}^N \big\rangle.
\end{aligned}$$

For any $\sigma \in \Delta$,

$$[\overline{\mathcal{O}}_\sigma] = D_{\ell_1} \cdots D_{\ell_r}$$

if $\sigma = \mathrm{span}_{\geq 0}(\ell_1, \ldots, \ell_r)$.

4.14. Computation of Chern classes Let us denote $D_\ell = [\overline{\mathcal{O}}_\ell]$ for $\ell \in \Delta(1)$. Recall that we have a short exact sequence

$$0 \longrightarrow \mathcal{O}(-D_\ell) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\overline{\mathcal{O}}_\ell} \longrightarrow 0.$$

As a result, by Whitney property

$$c(\mathcal{O}_{\overline{\mathcal{O}}_\ell}) = \frac{c(\mathcal{O}_X)}{c(\mathcal{O}(-D_\ell))} = \frac{1}{1-D_\ell}.$$

Therefore,

$$c(\Omega_X) = \frac{c(\mathcal{O}_X^N)}{\prod \frac{1}{1-D_\ell}} = \prod_{\ell\in\Delta(1)} (1-D_\ell).$$

As a result,

$$c(\mathscr{T}_X) = \prod_{\ell\in\Delta(1)} (1+D_\ell) = \sum_{\sigma\in\Delta} [\overline{\mathcal{O}}_\sigma].$$

In particular,

$$\mathrm{Td}(\mathscr{T}_X) = \prod_{\ell\in\Delta(1)} \frac{D_\ell}{1-e^{-D_\ell}}.$$

Now we have a couple of applications of Riemann–Roch.

4.15. Degree of Todd class Apply to trivial bundle,

$$\begin{tikzcd}
\mathcal{O}_{X(\Delta)} \arrow[r, mapsto] \arrow[d, mapsto] & 1 \arrow[d, mapsto] \\
\mathrm{Td}(X(\Delta)) \arrow[r] & \deg(\mathrm{Td}(X))
\end{tikzcd}$$

This shows

$$\mathrm{Td}(X(\Delta)) = 1 + \frac{1}{2}\sum_{\ell\in\Delta(1)} D_\ell + \cdots + 1\cdot[\mathsf{point}] \in \mathrm{CH}(X(\Delta;\mathbb{Q})).$$

For example,

$$\mathrm{Td}(X(\Delta)) = 1 + [\mathsf{point}] \qquad N=1,$$

$$\mathrm{Td}(X(\Delta)) = 1 + \frac{1}{2}\sum_{\ell} D_\ell + [\mathsf{point}] \qquad N=2.$$

4.16. Volume Let D be a globally generated divisor. Recall that we know

$$\frac{1}{N!}\deg(D^N) = \mathrm{volume}(P_D).$$

Actually, this can be seen from Riemann–Roch

$$\begin{tikzcd}
\mathcal{O}(mD) \arrow[r, mapsto] \arrow[d, mapsto] & \#(mP_D \cap \mathbb{Z}^N) \arrow[d, mapsto] \\
(1+\text{constant}) \cdot \left(\frac{m^N D^N}{N!} + \cdots\right) \arrow[r, mapsto] & \frac{m^N}{N!}\deg(D^N) + \cdots .
\end{tikzcd}$$

Thus,

$$\frac{1}{N!}\deg(D^N) = \lim_{m\to\infty} \frac{\#(mP_D \cap \mathbb{Z}^N)}{m^N} = \mathrm{volume}(P_D).$$

4.17. Lower dimensional Volume For $\sigma \in \Delta(N-k)$, we denote P_σ the face of P_D parallel to $\sigma^\perp$. Actually $\overline{\mathcal{O}}_\sigma$ is a toric variety of dimension k, can apply the same trick, we will get

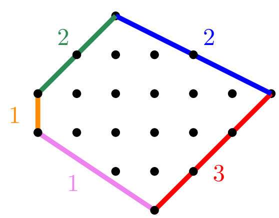
$$\frac{1}{k!}\deg(D^k \cap [\overline{\mathcal{O}}_\sigma]) = \frac{1}{k!}\deg_{\overline{\mathcal{O}}_\sigma}\left((D|_{\overline{\mathcal{O}}_\sigma})^k\right) = \mathrm{volume}_\sigma(P_\sigma).$$

Here, the volume is taken inside the space parallel to $\sigma^\perp$ normalized by

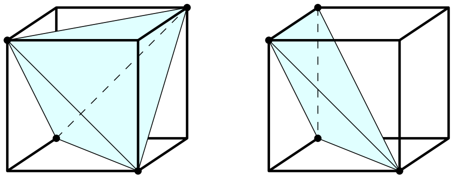
$$\mathrm{volume}_\sigma(\text{lattice cubic}) = 1.$$

In the degenerate case i.e. when $k = 0$, the volume of a single point is 1.

4.18. Example Here is an example.



Here are two other examples



where each face has area $1/2$.

4.19. Precise Riemann–Roch Assume

$$\mathrm{Td}(X(\Delta)) = \sum_{\sigma\in\Delta} r_\sigma[\overline{\mathcal{O}}_\sigma].$$

Note that this expansion is not unique in general. We can now conclude

$$\begin{aligned}\#(P_D \cap \mathbb{Z}^N) &= \deg\Big(\big(\textstyle\sum_{\sigma\in\Delta} r_\sigma[\overline{\mathcal{O}}_\sigma]\big)\big(\textstyle\sum_k \frac{D^k}{k!}\big)\Big)\\ &= \sum_{\sigma\in\Delta} r_\sigma \,\mathrm{volume}_\sigma(P_\sigma)\\ &= \mathrm{Vol}(P_D) + (\text{middle volumes}) + 1.\end{aligned}$$

This is a generalization of **Pick theorem**.

4.20. Projective Line When $N = 1$, then only possibility is $\mathbb{P}^1$. Note that

$$\mathrm{Td}(X(\Delta)) = 1 + [\mathsf{point}].$$

Applying the formula for $D = mD_0 + nD_\infty$ when $m < n$, we see

$$\#\big([m,n] \cap \mathbb{Z}\big) = \mathrm{length}([m,n]) + 1.$$

4.21. Toric Surfaces When $N = 2$. We have

$$\mathrm{Td}(X(\Delta)) = 1 + \frac{1}{2}\sum_\ell D_\ell + [\mathsf{point}].$$

We have

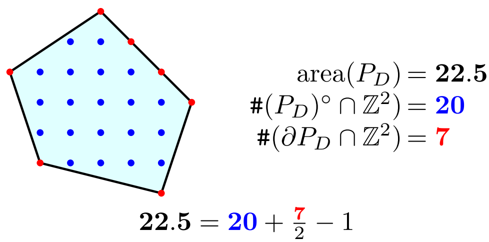
$$\begin{aligned}
\#(P_D \cap \mathbb{Z}^2) &= \operatorname{area}(P_D) + \frac{1}{2} \sum_{\ell \in \Delta(1)} \operatorname{length}_\ell(P_\ell) + 1 \\
&= \operatorname{area}(P_D) + \frac{1}{2} \sum_{\ell \in \Delta(1)} \big(\#(P_\ell \cap \mathbb{Z}^2) - 1\big) + 1 \\
&= \operatorname{area}(P_D) + \frac{1}{2} \#(\partial P_D \cap \mathbb{Z}^2) + 1.
\end{aligned}$$

We shall write

$$\operatorname{area}(P_D) = \#\big((P_D)^\circ \cap \mathbb{Z}^2\big) + \frac{1}{2}\#(\partial P_D \cap \mathbb{Z}^2) - 1.$$

This is known as **Pick theorem**.

4.22. Example Here is an example of computing area by point-counting



Exercise

4.23. Ehrhart Polynomials Let P be an integer polyhedron in $\mathbb{R}^N$. Show that

$$\#(mP \cap \mathbb{Z}^N)$$

is a polynomial in m. This polynomial is known as **Ehrhart polynomial**.

Hint: on the geometric side, we can find a smooth fan such that $P = P_D$ for some globally generated divisor D.

4.24. Counterexample of higher Pick theorem For higher dimensions, there are no **uniform** "Pick theorem", i.e. different "shapes" have different version of Pick theorem. Try to find a counterexample such that the underlying hypergraph are the isomorphic, of the same distribution of integer points on each face, but with different volume.

Next Time

We will discuss some Hodge theory and apply it to Stanley's theorem which characterizes the restriction of numbers of faces of different dimensions needed to build a simplicial polyhedron.

5 Hard Lefschetz and Stanley's Theorem

An introduction to Hard Lefchetz

5.1. Hodge decomposition Let X be a projective, nonsingular variety of (complex) dimension n (or more general, a Kälher manifold of real dimension $2n$). Hodge theory tells us that we have the decomposition

$$\begin{array}{c|ccccccc}
H^{2n}(X;\mathbb{C}) & & & & H^{n,n}(X) & & & \\
H^{2n-1}(X;\mathbb{C}) & & & H^{n-1,n}(X) & & H^{n,n-1}(X) & & \\
H^{2n-2}(X;\mathbb{C}) & & H^{n-2,n}(X) & & H^{n-1,n-1}(X) & & H^{n,n-2}(X) & \\
\cdots & \cdots\cdots\cdots & & \cdots\cdots\cdots & & \cdots\cdots\cdots & & \cdots\cdots\cdots \\
H^{2}(X;\mathbb{C}) & & H^{2,0}(X) & & H^{1,1}(X) & & H^{0,2}(X) & \\
H^{1}(X;\mathbb{C}) & & & H^{0,1}(X) & & H^{1,0}(X) & & \\
H^{0}(X;\mathbb{C}) & & & & H^{0,0}(X) & & &
\end{array}$$

where $H^{p,q}(X)$ is the Dolbeault cohomology

$$H^{p,q}(X) := H^q(X, \Omega_X^p) \qquad \text{(cohomology of coherent sheaves).}$$

5.2. Properties Let us denote $h^k = \dim_{\mathbb{C}} H^k(X;\mathbb{C})$ and $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$. We have

Horizontal	$h^{p,q} = h^{q,p}$	$\forall p, q$
Each Row	$h^k = \sum_{p+q=k} h^{p,q}$	$\forall 0 \le k \le 2n$
Vertical	$h^{p,q} = h^{p',q'}$	$\forall p + p' = q + q' = n$
Each Column	$h^{p,q} \le h^{p+1,q+1}$	$\forall p + q < n$

These properties are too strong, even after being folded up

Vertical	$h^k = h^\ell$	$\forall k + \ell = 2n$	Poincaré Duality
Each Column	$h^k \le h^{k+2}$	$\forall k < n$	Hard Lefschetz

5.3. Example For example, $\mathbb{C}P^n$.

h^0	h^1	h^2	h^3	$\cdots$	h^{2n-1}	h^{2n}
1	0	1	0	$\cdots$	0	1

For $\mathbb{C}P^2 \times \mathbb{C}P^2$.

h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8
1	0	2	0	3	0	2	0	1

For $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^2$.

h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8
1	0	3	0	4	0	3	0	1

5.4. Lefschetz Operator The above statement is numerical. Actually, the inequality can be realized by **Lefschetz operator**. Since we can embed $X \subseteq \mathbb{P}^n$, we have a natural map

$$H^\bullet(X;\mathbb{Q}) \longrightarrow H^{\bullet+2}(X;\mathbb{Q})$$

given by cup product with the hyperplane section from $\mathbb{P}^n$. **Hard Lefschetz** tells us the composition

$$H^{n-p}(X;\mathbb{C}) \xrightarrow{L} \cdots \xrightarrow{L} H^{n+p}(X;\mathbb{C})$$

is an isomorphism for any p. In particular, L is injective when $\bullet < n$, and surjective when $\bullet \geq n$.

5.5. Example A typical way to illustrate them is (when $H^{\mathrm{odd}} = 0$)

$$\begin{array}{c}
\bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet
\end{array}$$

Actually, this is a good exercise of linear algebra to show that we can actually pick a set of basis as above.

5.6. More general Now, let X be complete and non-singular. For any ample line bundle $\mathcal{O}(D)$, we can define

$$L : H^\bullet(X;\mathbb{Q}) \longrightarrow H^{\bullet+2}(X;\mathbb{Q})$$

by cup product with the divisor D. Then L also holds hard Lefschetz. Actually, this follows from Serre's Theorem that mD is very ample for $m \gg 0$.

5.7. Remark We remark that in particular, the Betti number is **unimodal**

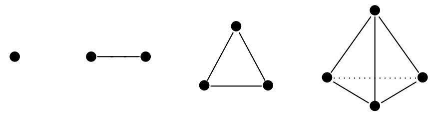
$$h^0 \le h^2 \le \cdots \ge h^{2n-1} \ge h^{2n}.$$

But this is not how Huh shows the Read's conjecture.

Simplicial Polytopes

5.8. When we say a **polytope**, we mean a convex hull of finite many points **with interior** (usually it is assumed).

5.9. Simplicial Polytopes A polytopes $K \subseteq \mathbb{P}^N$ is called **simplicial** if each face of it is a simplex.

simplex =  $\cdots$

point segment triangle tetrahedron $\cdots$

5.10. Face Vector We denote

$$f_i = \#\left\{ i\text{-faces on } K \right\}.$$

We call (f_i) the **face vector** of K. We will characterize the exact conditions for (f_i) to be a face vector of some simplicial polyhedron.

5.11. Example When $N = 1$ (resp., $N = 2$), the only simplicial polyhedrons can be a segment (resp., a triangle).

5.12. Example It is clear that we should have

$$f_0 \ge N + 1 \qquad \because K \text{ has some interior points.}$$

Actually, any N points are in a plan of dimension $N - 1$, so having no interior points. We also have

$$f_{N-1} \ge N + 1 \qquad \because K \text{ is bounded.}$$

Actually, to get a bounded domain, we need at least $N + 1$ half space.

5.13. Example When $N = 3$. Then, we see by the famous Euler formula,

$$f_0 - f_1 + f_2 = 2. \tag{R1}$$

Moreover, every edge is shared exactly twice by a triangle.

$$3f_2 = 2f_1. \tag{R2}$$

We also need to require

$$f_0 \geq 4 \tag{R3}$$

Actually, the mentioned three relations are also sufficient — since f_1 and f_2 are both determined by f_0,

$$f_1 = 3(f_0 - 2), \qquad f_2 = 2(f_0 - 2),$$

and it is easy to construct a polytope with given number of vertices (say, by attaching small tetrahedron on any face).

5.14. Constuction Now, assume we have a simplicial polyhedron K. We can assume $0 \in K^\circ$ and the vertices K are all rational (thus integer) points. We can construct a fan

$$\Delta = \left\{ \operatorname{span}_{\geq 0}(F) : F \text{ is a face of } K \right\}.$$

Note that the face polynomial is

$$F_K(t) := F_\Delta(t) = 1 + f_0 t + \cdots + f_{N-1} t^n.$$

It is clear that $X(\Delta_P)$ is complete since

$$|\Delta_P| = \mathbb{R}^N.$$

Moreover, $X(\Delta_P)$ is projective, i.e. admits a very ample divisor. Actually, we can pick the divisor

$$D = \sum c_\ell D_\ell$$

with $c_\ell \mathbf{u}_\ell$ the vertices on P. In general, $X(\Delta)$ is not smooth. But $X(\Delta)$ is **rational smooth**, actually it is always an orbifold, i.e. locally a quotient by a finite group.

5.15. Note that for projective smooth toric variety $X(\Delta)$, its coomology ring $H^\bullet(X(\Delta);\mathbb{Q})$ is generated by degree 2 elements holding hard Lefschetz. Actually, the same is true for projective and rational smooth toric varieties. In particular, the Poincaré polynomial

$$P_K(t) = P_\Delta(t) = (1-t)^N F_K\left(\tfrac{t}{1-t}\right).$$

Assume

$$P_K(t) = \sum h_i t^i.$$

We usually call (h_i) the **h-vector** of a polytope.

5.16. Example For $N = 3$, we have

Octahedron		
vertices	edges	faces
6	12	8

Icosahedron		
vertices	edges	faces
12	30	20

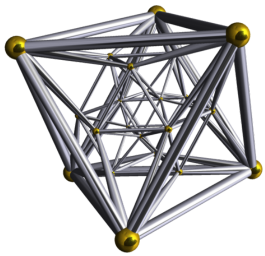
$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 6 & & & \\
 & & 1 & & 5 & & 12 & & \\
 & 1 & & 4 & & 7 & & 8 & \\
1 & & 3 & & 3 & & 1 & & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 12 & & & \\
 & & 1 & & 11 & & 30 & & \\
 & 1 & & 10 & & 19 & & 20 & \\
1 & & 9 & & 9 & & 1 & & 0
\end{array}$$

5.17. Example For simplex, we

$(n-1)$-dimensional simplex			
vertices	edges	$\cdots$	$(n-1)$-cells
$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & \binom{n}{1} & & & & \\
 & & & 1 & & \binom{n-1}{1} & & \binom{n}{2} & & & \\
 & & 1 & & \cdots & & \binom{n-1}{2} & & \cdots & & \\
 & 1 & & 2 & & \cdots & & \cdots & & \binom{n}{n-1} & \\
1 & & 1 & & 1 & & 1 & & 1 & & 0
\end{array}$$

5.18. Example Here is an example when $N = 4$,



from wikipedia

24-cells

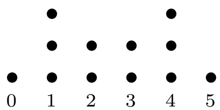
vertices	edges	faces	cells
24	96	96	24

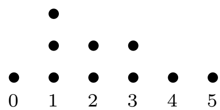
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 24 & & & & \\
 & & & 1 & & 23 & & 96 & & & \\
 & & 1 & & 22 & & 73 & & 96 & & \\
 & 1 & & 21 & & 51 & & 23 & & 24 & \\
1 & & 20 & & 30 & & 20 & & 1 & & 0
\end{array}
$$

5.19. Restrictions A By hard Lefschetz, we have

$$h_i = h_{N-i}, \qquad \begin{array}{l} i < N/2 \Rightarrow h_i \le h_{i+1}, \\ i > N/2 \Rightarrow h_i \le h_{i+1}. \end{array} \tag{\#HL}$$

For example (h_i) cannot be

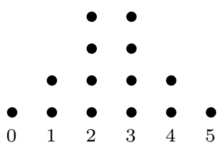


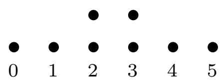


5.20. Restriction B Since $H^\bullet(X(\Delta); \mathbb{Q})$ is generated by degree 2 element, so

$$\text{$P_K(t)$ is the Hilbert series of a graded algebra generated by degree 1 element.} \tag{Mac}$$

For example (h_i) cannot be





Actually, the condition (`Mac`) can be described explicitly by Macaulay using Gröbner basis. But let us skip this since it is too technical.

5.21. Stanley Theorem The vector $(f_0, \ldots, f_{N-1})$ appears as a face vector of simplicial polyhedron if and only if the coefficients of

$$P_K(t) = (1-t)^N F_K\left(\tfrac{t}{1-t}\right)$$

satisfies (`#HL`) and (`Mac`).

Proof We have seen the necessity by Hodge theory. The sufficiency is given by direct construction by Billera and Lee.

Exercises

5.22. Show that the face vector of a simplicial polytope satisfies

$$f_p \geq \binom{N+1}{p+1}.$$

Next time

Now, we finished the part from Fulton's book. We will turn to our main theme, the proof of Read's conjecture. We will be of research level from the next time — it takes more time to solve a single problem.

6 Hodge Index and Mixed Volume

Hodge Theory

6.1. Hodge–Riemann relation Let X be a projective, non-singular variety of dimension n. Recall that **hard Lefschetz theorem** implies the iterated Lefschetz operator

$$H^{n-p}(X;\mathbb{Q}) \xrightarrow{L^p} H^{n+p}(X;\mathbb{Q})$$

is an isomorphism. Now, with Poincaré pairing (=intersection pairing), we can introduce Lefschetz pairing on $H^k(X;\mathbb{Q})$ by

$$\langle \alpha, \beta \rangle_{\text{Lefschetz}} = \langle \alpha, L^{n-k}\beta \rangle_{\text{Poincaré}} = \deg(L^{n-k}\alpha\beta).$$

The famous **Hodge–Riemann** relation claims the index of this pairing. We will use a typeical diagram to illustrate the index when $H^{pq}(X) = 0$ when $p \neq q$. The index is typically

$$\begin{array}{c} + \to + \\ - \to - \to - \to - \\ - \to - \to - \to - \\ + \to + \to + \to + \to + \to + \end{array}$$

$+$ positive definite

$-$ negative definite

$\to$ Lefschetz operator

6.2. Example When X is a surface, it is known that

$$\mathrm{CH}^1(X)_{\mathbb{C}} \longrightarrow H^{1,1}(X)$$

is surjective. Since the Lefschetz pairing over $\mathrm{CH}^1(X;\mathbb{Q})$ is now nothing but the intersection pairing, this gives the following **Hodge index theorem** on surface.

Hodge Index Theorem Let S be a projective non-singular surface with ample divisor H. If a divisor D with $D \cdot H = 0$, then the self-intersection $D^2 < 0$.

(see Hartshorne for a pure algebro geometric proof)

6.3. Example Consider a cubic surface $X \subset \mathbb{P}^3$. Its Hodge diamond look like

$$
\begin{array}{lccccc}
H^4 & & & 1 & & \\
H^3 & & 0 & & 0 & \\
H^2 & 0 & & 7 & & 0 \\
H^1 & & 0 & & 0 & \\
H^0 & & & 1 & &
\end{array}
$$

It is well-known that there are exactly 27 lines over X. They can be parametrized by root system of E_6 with

$$\langle \ell_\alpha, \ell_\beta \rangle = -\langle \alpha^\vee, \beta \rangle.$$

So they span a 6-dimensional negative definite subspace. Moreover, they are orthogonal to the canonical divisor $\kappa_X^\perp$.

6.4. Lorentzian Let us call a symmetric pair **Lorentzian** if it has at most one positive eigenvalue. For example, the Lefschetz pairing over $H^2(X;\mathbb{Q})$ is nondegenerate Lorentzian.

6.5. A linear algebra If a symmetric matrix $A = (a_{ij})$ is Lorentzian with nonnegative entries, then any principal 2-minor cannot be positive

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = a_{ii}a_{jj} - a_{ij}^2 \le 0.$$

Proof Note that $A - \epsilon I$ for small enough ϵ is nondegenerate Lorentzian. If any principal minor is positive, then $V = \operatorname{span}(\mathbf{e}_i, \mathbf{e}_j)$ is positive-definite. Thus $V \oplus V^\perp$ cannot be Lorentzian. Then we can take limit $\epsilon \to 0$ to conclude. Q.E.D.

6.6. Amplitude and Nefness We denote

$$\text{💡}^k(X) = \sum_{\substack{Y \text{ effective cycle} \\ \text{of codimension } k}} \mathbb{R}_{>0} \cdot [Y] \in H^{2k}(X;\mathbb{R}).$$

We define two cones in $H^2(X;\mathbb{R})$

$$\text{🍎}(X) = \sum_{D \text{ ample}} \mathbb{R}_{>0} \cdot D \in H^2(X;\mathbb{R})$$

$$\text{🔪}(X) = \sum_{D \text{ nef}} \mathbb{R}_{\ge 0} \cdot D \in H^2(X;\mathbb{R})$$

Then we have

🔪 $=$ cls 🍎
🍎 $=$ int 🔪

Nef cone is the closure of ample cone. Ample cone is the interior of ample cone.

🔪 $\cap$ ($-$ 🔪) $= 0$
🔪 $+$ ($-$ 🔪) $= H^2$

For any divisor D and ample class A, the divisor $D + mA$ is ample for $m \gg 0$.

cls 💡$^{n-1}$ $=$ pol 🔪

closure of effective cone and nef cones are polarizations to each other

💡 $\cdot$ 💡 $\nsubseteq$ 💡

Two effective classes might intersect negatively.

🍎 $\cdot$ 💡 $\subseteq$ 💡

Ample class intersects nonzero effective classes **positively**.

🍎 $+$ 🍎 $\subseteq$ 🍎

Sum of two ample classes is still ample.

🍎 $\subseteq$ 🔪 $\cap$ 💡

Ample class is nef. Very ample class is effective.

🔪 $\cdot$ 💡 $\subseteq$ 💡

Nef class intersects effective classes **non-negatively**.

🔪 $+$ 🔪 $\subseteq$ 🔪

Sum of two nef classes is still nef.

Note that usually people do not say a class in class group is effective. When I say this, it means it can be represented by an effective divisor.

6.7. Logarithm concave For a series of number $\mu_0, \mu_1, \ldots, \mu_k$, we say it is **logarithm concave** if

(i) $\mu_0, \mu_1, \ldots, \mu_k \geq 0$ without internal zeros;

(ii) $\mu_i^2 \geq \mu_{i-1}\mu_{i+1}$ for $i = 1, \ldots, k-1$.

In particular, it is **unimodal**

$$\mu_0 \leq \mu_1 \leq \cdots \leq \mu_r \geq \ldots \geq \mu_{k-1} \geq \mu_k,$$

for some r.

6.8. Remark Since unimodal series has no internal zeros, and being unimodal is closed, being logarithm concave is a closed property.

6.9. Khovanskii–Teissier For two nef divisors α, β, and an irreducible subvariety Y of dimension k, then

$$\mu_i = \deg\left((\alpha^{n-i} \smile \beta^i) \frown [Y]\right) = \int_Y \alpha^i \beta^{n-i}$$

is logarithm concave.

Proof Note that unimodality is a closed property, it suffices to show when α and β are very ample. In this case, all $\mu_i > 0$. For $\dim Y \geq 3$ and any very ample class α, by Bertini argument, we can always pick a smooth irreducible subvariety $Z \subseteq Y$ such that

$$\alpha \frown [Y] = [Z].$$

By iterated using above argument, we can pick a smooth irreducible subvariety $Y' \subseteq Y$ such that

$$(\alpha^{n-i-1} \smile \beta^{i-1}) \frown [Y] = [Y'].$$

By replacing Y by Y' we can assume Y to be of dimension 2, i.e. a surface. Consider the quadratic form

$$p(x,y) = \int_Y (x\alpha + y\beta)^2 = \mu_0 x^2 + 2\mu_1 xy + \mu_2 y^2.$$

By Hodge index theorem, $p(x,y)$ is not (positively) definite, so that the determinant

$$\begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix} = \mu_0\mu_2 - \mu_1^2 \leq 0,$$

which is exactly what we want to show. Q.E.D.

Alexandrov–Fenchel inequality

6.10. Mixed volume Let $K_1, \ldots, K_r$ be compact convex sets in $\mathbb{R}^N$. The function for $x_1, \ldots, x_r \geq 0$

$$f(x_1, \ldots, x_r) = \mathrm{Vol}(x_1 K_1 + \cdots + x_r K_r)$$

is known to be a homogeneous polynomial (known as **Minkowski's Theorem**). We define **mixed volume**

$$\mathrm{Vol}(K_{i_1}, \ldots, K_{i_r})$$

by the coefficients in

$$\begin{aligned} f(x_1,\ldots,x_r) &= \sum_{i_1,\ldots,i_r} \mathrm{Vol}(K_{i_1},\ldots,K_{i_r})x_{i_1}\cdots x_{i_r} \\ &= \sum_{a_1+\ldots+a_r=N} \mathrm{Vol}(K_1,\overset{a_1}{\cdots},K_1,\cdots,K_r,\overset{a_r}{\cdots},K_r)\binom{N}{a_1\cdots a_r}x_1^{a_1}\cdots x_r^{a_r} \end{aligned}$$

6.11. Example We have

$$\mathrm{Vol}(K,\ldots,K) = \mathrm{Vol}(K).$$

In general, one can check

$$\mathrm{Vol}(K_{i_1},\ldots,K_{i_r})$$

only depends on $K_{i_1},\ldots,K_{i_r}$ (a prior, it depends on all all K_i's).

6.12. Example For two convex bodies A, B in $\mathbb{R}^2$. We have

$$\frac{1}{2}\mathrm{Vol}(A,A) = \mathrm{Vol}(A), \qquad \frac{1}{2}\mathrm{Vol}(B,B) = \mathrm{Vol}(B)$$

So

$$\mathrm{Vol}(A,B) = \frac{1}{2}\big(\mathrm{Vol}(A+B) - \mathrm{Vol}(A) - \mathrm{Vol}(B)\big).$$

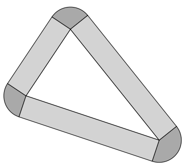
6.13. Example For any convex body $K \in \mathbb{R}^N$, **Steiner** studied

$$\mathrm{Vol}(K+\lambda D) = W_0(C) + \binom{N}{1}W_1(C)\lambda + \cdots + \binom{N}{N}W_N(C)\lambda^N,$$

where D is the unit disc of $\mathbb{R}^N$. Then

$$W_i(C) = \mathrm{Vol}(K,\overset{N-i}{\cdots},K,D,\overset{i}{\cdots},D)$$

We can read $W_i(C)$ from the diagram



$$\begin{aligned} W_0(C) &= \mathrm{Vol}(\text{white triangle}) \\ W_1(C) &= \tfrac{1}{2}\mathrm{Vol}(\text{three lightgray rectangles}) \\ W_2(C) &= \mathrm{Vol}(\text{three sectors}) \end{aligned}$$

We have

$$W_0(C) = \mathrm{Vol}(C), \qquad W_1(C) = \frac{1}{N}\mathrm{Vol}(\partial C), \qquad W_N(C) = \mathrm{Vol}(D).$$

6.14. Symmetric and multilinear We have

$$\mathrm{Vol}(\dots, A + B, \dots) = \mathrm{Vol}(\dots, A, \dots) + \mathrm{Vol}(\dots, B, \dots).$$

$$\mathrm{Vol}(\dots, A, B, \dots) = \mathrm{Vol}(\dots, B, A, \dots).$$

6.15. Example Let us consider a smooth projective toric variety $X(\Delta)$. Assume we have effective divisors $D_1, \dots, D_N$. We already know that

$$\frac{\langle D^N \rangle}{N!} = \mathrm{Vol}(P_D), \qquad P_D = \left\{ \mathbf{u} \in \mathbb{R}^N : \begin{array}{l} \forall \ell \in \Delta(1) \\ \langle \mathbf{u}, \mathbf{v}_\ell \rangle + c_\ell \geq 0 \end{array} \right\}.$$

Note this is also true for effective divisors with $\mathbb{Q}$-coefficients by consider mD for some $m > 0$, and thus is also true for effective divisors with $\mathbb{R}$-coefficients. Let us denote $P_i = P_{D_i}$ for $i = 1, \dots, N$. Note that

$$\frac{\langle (x_1 D + \cdots + x_N D_N)^N \rangle}{N!} = \mathrm{Vol}(x_1 P_1 + \cdots + x_N P_N).$$

We get immediately

$$\mathrm{Vol}(P_{D_1}, \dots, P_{D_N}) = \frac{1}{N!} \int_{X(\Delta)} D_1 \cdots D_N = \frac{1}{N!} \deg(D_1 \cdots D_N).$$

When $D_1, \dots, D_N$ are ample, this number is positive. So when $D_1, \dots, D_N$ are effective, the number is non-negative.

6.16. Theorem The mixed volume

$$\mathrm{Vol}(K_1, \dots, K_N) \geq 0.$$

When $K_1, \dots, K_N$ have interior, this is positive.

Proof We can approximate every convex body by polytopes, then we can apply above argument on toric varieties. Q.E.D.

6.17. Example Let us apply Khovanskii–Teissier theorem on toric varieties. Assume $D_1, D_2, \dots, D_N$ are ample divisors. Denote $P_i = P_{D_i}$ for $1 \leq i \leq N$. Then by Bertini argument, we can pick a smooth irreducible subvariety Y such that

$$D_3 \cap \cdots \cap D_N = [Y].$$

Then by Khovanskii–Teissier theorem

$$\left(\int_Y D_1 D_2\right)^2 \geq \left(\int_Y D_1^2\right)^2 \left(\int_Y D_2^2\right)^2.$$

That is

$$\operatorname{Vol}(P_1, P_2, P_3, \cdots, P_N)^2 \geq \operatorname{Vol}(P_1, P_1, P_3, \cdots, P_N) \operatorname{Vol}(P_2, P_2, P_3, \cdots, P_N).$$

This is also true when $D_1, \ldots, D_N$ are effective.

6.18. Alexandrov–Fenchel inequality We have

$$\operatorname{Vol}(A, B, \cdots)^2 \geq \operatorname{Vol}(A, A, \cdots) \operatorname{Vol}(B, B, \cdots).$$

Proof By the same reason, we can approximate convex body by polytopes. Q.E.D.

6.19. Corollary We have

$$\operatorname{Vol}(A, \overset{k}{\cdots}, A, B, \overset{N-k}{\cdots}, B)^N \geq \operatorname{Vol}(A)^k \operatorname{Vol}(B)^{N-k}$$

Proof If we denote

$$c_k = \operatorname{Vol}(A, \overset{k}{\cdots}, A, B, \overset{N-k}{\cdots}, B).$$

Then

$$c_0, c_1, \ldots, c_N$$

is logarithm concave by Alexandrov–Fenchel inequality. Equivalently,

$$\log c_0, \log c_1, \ldots, \log c_N$$

is concanve. So

$$\log c_k \geq \frac{k}{N} \log c_0 + \frac{N-k}{N} \log c_N,$$

This is desired. Q.E.D.

6.20. Brunn–Minkowski Inequality We have

$$\operatorname{Vol}(A+B)^{1/N} \geq \operatorname{Vol}(A)^{1/N} + \operatorname{Vol}(B)^{1/N}.$$

Proof We have

$$\begin{aligned}\mathrm{Vol}(A+B) &= \sum_{k=0}^{N}\binom{N}{k}\mathrm{Vol}(A,\overset{k}{\cdots},A,B,\overset{N-k}{\cdots},B)\\ &\geq \sum_{k=0}^{N}\binom{N}{k}\mathrm{Vol}(A)^{\frac{k}{N}}\,\mathrm{Vol}(B)^{\frac{N-k}{N}}\\ &= (\mathrm{Vol}(A)^{1/n}+\mathrm{Vol}(B)^{1/n})^n.\end{aligned}$$

Q.E.D.

Newton Polytopes

6.21. General setting We want to study the number of solutions of a system of Laurant polynomials over

$$(\Bbbk^\times)^N = \{(x_1,\dots,x_N) : x_i \neq 0, 1 \leq i \leq N\} \subset \Bbbk^N.$$

6.22. Example When $N = 1$. Assume we have an equation

$$a_n x^n + \cdots + a_k x^k = 0, \qquad a_n \neq 0 \neq a_k.$$

Then the number of non-zero zero is $n-k$.

6.23. Newton polytope For a Laurant polynomial $f \in \Bbbk[x_1^{\pm 1},\dots,x_N^{\pm 1}]$, we define

$$\mathrm{supp}(f) = \{m \in \mathbb{Z}^N : \text{the coefficient of } x^m \text{ in } f \text{ is nonzero}\} \subset \mathbb{R}^N.$$

The **Newton polytope** is

$$\mathrm{Newton}(f) = \mathrm{Conv}(\mathrm{supp}(f)).$$

6.24. Properties It is easy to see

$$\mathrm{Newton}(fg) = \mathrm{Newton}(f) + \mathrm{Newton}(g)$$

$$\mathrm{Newton}(f+g) \subseteq \mathrm{Newton}(f) \cup \mathrm{Newton}(g)$$

6.25. Minding–Kouchnirenko–Bernstein Given integral polytopes $P_1, \ldots, P_N$ in $\mathbb{R}^N$, consider generic Laurant polynomials $f_1, \ldots, f_N$ with

$$\mathrm{Newton}(f_i) = P_i, \qquad 1 \le i \le N.$$

Then the number of solutions of the equations $f_1 = \ldots = f_N = 0$ is

$$N!\ \mathrm{Vol}(P_1, \ldots, P_N).$$

Proof We can construct a smooth proper toric variety $X(\Delta) \supset (\Bbbk^\times)^N$ and effective divisors $D_1, \ldots, D_N$ such that

$$P_i = P_{D_i}, \qquad 1 \le i \le N.$$

Since $\mathrm{supp}(f_i) \subseteq P_{D_i}$, we can view $f_i \in \Gamma(\mathcal{O}(D_i))$. As f_i is chosen generically, the zero locus $Z(f_i)$ is the closure of $\{x \in (\Bbbk^\times)^N : f_i(x) = 0\}$ in $X(\Delta)$ and we have $[Z(f_i)] = D_i$. Since we choose f_i's generically, the intersection

$$Z(f_1) \cap \cdots \cap Z(f_N)$$

is transversal and can be assumed to be inside any nonempty open subset, e.g. T. This implies

$$\begin{aligned}
\#\{f_1 = \cdots = f_N = 0\} &= \#(Z(f_1) \cap \cdots Z(f_N)) \\
&= \deg([Z(f_1)] \cap \cdots [Z(f_N)]) \\
&= \deg(D_1 \cdots D_N) \\
&= N!\ \mathrm{Vol}(P_1, \ldots, P_N).
\end{aligned}$$

As desired. Q.E.D.

6.26. Example Assume f is a polynomial of degree d, then generically

$$\mathrm{Newton}(f) = \left\{ (m_1, \ldots, m_N) : \begin{array}{l} m_1 \ge 0, \cdots, m_N \ge 0, \\ m_1 + \cdots + m_N \le d \end{array} \right\} =: d\,\Delta^N$$

a simplex. If each f_i is generically of degree d_i. Then it is not hard to show

$$N!\ \mathrm{Vol}(d_1\,\Delta^N, \ldots, d_N\,\Delta^N) = d_1 \cdots d_N N!\ \mathrm{Vol}(\Delta^N) = d_1 \cdots d_N.$$

This is Bézout theorem. Actually the $X(\Delta)$ in the proof can be chosen to be $\mathbb{P}^N$.

Exerices

6.27. Exercise Let us denote

$$A - B = \{v \in \mathbb{R} : v + B \subseteq A\}.$$

Prove that for two compact convex sets A, B,

$$(A + B) - B = A.$$

In particular, Minkowski sum has cancelation for compact convex sets.

Hint Assume there is some element $v \notin A$ but with $v + B \subseteq A + B$, then by picking a hyperplane seperating v and A, we can find a contradiction.

6.28. Exercise If $K_i' \subseteq K_i$, show that

$$\mathrm{Vol}(K_1', \ldots, K_N') \leq \mathrm{Vol}(K_1, \ldots, K_N).$$

7 Proof of Read's Conjecture (I)

7.1. To proof Read's conjecture, we need

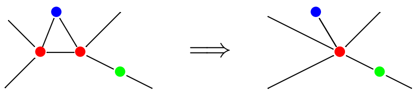
- more knowledge from Graph theory.
- combinatorial formula on permutohedral varieties.

Graph Theory Revised

7.2. Chromatic Polynomials Let $G = (V, E)$ be a graph. Recall that the chromatic polynomial χ_G is the unique polynomial such that

$$\chi_G(q) = \#\{\text{vertex } q\text{-colorings of } G\}.$$

To compute the coefficients, we can first color G by q colors arbitrarily. Now that we can always merge edges such that it is a vertex q-colorings



Let us denote $g \leq G$ when g is obtained by contracting edges from G. This leads to

$$\sum_{g \leq G} \chi_g(q) = q^{\#G}.$$

Thus by induction, we see

$$\chi_G(q) = q^{\#G} - \sum_{g < G} \chi_g(q)$$

is a polynomial in q. It will be more precise after introducing Möbius function.

7.3. Linear Space formulation Let $G = (V, E)$ be a graph. We assume the vertices set is $\{1, \ldots, m\}$. For each edge $i \overset{e}{\text{---}} j$ for $i < j$, we define

$$v_e = \mathbf{e}_i - \mathbf{e}_j \subseteq \mathbb{R}^m.$$

7.4. Proposition For a subset $\phi \subseteq E$, the following two statements are equivalent

(i) the set $\{v_e\}_{e\in\phi}$ is linearly independent;

(ii) the graph (V, ϕ) is a forest (i.e. not cycles).

(A good linear algebra exercise)

7.5. Lattice of flats For $\phi \subseteq E$, we denote

$$F(\phi) = \operatorname{span}_{e\in\phi}(v_e).$$

We allow $F(\varnothing) = 0$. Let us denote $\mathcal{F}_G = \{F(\phi)\}_{\phi\subseteq E}$ the set of **flats**. Note that $\mathcal{F}_G$ is closed under sum, but might not closed under the usual intersection. But there exists a unique maximal flat covered by two given flats. To be exact, for two flats F_1 and F_2 of $\mathcal{F}_G$, it is a **bounded lattice** under

$$F_1 \vee F_2 := F_1 + F_2, \qquad \mathbf{1} = F(E),$$
$$F_1 \wedge F_2 := \sum_{\substack{F\subseteq F_1\in\mathcal{F}_G\\ F\subseteq F_2\in\mathcal{F}_G}} F, \qquad \mathbf{0} = F(\varnothing).$$

Note that we defined a map

$$F : 2^E \longrightarrow \mathcal{F}_G, \qquad \phi \longmapsto F(\phi).$$

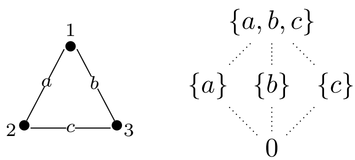
We will call $\phi \subseteq E$ a **flat** if it is the **unique** maximal element among all ϕ' such that $F(\phi') = F(\phi)$. We denote $\Phi_G \subseteq 2^E$ the set of flags of G. Note that F restricts to an isomorphism

$$\Phi_G \xrightarrow{\sim} \mathcal{F}_G.$$

That is, Φ_G can be equipped with a structure of bounded lattice under

$$F(\phi_1 \vee \phi_2) = F(\phi_1) \vee F(\phi_2), \qquad \mathbf{1} = E,$$
$$F(\phi_1 \wedge \phi_2) = F(\phi_1) \wedge F(\phi_2), \qquad \mathbf{0} = \varnothing.$$

7.6. Example Consider a triangle



Note that for example $\{a, b\}$ is not a flat, since it spans the same space as $\{a, b, c\}$.

7.7. Proposition Denote

$$\mathcal{G}_G = \{g \leq G\} = \left\{ \begin{matrix} \text{graphs that can be obtained} \\ \text{by contracting edges of } G \end{matrix} \right\}$$

We have an anti-isomorphism

$$\gamma : \Phi_G \xrightarrow{\sim} \mathcal{G}_G, \qquad \phi \longmapsto \begin{pmatrix} \text{the graph obtained by} \\ \text{contracting all edges in } \phi \end{pmatrix}.$$

Moreover, we have

$$\#\mathsf{Vertex}(\gamma(\phi)) = \#\mathsf{Vertex}(G) - \mathrm{rk}(\phi)$$

where $\mathrm{rk}(\phi) = \dim(\mathrm{span}(v_e)_{e\in\phi})$ is the number of edges of supporting forests.

7.8. Möbius inversion Over a finite paritial ordered set $(P, \leq)$, we can always define a **Möbius function** μ, such that for any $x, z \in P$,

$$\sum_{y : x \leq y \leq z} \mu(y, z) = \delta_{xz}, \quad \text{and} \quad \sum_{y : x \leq y \leq z} \mu(x, y) = \delta_{xz}.$$

Actually $(\mu(y, z))_{y,z\in P}$ is the inverse matrix of $(\delta_{x\leq y})_{x,y\in P}$. In particular, for two functions f and g over P, we have the following **Möbius inversion formula**

$$\sum_{x : x\leq y} f(x) = g(y) \iff \sum_{x : x\leq y} g(x)\mu(x, y) = f(y).$$

7.9. On Graphs Now, we have

$$\sum_{g\leq G} \chi_g(q) = q^{\#G}.$$

But Möbius inversion, we have

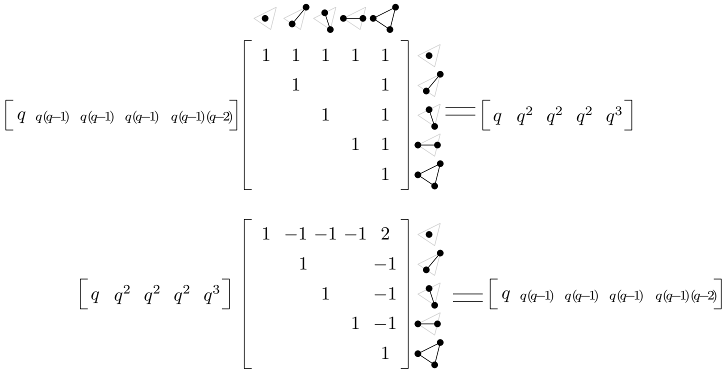
$$\chi_G(q) = \sum_{g\leq G} \mu(g, G) \cdot q^{\#g} = \sum_k \Bigg(\sum_{\substack{g\leq G, \\ \#g=k}} \mu(g, G) \Bigg) q^k.$$

This computes $\chi_G(q)$ explicitly — the coefficients are determined by its Möbius function. We can equivalently translate it to Φ_G, say

$$\tilde{\chi}_G(q) := q^{\#G}\chi_G(q^{-1}) = \sum_{\phi\in\Phi_G} \mu(\phi)q^{\mathrm{rk}(\phi)},$$

where $\mu(\phi) = \mu(\mathbf{0}, \phi)$.

7.10. Example Here is an example



$$\begin{bmatrix} q & q(q-1) & q(q-1) & q(q-1) & q(q-1)(q-2) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} q & q^2 & q^2 & q^2 & q^3 \end{bmatrix}$$

$$\begin{bmatrix} q & q^2 & q^2 & q^2 & q^3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} q & q(q-1) & q(q-1) & q(q-1) & q(q-1)(q-2) \end{bmatrix}$$

7.11. Recursion formula Now, we need to find some some formula of $\mu(\phi)$. Note that by definition $\mu(\phi)$ is determined by the following relations

(i) $\mu(\mathbf{0}) = 1$;

(ii) $\sum_{\psi\leq\phi} \mu(\psi) = 0$ for $\phi \neq \mathbf{0}$.

Actually, since Φ_G is a geometric lattice, we can a "shortening recrusion" form of (ii). Here, geometric lattice is equivalent to say

If x and y covers $x \wedge y$, then $x \vee y$ covers x and y.

Actually, by tracing back to $\mathcal{F}_G$, we have

$$\mathrm{rk}(x \vee y) \leq \mathrm{rk}(x) + \mathrm{rk}(y) - \mathrm{rk}(x \wedge y) = \mathrm{rk}(x) + 1,$$

Note that $x \vee y \neq x$.

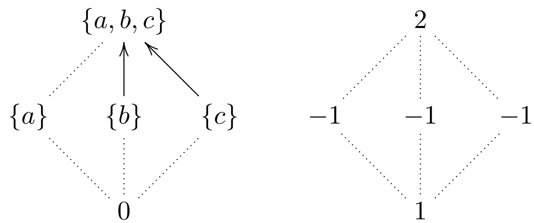
7.12. Weisner Theorem For a bounded geometric lattice L, and any nonzero $a \leq x \neq 0$, we have

$$\sum_{y \vee a = x} \mu(y) = 0.$$

Proof This follows from direct computation

$$\begin{aligned}
\sum_{y \vee a = x} \mu(\mathbf{0}, y) &= \sum_{y} \mu(\mathbf{0}, y) \cdot \delta_{y \vee a = x} \\
&= \sum_{y} \mu(\mathbf{0}, y) \sum_{z : y \vee a \leq z \leq x} \mu(z, x) \\
&= \sum_{z : a \leq z \leq x} \mu(z, x) \sum_{y : y \leq z} \mu(\mathbf{0}, y) \\
&= \sum_{z : a \leq z \leq x} \mu(z, x) \cdot \delta_{\mathbf{0} = z} = 0.
\end{aligned}$$

7.13. Example For example,



$$\begin{array}{llll}
2 + (-1) + (-1) + (-1) + 1 &= 0 & \text{Using Definition} \\
2 \qquad\quad + (-1) + (-1) &= 0 & \text{Using Weisner Theorem}
\end{array}$$

7.14. Proposition For any edge $e \in E$, the number $\mu(\phi)$ is determined by the following relations

(i) $\mu(\mathbf{0}) = 1$;

(ii) $\sum_{\psi \vee \{e\} \leq \phi} \mu(\psi) = 0$ for any edge $e \in \phi$.

7.15. Alternating properties In particular,

$$\mu(\phi) = -\sum_{\psi \vee \{e\} = \phi} \mu(\psi)$$

As a result, $(-1)^{\operatorname{rk}\phi}\mu(\phi) \geq 0$.

7.16. Computation Let us linearly order E, and fix the choice of e by assuming $e = \min(\phi)$. We have

$$|\mu(\phi)| = \sum_{\psi \vee \min(\phi) = \phi} |\mu(\psi)| = \sum_{\substack{\psi \vee \min(\phi) = \phi \\ \pi \vee \min(\psi) = \pi}} |\mu(\pi)| = \cdots = \#S_k(\phi)$$

where $k = \operatorname{rk}(\phi) = \dim \operatorname{span}(v_e : e \in \phi)$ and

$$S_k = \left\{ \varnothing \subsetneqq \phi_1 \subsetneqq \cdots \subsetneqq \phi_{k-1} \subsetneqq \phi_k \subsetneqq E : \begin{array}{l} \forall i = 1, \ldots, k \\ \phi_{i-1} \vee \{\min(\phi_i)\} = \phi_i \end{array} \right\}$$

with $S_k(\phi) = \{\phi \in S_k : \phi_k = \phi\}$. In particular,

$$\mu_k = \sum_{\operatorname{rk}\phi = k} |\mu(\phi)| = \#S_k.$$

Equivalently, $\phi \in S_k$ if and only if

(o) ϕ is a flag of G length k;

(i) $\operatorname{rk}(\phi_i) = \dim \operatorname{span}(v_e : e \in \phi_i) = i$, which is called **initial**;

(ii) $\min(\phi_k) < \min(\phi_{k-1}) < \cdots < \min(\phi_1)$, which is called **descending**.

As a result,

$$\mu_k = \#\left\{ \begin{array}{c} \text{initial, descending} \\ \text{flags of } G \text{ of length } k \end{array} \right\}$$

If moreover

$$\min(E) < \min(\phi_k) < \min(\phi_{k-1}) < \cdots < \min(\phi_1).$$

we say ϕ is **strictly descending**. Denote

$$\bar{\mu}_k = \#\left\{ \begin{array}{c} \text{initial, strictly descending} \\ \text{flags of } G \text{ of length } k \end{array} \right\}.$$

Then we have

$$\bar{\mu}_k + \bar{\mu}_{k-1} = \mu_k.$$

Since when $\min(E) = \min(\phi_k)$ for $\phi \in S_k$, then $\phi_k = \phi_{k-1} \vee \{\min(E)\}$. The details are left to readers. Actually, when $E \neq \varnothing$, we have

$$(1+q)\sum |\bar{\mu}_k| q^k = \sum |\mu_k| q^k.$$

It is easy to see

$$\{\bar{\mu}_k\} \text{ is unimodal} \Longrightarrow \{\mu_k\} \text{ is unimodal.}$$

8 Proof of Read's Conjecture (II)

Permutohedral variety

8.1. A reformulation We developed the theory over $\mathbb{Z}^N$. Actually, it would be useful to us a coordinate-free notation, which is benefit to our application. We prefer a geometric notation, so we start from a torus T. We have two lattices

$$\mathbf{ch}(T) = \mathrm{Hom}_{\mathsf{AlgGrp}}(T, \mathbb{G}_m), \qquad \mathbf{1PS}(T) = \mathrm{Hom}_{\mathsf{AlgGro}}(\mathbb{G}_m, T).$$

They are dual under the natural pairing

$$\mathbf{1PS}(T) \times \mathbf{ch}(T) \longrightarrow \mathrm{Hom}_{\mathsf{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z} \cdot \mathrm{id}\,.$$

We can recover T by

$$T = \mathrm{Spec}(\Bbbk[\mathbf{ch}(T)]).$$

We denote

$$\mathfrak{t} := \mathbb{R} \otimes \mathbf{1PS}(T).$$

In this case, we should take the following convention of coordinate-free description of toric variety

$$\text{a cone } \sigma \in \Delta \text{ lies in } \mathfrak{t},$$
$$\text{a monoid } Q_\sigma \text{ lies in } \mathbf{Ch}(T).$$

8.2. Projective torus We will mainly use the maximal torus of projective lineear group. To be exact,

$$T = \mathbb{G}_m^n / \Delta\mathbb{G}_m = \frac{\{(z_1, \dots, z_n) : z_i \in \mathbb{G}_m\}}{\{(z, \dots, z) : z \in \mathbb{G}_m\}}.$$

Then

$$\begin{aligned}
\mathbf{1PS}(T) &= \mathbb{Z}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_n / \mathbb{Z}(\mathbf{e}_1 + \cdots + \mathbf{e}_n) \\
\mathbf{ch}(T) &= \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_1 + \cdots + \lambda_n = 0\} \\
&= \mathbb{Z}(x_1 - x_2) \oplus \mathbb{Z}(x_2 - x_3) \oplus \cdots \mathbb{Z}(x_{n-1} - x_n).
\end{aligned}$$

As a result,

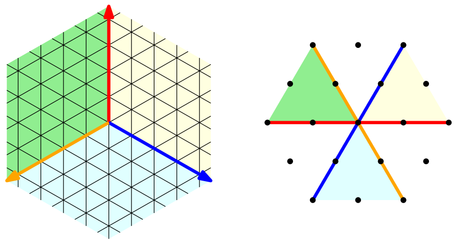
$$\mathscr{K}(T) = \Bbbk(\tfrac{z_1}{z_2}, \dots \tfrac{z_{n-1}}{z_n}) \subseteq \Bbbk(z_1, \dots, z_n).$$

8.3. Projective Space We define a complete fan $\Delta_\circ$ over $\mathfrak{t}$ with rays

$$\mathbb{R}_{\geq 0}\mathbf{e}_1, \dots, \mathbb{R}_{\geq 0}\mathbf{e}_n.$$

We know $X(\Delta_\circ) = \mathbb{P}^{n-1}$.

8.4. Example When $n = 3$, we see



8.5. Permutahedron Varieties Let us simply denote $[n] = \{1, \dots, n\}$. For any $S \subseteq [n]$, we denote

$$\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i.$$

Note that

$$\mathbf{e}_{[n]} = 0, \qquad \mathbf{e}_S = -\mathbf{e}_{[n] \setminus S}.$$

We define a complete fan Δ of $\mathfrak{t}$ with rays spanned by $\mathbf{e}_S$ for all $\varnothing \subsetneqq S \subsetneqq [n]$. For a flag of subset

$$\phi : \varnothing \subsetneqq \phi_1 \subsetneqq \cdots \subsetneqq \phi_k \subsetneqq [n]$$

if we denote

$$\sigma_\phi = \mathrm{span}_{\geq 0}(\mathbf{e}_{\phi_1}, \dots, \mathbf{e}_{\phi_k}),$$

then

$$\Delta = \{\sigma_\phi\}_{\phi \text{ is a flag of subset}}.$$

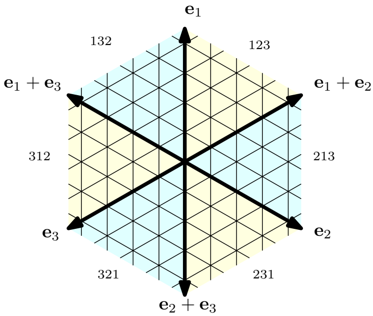
We define **permutahedron variety** to be toric variety $X(\Delta)$.

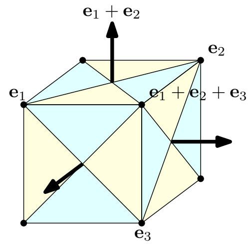
8.6. Remark Note that we have a bijection

$$\mathfrak{S}_n \xrightarrow{\sim} \Delta(N-1)$$

by sending w to the flag with $\phi_i = \{w(1), \dots, w(i)\}$. That's the reason it is called permutahedron variety.

8.7. Example Here are two examples





8.8. Cohomology We can compute the cohomology of permutahedron easily

$$H^\bullet(X(\Delta),\mathbb{Q}) = \frac{\mathbb{Q}[x_S]_{\varnothing\subsetneq S\subsetneq[n]}}{\left\langle \begin{array}{ll} x_{S_1}x_{S_2}=0 & \text{if } S_1\not\subseteq S_2 \text{ and } S_2\not\subseteq S_1, \\ \sum_{S\ni i}x_S=\sum_{S\ni j}x_S & \forall i,j\in[n]. \end{array}\right\rangle}$$

Actually, the first condition is enough to generate $\mathcal{I}_\Delta$, since

> a family of subsets does not forming a flag if and only if there are two of them not comparable (or incident)

and the second condition generates $\mathcal{J}_\Delta$, since

$$\sum_{S\ni i}x_S - \sum_{S\ni j}x_S = \operatorname{div}\frac{z_i}{z_j} \quad \text{and} \quad \operatorname{ch}(T)=\operatorname{span}_{\mathbb{Z}}(x_i-x_j)_{i,j=1}^n.$$

8.9. Two nef classes Define

$$\alpha=\sum_{S\ni i}x_S\in H^2(X(\Delta)) \qquad\qquad \beta=\sum_{S\not\ni i}x_S\in H^2(X(\Delta))$$

for any $i\in[n]$. They are not dependent on the choice of i. One can check they are both nef.

8.10. Remark Actually, α is the pullback of hyperplane section under the following induced morphism

$$\pi_\circ : X(\Delta)\longrightarrow X(\Delta_\circ)=\mathbb{P}^{n-1}.$$

To be exact, the inclusion gives a morphism induces $\Delta \to \Delta_\circ$. Similarly, β is the pullback of hyperplane section under the following induced morphism

$$\pi_\circ : X(\Delta) \longrightarrow X(\Delta_\bullet) = \mathbb{P}^{n-1},$$

where $\Delta_\bullet = -\Delta_\circ$. These can be seen from the fact that H is the zero of $x_i \in \Gamma(\mathcal{O}(H))$ whose pull back on $X(\Delta)$ has zero $\sum_{S \ni i} x_S$.

8.11. Chevalley formula Note that

$$\Delta(k) = \Big\{\phi : \varnothing \subsetneqq \phi_1 \subsetneqq \cdots \subsetneqq \phi_k \subsetneqq [n]\Big\}.$$

In particular, $\Delta(n-1)$ is the set of full flags. Let us denote

$$D_\phi = D_{\sigma_\phi} = x_{\phi_1} \cdots x_{\phi_k} \in \mathrm{CH}^k(X(\Delta)).$$

We will study the Chevalley formula for α and β respectively. To be exact, we will expand cap/cup products

$$\alpha \frown (\text{homology classes}), \qquad \beta \smile (\text{cohomology classes}).$$

Due to its analogue in Schubuert calculus, we call such rules Chevalley formulas. The following cohomology Chevalley formula is obvious.

8.12. Chevalley formula in cohomology For any $i \in \psi_1$, we have

$$\beta \smile D_\psi = \sum_\phi D_\phi$$

with the sum over

$$\phi : \varnothing \subsetneqq S \subsetneqq \psi_1 \subsetneqq \cdots \subsetneqq \psi_{k-1} \subsetneqq [n] \quad \text{and} \quad i \notin S.$$

8.13. Generating flags We apply Chevallay formula by choosing i to be minimal at each step, we obtain

$$\beta^k = \beta^{k-1} \sum_{\phi_k \not\ni 1} D_{\phi_k} = \beta^{k-2} \sum_{\substack{1 \notin \phi_k \subsetneqq [n] \\ \min(\phi_k) \notin \phi_{k-1} \subsetneqq \phi_k}} D_{\phi_k} D_{\phi_{k-1}} = \cdots = \sum_\phi D_\phi$$

where the sum over all flag of subsets ϕ

$$\varnothing \subsetneqq \phi_1 \subsetneqq \cdots \subsetneqq \phi_{k-1} \subsetneqq \phi_k \subsetneqq [n]$$

with

$$1 < \min(\phi_k) < \min(\phi_{k-1}) < \cdots < \min(\phi_1).$$

That is, it is a flag by a **strictly descending flag of subsets of length k**.

8.14. Homology The homology

$$H_\bullet(X(\Delta), \mathbb{Q}) = H^\bullet(X(\Delta), \mathbb{Q})^*.$$

Since

$$H^{2k}(X(\Delta), \mathbb{Q}) = \sum_{\sigma \in \Delta(k)} \mathbb{Q} \cdot [D_\sigma],$$

We can think

$$H_k(X(\Delta), \mathbb{Q}) = \left\{ \Delta(k) \xrightarrow{f} \mathbb{Q} : \begin{array}{c} \text{some} \\ \text{conditions} \end{array} \right\}$$

The conditions can be explicitly described, and the functions satisfying the condition is called a **Minkowski weight**.

8.15. Chevalley formula in homology Let

$$\psi : \varnothing \subsetneqq \psi_1 \subsetneqq \cdots \subsetneqq \psi_{k-1} \subsetneqq [n]$$

be an element in $\Delta(k-1)$. For any $i \notin \psi_{k-1}$, we have

$$(\alpha \frown f)(\psi) = \textstyle\sum_\phi f(\phi)$$

with

$$\phi : \varnothing \subsetneqq \psi_1 \subsetneqq \cdots \subsetneqq \psi_{k-1} \subsetneqq S \subsetneqq [n], \quad \text{and} \quad i \in S.$$

Say,

$$\alpha \frown - = \begin{array}{l} \text{by extending the flag by a bigger subset} \\ \text{containing a fixed element out of the flat.} \end{array}$$

Proof By direct computation

$$(\alpha \frown f)(\psi) = (\alpha \frown f)(D_\psi) = f(D_\psi \alpha) = \sum_{S \ni i} f(D_\psi x_S) \overset{(*)}{=} \sum_\phi f(\phi)$$

where $(*)$ is true since $S \neq \phi_i$ for all i, and when $\{S\} \cup \psi$ forms a flag, S must be the biggest one. Q.E.D.

8.16. Truncation Denote $\delta^k : \Delta(k) \to \mathbb{Q}$ by

$$\delta^k(\phi) = \begin{cases} 1, & \phi \text{ is an initial flag,} \\ 0, & \text{otherwise.} \end{cases}$$

Here ϕ is an **initial flag** means $|\phi_i| = i$ for $1 \le i \le k$.

For example,

- when $k = n-1$, we have $\delta^{n-1}(\phi) = 1$ for any full flags $\phi \in \Delta(n-1)$. So it is the fundamental class

$$\delta^{n-1} = [X(\Delta)] \in H^{2(n-1)}(X(\Delta), \mathbb{Q}).$$

- When $k = 1$, we have $\delta(S) = 1$ for any 1-subset $S \in \Delta(1)$.

We have

$$\alpha \frown \delta^k = \delta^{k-1}.$$

Actually, the only S allowed in the Chevalley formula is $\psi_{k-1} \cup \{i\}$. In particular,

$$\alpha^{n-1-k} \frown [X(\Delta)] = \delta^k.$$

So

$$\deg(\alpha^{n-1-k} \cdot \beta^k) = \#\left\{\begin{array}{c} \text{initial, strictly descending} \\ \text{flags of } G \text{ of length } k \end{array}\right\}$$

By Khovanskii–Teissier, this sequence is logarithm concave. We will generalize this to any graph.

8.17. Bergman Fans Now let us work with grpah $G = (V, E)$. We assume $E = [n]$. Denote $\delta^k_G : \Delta(k) \to \mathbb{Q}$ by

$$\delta^k_G(\phi) = \begin{cases} 1, & \phi \text{ is an initial flag of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Recall ϕ is an **initial flag of G** means $\mathrm{rk}(\phi_i) = i$ for $1 \le i \le k$. Assume we know

$$\delta^{\mathsf{top}}_G = \delta^r_G \in H_{2r}(X(\Delta), \mathbb{Q}), \qquad r = \mathrm{rk}(G).$$

Then we can easily check that

$$\alpha \frown \delta^k = \delta^{k-1}.$$

Actually, the only S allowed in the Chevalley formula is $\psi_{k-1} \vee \{i\}$.

8.18. Conclusion Now, we can conclude

$$(\beta^k \smile \alpha^{r-k}) \frown \delta_G^{\mathsf{top}} = \delta_G^k(\beta^k) = |\bar{\mu}_k|.$$

Recall that

$$|\bar{\mu}_k| = \#\left\{\begin{array}{c}\text{initial, strictly descending}\\ \text{flags of } G \text{ of length } k\end{array}\right\}.$$

By Khovanskii–Teissier, this sequence is logarithm concave, once we show

$$\delta_G^{\mathsf{top}} \in H_{2r}(X(\Delta); \mathbb{Q})$$

is represented by some irreducible subvariety,

Blow-up

8.19. Blow-up Let X be a variety, and Y a subvariety. We have blowup

$$\pi : \mathrm{Bl}_Y X \longrightarrow X.$$

We can understand π by understanding its fibre when X and Y are both non-singular,

$$\text{fibre at } x \in X = \begin{cases}\text{a singular point}, & x \notin Y,\\ \mathbb{P}N_{Y/X}(x), & x \in Y.\end{cases}$$

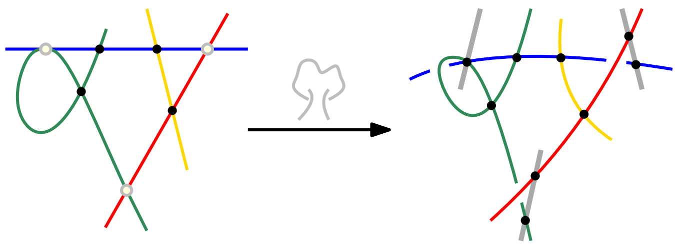
For any closed $Z \subseteq X$, we call the **monoidal transformation**

$$\tilde{Z} = \overline{\pi^{-1}(Z \setminus Y)} \xlongequal[\text{is non-singular}]{\text{when } Z \cap Y} \mathrm{Bl}_{Y \cap Z} Y.$$

We remark that at the cohomology level,

$$\mathrm{CH}(X) \ni [Z] \overset{\pi^*}{\not\mapsto} [\tilde{Z}] \in \mathrm{CH}(\mathrm{Bl}_Y X).$$

8.20. Example Here is an example to analyse



8.21. Blowups for Toric Varieties Blowup of toric varieties $X(\Delta)$ along any $\overline{\mathcal{O}}_\sigma$ for $\sigma \in \Delta$ is also a toric variety $X(\Delta')$. To be exact, Δ' is obtained by refining σ by the ray

$$\mathbb{R}_{\geq 0}(\mathbf{u}_{\ell_1} + \cdots + \mathbf{u}_{\ell_k}),$$

if $\sigma = \mathrm{span}_{\geq 0}(\ell_1, \ldots, \ell_k) \in \Delta(k)$. This can be checked locally, we refer Fulton's book for the proof.

8.22. Permutahedron variety as an Iterative Blow-up We have

$$X(\Delta) = X_{n-2} \to \cdots \to X_1 \to X_0 = \mathbb{P}^{n-1},$$

where $X_i = X(\Delta_i)$ with Δ_i the fan obtained by adding rays

$$\mathbb{R}_{\geq 0}\mathbf{e}_S, \qquad \#S = n - i$$

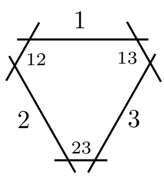
to Δ_{i-1}. More exactly, $X_i \to X_{i-1}$ is the blowup along the proper transforms of

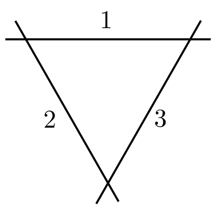
$$\mathbb{P}(\mathbb{C}^{[n]\setminus S}) = \{x_i = 0\}_{i \in S}, \qquad \#S = n - i,$$

i.e. the space defined by coordinate of S. So

$$\begin{aligned} X_1 &= \text{blowup of } X_0 \text{ along } n \text{ points} \\ X_2 &= \text{blowup of } X_1 \text{ along } \binom{n}{2} \text{ lines} \\ \cdots &= \cdots \end{aligned}$$

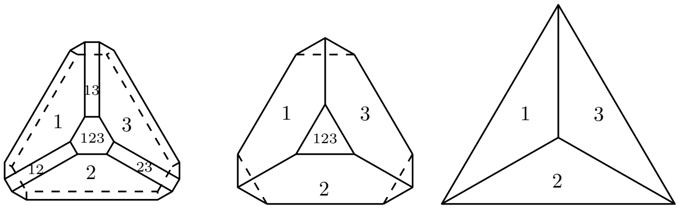
8.23. Example Here we illustrate the permutahedron variety of dimension 2.





Note that each line is a divisor.

8.24. Example Here we illustrate the permutahedron variety of dimension 3.



Note that each face is a divisor.

8.25. Representability Let $G = (V, E)$ with $E = [n]$. Recall that we define for each $e \in E$, a vector $v_e \in \mathbb{R}^{\#V}$ such that for any $\phi \subseteq E$,

$$\mathrm{rk}(\phi) = \dim \mathrm{span}(v_e : e \in E).$$

It can be formally written as

$$\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{\#V}, \qquad \mathbf{e}_e \longmapsto v_e.$$

Let $K = \ker \pi^{\perp} = \{(x_e)_{e \in E} : x_e = 0\}$. Then

$$\begin{aligned} \mathrm{rk}(\phi) &= \dim \mathbb{C}^{\phi}/(\mathbb{C}^{\phi} \cap \ker \pi), \qquad \mathbb{C}^{\phi} = \text{coordinate plane of } \phi \\ &= \dim(\mathbb{C}^{\phi} + \ker \pi)/\ker \pi \\ &= \dim K - \dim(K \cap \mathbb{C}^{E \setminus \phi}). \end{aligned}$$

We will focus on $\mathbb{P}(K) \subseteq \mathbb{P}^{n-1}$. We will call

$$\mathbb{P}(K) \cap \mathbb{P}(\mathbb{C}^{E \setminus \phi}) = \mathbb{P}(K \cap \mathbb{C}^{E \setminus \phi}) \subseteq \mathbb{P}(K)$$

the flat over K. It is clear that they are in bijection to flats of G.

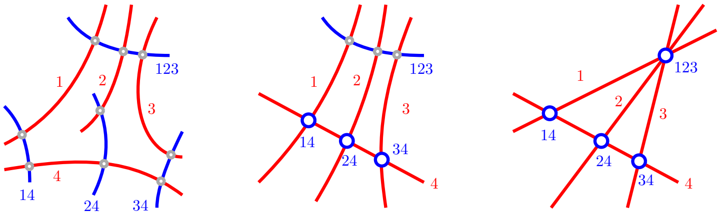
8.26. Realization If we restrict to $\mathbb{P}(K)$, we have

$$\begin{tikzcd} \widetilde{\mathbb{P}(K)} = P_{n-2} \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 = \mathbb{P}(K) \arrow[d] \\ X(\Delta) = X_{n-2} \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 = \mathbb{P}^{n-1}. \end{tikzcd}$$

More exactly, $P_i \to P_{i-1}$ is the blowup along the proper transforms of

$$\mathbb{P}(\mathbb{K} \cap \mathbb{C}^{E\setminus\phi}), \qquad \#\phi = n - i.$$

The typical picture looks like



For example, we do not need to blow up the flat for $\{1,2\}$ after blowing up the flag for $\{1,2,3\}$.

By construction,

$$H^2(X(\Delta);\mathbb{Q}) \ni D_\phi \overset{\text{res}}{\longmapsto} \begin{cases} D_\phi^K, & \phi \text{ is a flat of } G \\ 0, & \text{otherwise} \end{cases} \in H^2(\widetilde{\mathbb{P}(K)};\mathbb{Q}).$$

We see, flags of K is just (reversed) flags of G, and

$$\left\langle \widetilde{\mathbb{P}(K)}, D_\phi \right\rangle = \int_{\widetilde{\mathbb{P}(K)}} D_\phi = \begin{cases} 1, & \phi \text{ is a (complete) flag of } G \\ 0, & \text{otherwise.} \end{cases}$$

In other words,

$$[\widetilde{\mathbb{P}(K)}] = \delta_G^{\mathsf{top}}.$$

This finishes the proof of Read's conjecture.

8.27. Huh Read's conjecture is true.

8.28. Remark Our proof is mainly modified from due to Huh and Katz. Actually, our proof works for representable matroids.