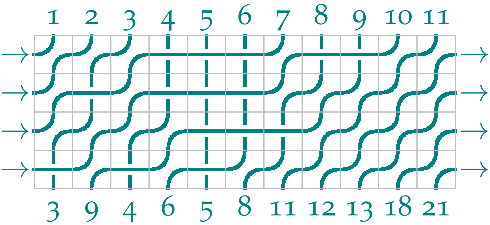
a Combinatorial Adventure in Inverse Grassmannian Permutations

(Joint with Yiming Chen, Neil J.Y. Fan and Ming Yao)

Rui Xiong



Schubert Calculus

The central problem in (equivariant) Schubert calculus is to find manifestly positive combinatorial models of the coefficients $c_{u,v}^{w}(t)$ in the expansion

$$\mathfrak{S}_u(x;t) \cdot \mathfrak{S}_v(x;t) = \sum_{w \in S_\infty} c_{u,v}^{w}(t) \cdot \mathfrak{S}_w(x;t).$$

The coefficients are known in very limited cases — for example:

- When one of u, v is $s_{k-r+1} \cdots s_{k-1} s_k$ for some $1 \leq r \leq k$, the expansion is known as the equivariant **Pieri rule** [1, 2];

S. Robinson, A Pieri-type formula for $H_T^*(SL_n(\mathbb{C})/B)$, J. Algebra 249 (2002), 38–58.

C. Li, V. Ravikumar, F. Sottile and M. Yang, A geometric proof of Forum Math. 31 (2019), 779–783.

Grassmannian Permutations and Beyond

The biggest breakthrough in recent years is the study of puzzles.

- When u, v are both **Grassmannian permutations** [4], these expansions admit a puzzle rule by Knutson and Tao [1].

There are also

- **two-step** puzzles by Buch [1] and
- **separated descents** puzzles by Knutson and Zinn-Justin [3].

A.S. Buch. Mutations of puzzles and equivariant cohomology of
Annals of Mathematics 182 (2015), 173-220.

A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of
Duke Math. J. 119 (2003), 221–260.

A. Knutson and P. Zinn-Justin, Schubert puzzles and integrabilit
arXiv:2306.13855.

A.I. Molev and B.E. Sagan, A Littlewood–Richardson rule for fac
Trans. Amer. Math. Soc. 351 (1999), 4429–4443.

Inverse Grassmannians Permutations

In this talk, I will explain a new family of coefficients we obtained.

- When u, v are both **inverse Grassmannian permutations**, there is a combinatorial formula.

The expansions for **single** Schubert polynomials under this condition are obtained by Pechenik and Weigandt [1] based on the result of Wyser [2].

However, the method does not directly apply to the double case. We used the theory of **spherical subgroups**, **quiver representations**, **positroid varieties**, **affine Stanley symmetric polynomials**, **pipe dreams**, ...

O. Pechenik, A. Weigandt. An inverse Grassmannian Littlewood-
Forum Math., Sigma (2024), Vol. 12:e114 1–24

B. Wyser. Schubert calculus of Richardson varieties stable under
J. Algebraic Combin. **38**(4) (2013), 829–850.

Chapter 1

Nonequivariant Coefficients

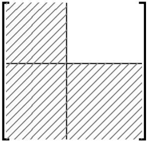
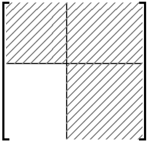
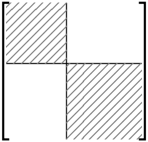
&

the Symmetric Subgroup K

— the difficulty of generalization —

Nonequivariant Coefficients

Let $n = p + q$. Consider the following subgroups of $G = GL_n$.

$P =$ , $Q =$ , $K =$ ,

Assume $v \in S_n$ is p-inverse Grassmannian, and $u \in S_n$ is q-inverse Grassmannian.

- $\overline{B^- vB/B}$ is P-invariant and $\overline{Bw_\circ uB/B}$ is Q-invariant.
- $R_{v,w_\circ u} = \overline{Bw_\circ uB/B} \cap \overline{B^- vB/B}$ is K-invariant.
- Since there are only finitely many K-orbits on G/B,
  - the Richardson variety $R_{u,w_\circ v}$ has to be a K-orbit closure;
  - its Schubert expansion is given by Brion [1].

The case $u, v \notin S_{p+q}$ follows from a (back) stability argument.

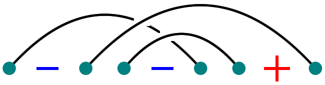
M. Brion. On orbit closures of spherical subgroups in flag varieti
Comment. Math. Helv. 76(2) 2001, 263–299.

Clans and K-orbits

A **clan** is a partial matching with all unmatched nodes colored by $+$ or $-$. We say a clan γ is a (p, q)-clan if

- the number of $+$'s and matchings is p;
- the number of $-$'s and matchings is q.

For example, the following diagram represents a $(4, 5)$-clan

$\gamma =$  .

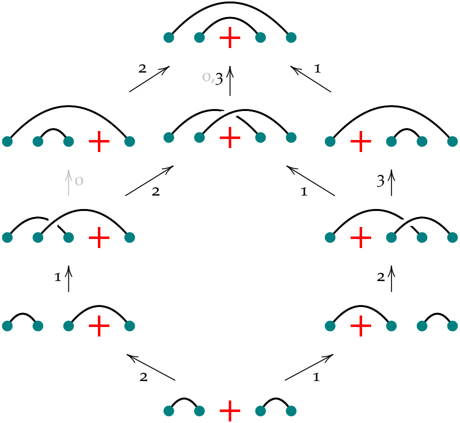
Theorem (Matsuki [1])

The K-orbits of G/B are finite and parametrized by (p, q)-clans.

T. Matsuki. The orbits of affine symmetric spaces under the actio
J. Math. Soc. Japan **31**(2) (1979), 331–357.

Example

The following is a small example of Pechenik–Weigandt [1].



$\gamma =$ 

$v = \ 0 \quad 3 \quad 1 \quad 2 \quad 4$

$u = \ 0 \quad 2 \quad 3 \quad 1 \quad 4$

$$\mathfrak{S}_{312}(x) = x_1^2$$
$$\mathfrak{S}_{231}(x) = x_1 x_2$$
$$\mathfrak{S}_{4213}(x) = x_1^3 x_2$$

$$\mathfrak{S}_{312}(x)\mathfrak{S}_{231}(x)$$
$$= \mathfrak{S}_{s_3 s_1 s_2 s_1}(x)$$

O. Pechenik, A. Weigandt. An inverse Grassmannian Littlewood-
Forum Math., Sigma (2024), Vol. 12:e114 1–24

The Difficulty

The equivariant Schubert expansion of K-orbit closure was obtained recently by us [1]. Unfortunately, it does not directly apply to this question.

The essential difficulty of generalizing to the equivariant setting is

the product of two double Schubert polynomials $\mathfrak{S}_u(x,t) \cdot \mathfrak{S}_v(x,t)$ does not represent a Richardson variety.

To overcome the difficulty, many new ideas have to be introduced.

Y. Chen, N.J.Y.Fan, R. Xiong, and M. Yao, Bumpless pipe dream f
arXiv:2511.00980.

Chapter 2

Triple Schubert Calculus
&
the Spherical Subgroup S

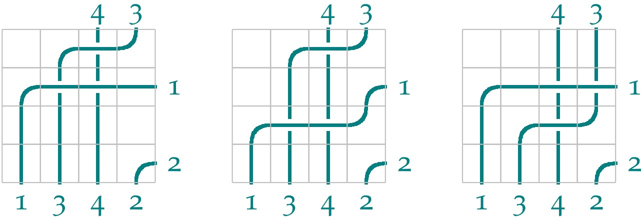
— quivers, preclans, polynomials —

Triple Schubert Calculus

In [1], Knutson and Tao pioneered the study of triple Schubert calculus, i.e., the product of two Schubert polynomials of two different secondary variables

$$\mathfrak{S}_u(x;t)\cdot\mathfrak{S}_v(x;y)=\sum_{w\in S_\infty} c_{u,v}^{w}(t;y)\cdot\mathfrak{S}_w(x;t).$$

In [1], we establish the puzzle for the separated descents case.



A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of
Duke Math. J. 119 (2003), 221–260.

N.J.Y. Fan, P.L. Guo, Rui Xiong, Bumpless pipe dreams meet puz
Advances in Mathematics 463 (2025), 110113.

A principle

An important philosophy of [1] is

> *While the problem of computing triple Schubert structure constants is broader, its proof could be simpler.*

Moreover, we also learnt a principle

> *Usually the double Schubert structure constants can be obtained combinatorially from the triple ones.*

The same principle also hold for the inverse Grassmannian case! **The triple coefficients will serve as a necessary step to get the double coefficients.**

N.J.Y. Fan, P.L. Guo, Rui Xiong, Bumpless pipe dreams meet puz
Advances in Mathematics 463 (2025), 110113.

Geometric meaning of Triple

Samuel [2] conjectured that the coefficients

$$c^w_{u,v}(t;y) \in \mathbb{N}[t_i - y_j]_{i,j\geq 0}.$$

In [1], we proved this positivity conjecture via establishing a new geometric explanation of the product. The key step of the proof is

$$\text{triple product} = \begin{array}{c}\text{equivariant Schubert expansion}\\ \text{of a (shifted) Richardson variety.}\end{array}$$

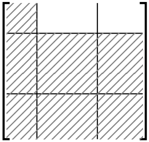
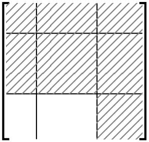
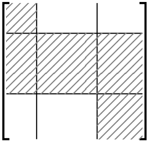
This is a nature double Schubert calculus does not have — **lifting to triple Schubert calculus will solve the difficulty above**.

Y. Gao and R. Xiong, Graham positivity of triple Schubert calculu
arXiv:2506.09421.

M.J. Samuel, Molev–Sagan type formula for double Schubert pol
J. Pure Appl. Algebra 228 (2024), Paper No. 107636, 39 pp.

Spherical Subgroup S

But at the same time we will have to replace the **back stability** by an **geometric argument**. That is, it is necessary to work with the following subgroups of GL_n for $n \gg 0$

$P =$ , $Q =$ , $S =$ ,

where the heights of rows and the width of the columns are both p, m, q $(m = n - p - q)$.

Assume $v \in S_n$ is p-inverse Grassmannian, and $u \in S_n$ is q-inverse Grassmannian.

- $\overline{B^- vB/B}$ is P-invariant and $\overline{Bw_0 uB/B}$ is Q-invariant.
- $R_{v,w_0 u} = \overline{Bw_0 uB/B} \cap \overline{B^- vB/B}$ is now S-invariant.

Spherical Subgroup S

It turns out the subgroup S is also a **spherical subgroup**, i.e.

$$\#\{S\text{-orbits of } G/B\} < \infty.$$

This can be seen from the following embedding

$$\begin{aligned}\{S\text{-orbits of } G/B\} &= \{G\text{-orbits of } G/S \times G/B\} \\ &\hookrightarrow \{G\text{-orbits of } G/P \times G/Q \times G/B\} \\ &\hookrightarrow \{\text{quiver representations of type } D\}.\end{aligned}$$

The last embedding is historically well-known, and pointed out in the study of multiple flag varieties of finite type by Magyar, Weyman and Zelevinsky [1].

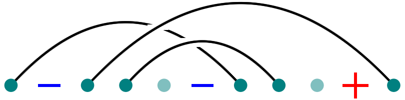
P. Magyar, J. Weyman, and A. Zelevinsky, Multiple flag varieties
Adv. Math. 141 (1999), no. 1, 97–118.

Preclans and S-orbits

A **preclan** is a clan with some unmatched nodes uncolored.
We say a preclan is a (p, m, q)-preclan if

- the number of $+$'s and matchings is p;
- the number of $-$'s and matchings is q;
- the number of uncolored nodes • 's is m.

For example, the following diagram is a $(4, 2, 5)$-preclan

$\gamma =$  .

We will always denote $n = p + m + q$.

Theorem (Chen–Fan–X.-Yao)

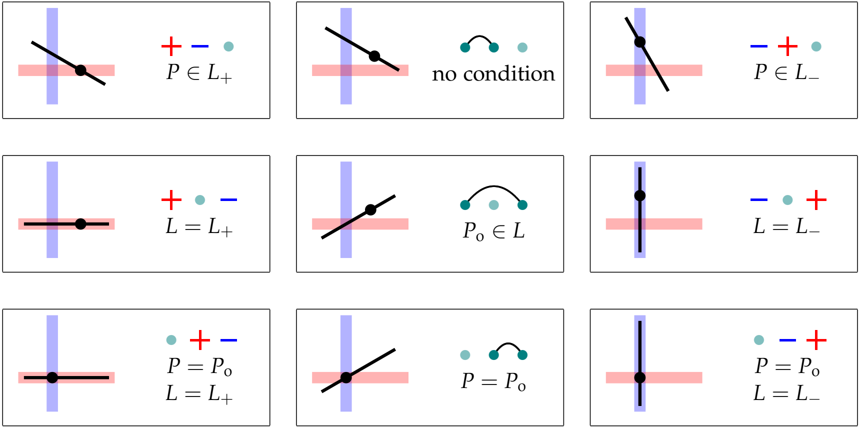
The S-orbits of G/B are finite and parametrized by (p, m, q)-preclans.

Example

When $n = 3$, we can identify

$$G/B = \big\{(P, L) : \overset{\text{point}}{P} \in \overset{\text{line}}{L} \subset \mathbb{P}^2\big\}.$$

For $p = q = m = 1$, the S-orbit closures are described as



Relation to Quiver Representations

The S-orbits can be described by the "relative position" with

$$K_+ = \mathrm{span}(\mathbf{e}_{p+1}, \ldots, \mathbf{e}_n), \qquad K_- = \mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_{n-q}).$$

Following Magyar, Weyman and Zelevinsky [1], for a given flag $V_\bullet$, we can construct a representation of quiver

$$\begin{tikzcd} & & & & \mathbb{C}^n/K_+ \\ V_1 \arrow[r, "\subset"] & V_2 \arrow[r, "\subset"] & \cdots \arrow[r, "\subset"] & \mathbb{C}^n \arrow[ur] \arrow[dr] & \\ & & & & \mathbb{C}^n/K_-, \end{tikzcd}$$

Its isomorphism class determines which S-orbit it belongs to.

P. Magyar, J. Weyman, and A. Zelevinsky, Multiple flag varieties
Adv. Math. 141 (1999), no. 1, 97–118.

Representations and Preclans

Each component of a preclan corresponds to an indecomposable representations of type D.

$$\begin{tikzcd}
 & & & & & & & & \mathbb{C} \\
{} \arrow[r] & 0 \arrow[r] & \overset{i}{\mathbb{C}} \arrow[r, "1"] & \cdots \arrow[r, "1"] & \overset{j-1}{\mathbb{C}} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"] & \mathbb{C}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \cdots \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \mathbb{C}^2 \arrow[ur, "{[1\ 0]}"] \arrow[dr, "{[0\ 1]}"'] & \\
 & & & & & & & & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \overset{i}{\mathbb{C}} \arrow[r, "1"] & \cdots \arrow[r, "1"] & \mathbb{C} \arrow[ur] \arrow[dr] & \\
 & & & & & & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & & \mathbb{C} \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \overset{i}{\mathbb{C}} \arrow[r, "1"] & \cdots \arrow[r, "1"] & \mathbb{C} \arrow[ur, "1"] \arrow[dr] & \\
 & & & & & & 0
\end{tikzcd}$$

$+$

$$\begin{tikzcd}
 & & & & & & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \overset{i}{\mathbb{C}} \arrow[r, "1"] & \cdots \arrow[r, "1"] & \mathbb{C} \arrow[ur] \arrow[dr, "1"'] & \\
 & & & & & & \mathbb{C}
\end{tikzcd}$$

$-$

Weak order on Orbits

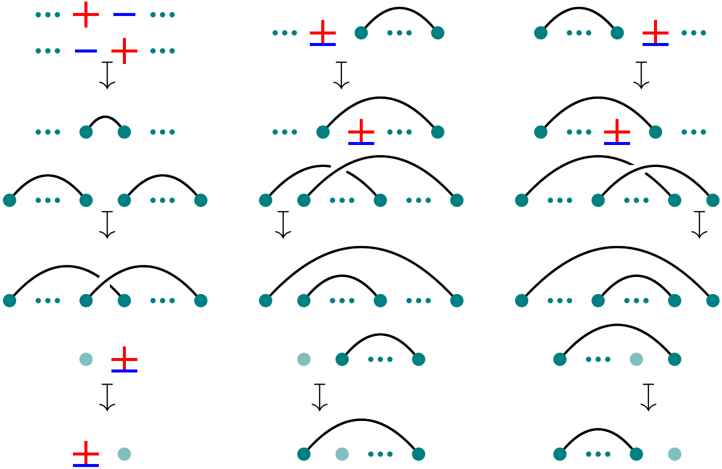
By Mars and Springer [1], there are seven types of behavior of an orbit $\mathbb{O}$ under push-pull with respect to $\pi_k : G/B \to G/P_k$

types	push-pull	fiber $\cong \mathbb{P}^1$	monoid action
(I)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O}$	$\mathbb{O}$	$s_k * \mathbb{O} = \mathbb{O}$
(IIa)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}'$	$\mathbb{O}$ $\mathbb{O}'$	$s_k * \mathbb{O} = \mathbb{O}'$
(IIb)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}'$	$\mathbb{O}'$ $\mathbb{O}$	$s_k * \mathbb{O} = \mathbb{O}$
(IIIa)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}' \cup \mathbb{O}''$	$\mathbb{O}$ $\mathbb{O}'$ $\mathbb{O}''$	$s_k * \mathbb{O} = \mathbb{O}'$
(IIIb)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}' \cup \mathbb{O}''$	$\mathbb{O}'$ $\mathbb{O}$ $\mathbb{O}''$	$s_k * \mathbb{O} = \mathbb{O}$
(IVa)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}'$	$\mathbb{O}$ $\mathbb{O}'$ $\mathbb{O}$	$s_k * \mathbb{O} = \mathbb{O}'$
(IVb)	$\pi_k^{-1}(\pi_k(\mathbb{O})) = \mathbb{O} \cup \mathbb{O}'$	$\mathbb{O}'$ $\mathbb{O}$ $\mathbb{O}'$	$s_k * \mathbb{O} = \mathbb{O}$

J. G. M. Mars and T. A. Springer, Hecke algebra representations r
Represent. Theory 2 (1998), 33–69.

Weak Order on Preclans

The induced monoid action on preclans is given by

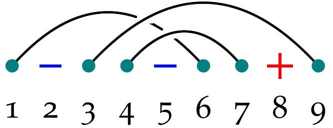


The first two rows are the same as clans.

Lengths

The length of a clan is

$$\sum\{\text{widths of mathchings}\} - \#\{\text{crossings}\}.$$

$\ell\Bigg($ 

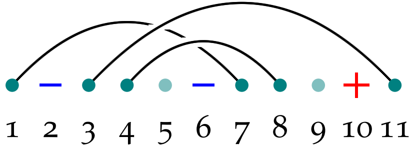
$\Bigg)$

$$\begin{aligned} &= (6-1) + (9-3) + (7-4) - 2 \\ &= 12 \end{aligned}$$

The length of a preclan is

$$\ell(\gamma) := \ell(\bar{\gamma}) + \ell\big(\{\text{indices of } \bullet \text{ 's}\}\big)$$

where the length of a subset $A = \{a_1 < \cdots < a_m\}$ is $\ell(A) = (a_1 - 1) + \cdots + (a_m - m)$.

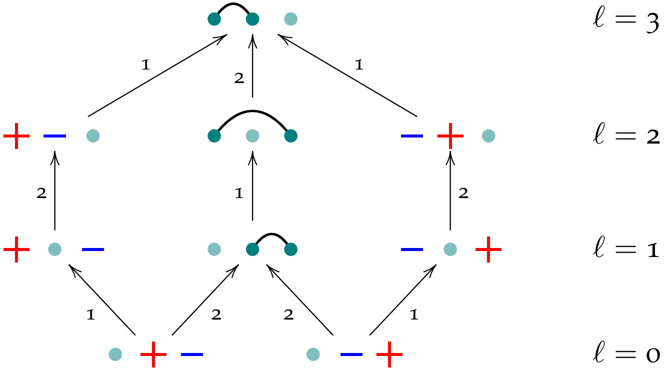
$\ell\Bigg($ 

$\Bigg)$

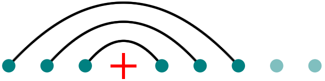
$$\begin{aligned} &= 12 + (5-1) + (9-2) \\ &= 23 \end{aligned}$$

Example

The following diagram shows the Hasse diagram of $(1,1,1)$-preclans.



Special Preclans (I)



open preclan

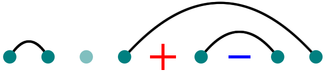
It is a rainbow (p, q)-clan followed by m uncolored nodes • .

Geometry: the open orbit

closed preclan

It is m uncolored nodes • followed by a matchless (p, q)-clan.

Geometry: closed orbits



Richardson preclan

If it is non-crossing and uncolored nodes • are uncovered.

Geometry: Richardson variety

More on Richardson

Let us define v_γ, u_γ

$$
\begin{array}{rcccccccccccc}
\gamma & = & \bullet & - & \bullet & \bullet & \bullet & - & \bullet & \bullet & \bullet & + & \bullet \\
v_\gamma & = & 1 & 5 & 2 & 3 & 6 & 7 & 8 & 9 & 10 & 4 & 11 \\
u_\gamma & = & 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 & 9 & 10 & 11
\end{array}
$$

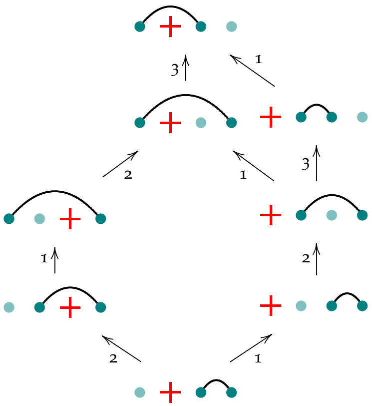
Theorem (Chen–Fan–X.-Yao)

For any (p, m, q)-preclan γ, we have

- $S\dot{\gamma}B/B \subseteq Pv_\gamma B/B \cap Qw_0 u_\gamma B/B$;
- $\overline{S\dot{\gamma}B/B} = \overline{B^- v_\gamma B/B} \cap \overline{Bw_0 u_\gamma B/B}$ *if γ is Richardson;*
- *For any p-inverse v and q-inverse u, there exists a (p, m, q)-preclan γ for $m \gg 0$ such that $v_\gamma = v$ and $u_\gamma = u$.*

Example

At this step, using the general theorem of Brion [1], we can recover Pechenik–Weigandt [1].



$\gamma =$ 

$v = 3\ 1\ 2\ 4$

$u = 2\ 3\ 1\ 4$

$$\mathfrak{S}_{312}(x) = x_1^2$$
$$\mathfrak{S}_{231}(x) = x_1 x_2$$
$$\mathfrak{S}_{4213}(x) = x_1^3 x_2$$

$$\mathfrak{S}_{312}(x)\mathfrak{S}_{231}(x) = \mathfrak{S}_{s_3 s_1 s_2 s_1}(x)$$

However, there are many steps to do for double and triple coefficients...

Polynomial Representatives

Theorem (Chen–Fan–X.-Yao)

There is a polynomial representative $\Upsilon_\gamma(x;t)$ *for* $[\overline{S\dot{\gamma}B/B}]_T$ *with*

$$\Upsilon_\gamma(x;t) = \mathfrak{S}_{v_\gamma}(x;t)\mathfrak{S}_{u_\gamma}(x;\overleftarrow{t}) \qquad \textit{when } \gamma \textit{ is Richardson}$$

$$\partial_k \Upsilon_\gamma(x;t) = \begin{cases} \Upsilon_{s_k * \gamma}(x;t), & \ell(s_k * \gamma) = \ell(\gamma) + 1, \\ 0, & \textit{otherwise.} \end{cases}$$

These polynomial generalizes those defined by Wyser and Yong [1] for clans.

B. Wyser and A. Yong. Polynomials for $GL_p \times GL_q$ orbit closures
Selecta Math. (N.S.) **20** (2014), 1083–1110.

Example

Let us compute for $(1,1,1)$-preclans

preclan γ	v_γ	u_γ	$Y_\gamma(x,t)$	Richardson?
• + −	213	231	$(x_1-t_1)\cdot(x_1-t_3)(x_2-t_3)$	Yes
• − +	231	213	$(x_1-t_1)(x_2-t_1)\cdot(x_1-t_3)$	Yes
	213	213	$(x_1-t_1)\cdot(x_1-t_3)$	Yes
+ • −	123	231	$(x_1-t_3)(x_2-t_3)$	Yes
− • +	231	123	$(x_1-t_1)(x_2-t_1)$	Yes
+ − •	123	213	(x_1-t_3)	Yes
− + •	213	123	(x_1-t_1)	Yes
	123	123	$x_1+x_2-t_1-t_3$	No
	123	123	1	Yes

Chapter 2

Schubert Expansion
&
Positroid Varieties

— localization and globalization —

Schubert to Localization

To determine the equivariant Schubert expansion of S-orbit closures, similar as [1], there is a classical trick:

Theorem (Chen–Fan–X.–Yao)

For a (p, m, q)-preclan γ, we have

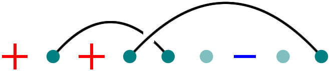
$$\Upsilon_\gamma(x;t) = \sum_w \Upsilon_{w*\gamma}(t;t) \cdot \mathfrak{S}_w(x;t),$$

where the sum over all $w \in S_n$ such that $\ell(w) + \ell(\gamma) = \ell(w * \gamma)$.

Thus the question can be reduced to computing the equivariant localization at the identity $1 \cdot B/B$.

Y. Chen, N.J.Y.Fan, R. Xiong, and M. Yao, Bumpless pipe dream f
arXiv:2511.00980.

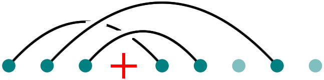
Special Preclans (II)



separated preclan

If $v_\gamma = e$.

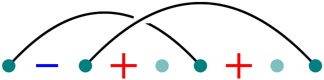
Geometry: contained in PB/B.



permutational preclan

If $v_\gamma = u_\gamma = e$

Geometry: in $PB/B \cap Qw_\circ B/B$.



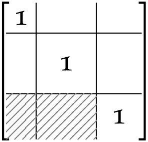
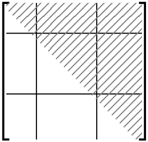
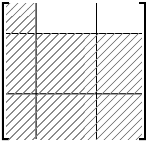
coseparated preclan

If $u_\gamma = e$

Geometry: contained in $Qw_\circ B/B$.

Localization

Denote and recall

$N =$ , $B =$ , $P =$ .

The localization at the identity can be decomposed into

$$\begin{tikzcd} G/B & P \cdot B/B \arrow[l, "f"'] & N \cdot B/B \arrow[l, "g"'] & 1 \cdot B \arrow[l, "h"'] \arrow[lll, bend right] \end{tikzcd}$$

Since

- f is an open embedding, and
- g intersects all S-orbits transversally,

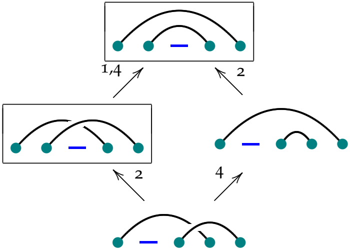
the question reduces to study the intersection of an S-orbit with $N \cdot B/B$.

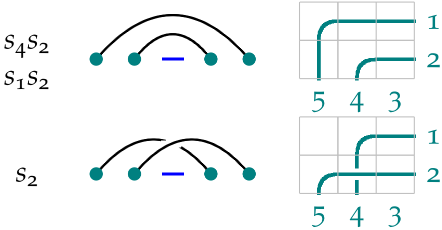
Example

When

$$\gamma \text{ is a clan} \quad \text{i.e.} \quad m = 0 \quad \text{i.e.} \quad S = K,$$

we [1] proved the intersection is a **matrix Schubert variety**.





$$Y_\gamma(x;t) = \mathfrak{S}_{s_4s_2}(x;t) + \mathfrak{S}_{s_1s_2}(x;t) + (t_1 - t_5) \cdot \mathfrak{S}_{s_2}(x;t).$$

Y. Chen, N.J.Y.Fan, R. Xiong, and M. Yao, Bumpless pipe dream f
arXiv:2511.00980.

Localization to Globalization

Surprisingly, for general S-orbits, the intersections can be **globalized** to **positroid varieties** [1] over Grassmannians.

Intersecting an S-orbit closure
with the transversal slide N

$$\Updownarrow$$

intersecting a positroid variety
with the open opposite Schubert cell

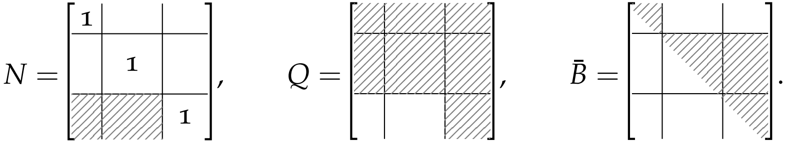
Positroids were introduced by Postnikov [2] to understand totally non-negative Grassmannians. Knutson, Lam and Speyer [1] defined positroid varieties.

A. Knutson, T. Lam, and D. Speyer. Positroid varieties: Juggling a
Compositio Mathematatica, 149:1710–1752, 2013.

A. Postnikov. Total positivity, Grassmannians, and networks.
arXiv:math/0609764, 2006.

Globalization

Recall and denote



Theorem (Chen–Fan–X.–Yao)

For any separated (p, m, q)-preclan γ, we have

$$S\dot{\gamma}B/B \cap N \cdot B/B \cong Q\backslash Q\dot{\gamma}\bar{B} \cap Q\backslash Q \cdot N.$$

Moreover,

$$Y_\gamma(t, t)|_{t_i \mapsto -t_i} = [\overline{Q\backslash Q\dot{\gamma}\bar{B}}]_T|_{Q\cdot 1},$$

The Diagram

The following is the commutative diagram for “globalization”

$$
\begin{tikzcd}
G/B & P\cdot B/B \arrow[l, "f"'] & N\cdot B/B \arrow[l, "g"'] & 1\cdot B \arrow[l, "h"'] \arrow[lll, bend right] \\
 & & N \arrow[u, "\wr"] \arrow[d, "\wr"] & \{1_n\} \arrow[l] \arrow[u] \arrow[d] \\
 & Q\backslash G & Q\backslash Q\cdot N \arrow[l, "\phi"'] & Q\cdot 1 \arrow[l, "\psi"'] \arrow[ll, bend left]
\end{tikzcd}
$$

Positroid Varieties

Positroid varieties over $Gr_k(\mathbb{C}^n)$ are parametrized by **affine bounded permutation**

$$\mathrm{Bound}(k,n) = \left\{ \begin{matrix} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{matrix} : \begin{matrix} f(i+n) = f(i) + n \\ i \leq f(i) \leq n+i \\ \frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k \end{matrix} \right\} \subseteq \tilde{S}_n.$$

For $f \in \mathrm{Bound}(k,m)$, the postroid variety

$$\Pi_f = \overline{GL_k \backslash \left\{ \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix} : \begin{matrix} f(i) = \min\{j > i : v_i \in \\ \mathrm{span}(v_{i+1}, \ldots, v_j)\} \end{matrix} \right\}}.$$

Here we extend v_i for $i \in \mathbb{Z}$ by $v_i = v_{i \bmod n}$. When $v_i = 0$ the minimum is understood as i

Example

The following element belongs to

$$\begin{array}{c} \begin{array}{cccc|ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc|ccccccc} 4 & 7 & 1 & 4 & 0 & 2 & 5 & 4 & 6 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 7 & 3 & 6 \\ 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 9 \end{array}\right] \end{array} \in \Pi_f$$

with

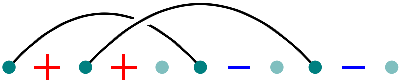
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 9 & 4 & 6 & 5 & 8 & 11 & 12 & 13 & 18 & 21 \end{pmatrix}.$$

For example,

$$\begin{aligned} v_1 &\in \operatorname{span}(v_2, v_3) \\ v_2 &\in \operatorname{span}(v_3, \ldots, v_9) \\ v_6 &\in \operatorname{span}(v_7, v_8) \\ v_8 &\in \operatorname{span}(v_9, v_{10}, v_{11}, v_1) \\ v_{10} &\in \operatorname{span}(v_{11}, v_1, \ldots, v_7) \end{aligned}$$

Bounded Affine Permutations

For a separated (p, m, q)-preclan γ, we can define a bounded affine permutation $f_\gamma \in \tilde{S}_n$.

$\gamma =$ 

$$\begin{array}{rccccccccccc} i = & 4 & 3 & 2 & 1 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ f_\gamma(i) = & 6 & 4 & 9 & 3 & 5 & 8 & 11 & 12 & 13 & 18 & 21 \end{array}$$

- $f_\gamma(1) = 3$, since the $(p+1) - 1 = 4$-th node is $+$, and there are 2 matchings over it; similarly $f_\gamma(3) = 4$;
- $f_\gamma(2) = 9$, since the $(p+1) - 2 = 3$-rd node is matched with the 9-th node; similarly $f_\gamma(4) = 6$;
- $f_\gamma(5) = 5$, $f_\gamma(6) = 8$, $f_\gamma(7) = 11$, since the indices of unmatched $\bullet$ are $5, 8, 11$;
- $f_\gamma(8) = 11 + 1$, $f_\gamma(9) = 11 + 2$, since there are two matches;
- $f_\gamma(10) = 11 + 7$, $f_\gamma(11) = 11 + 10$, since the indices of $-$ are $7, 10$.

Globalization to Positroid Varieties

Let us denote the permutation matrix

$$u_\circ = \left[\begin{array}{ccc|ccc|ccc} & & 1 & & & & & & \\ & \iddots & & & & & & & \\ 1 & & & & & & & & \\ \hline & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & & 1 & & & \\ \hline & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{array}\right] \in GL_{p+m+q}.$$

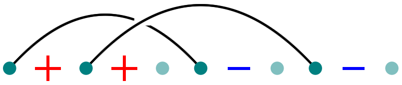
Theorem (Chen–Fan–X.–Yao)

Suppose γ is an separated (p,m,q)-preclan We have

$$\overline{Q\backslash Q\dot{\gamma}\overline{B}} \cdot u_\circ = \Pi_{f_\gamma}.$$

Example

For example, for the preclan above

$\gamma =$ 

a generic element of $\overline{Q\backslash Q\dot{\gamma}\overline{B}} \cdot u_0$ is represented by

$$
\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left[\begin{array}{cccc|ccccccc}
* & * & * & * & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{array}\right]
\end{array}
$$

We have

$$
f_\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 9 & 4 & 6 & 5 & 8 & 11 & 12 & 13 & 18 & 21 \end{pmatrix}.
$$

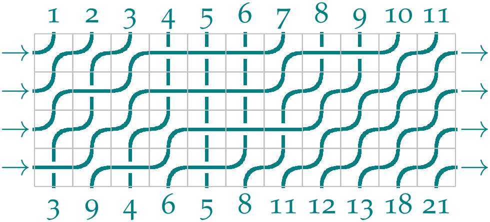
Chapter 3

Double Affine Stanley Symmetric Polynomials & Pipe Dreams

— statement of the main theorem(s) —

Double Affine Stanley Symmetric Polynomials

Stanley symmetric functions [2] were introduced by Stanley in the study of number of reduced words. Affine Stanley symmetric functions were introduced by Lam [1] as an affine generalization of it. By Shimozono and Zhang [3], it admits a pipe dream model:



Thomas Lam. Affine Stanley symmetric functions.
American Journal of Mathematics, 128(6):1553–1586, 2006.

R.P. Stanley. On the number of reduced decompositions of eleme
European Journal of Combinatorics, 5(4):359–372, December 1984.

M. Shimozono and S. Zhang, Pipe-dreams for affine double Schu
in preparation.

Main Result — Expansion of $\Upsilon_\gamma(x;t)$

It is known that the equivariant fundamental class of positroid varieties are represented by double affine Stanley symmetric polynomials [1, 2]. Using the above connection, we have the following Theorem.

Theorem (Chen–Fan–X.–Yao)

For any (p,m,q)-preclan γ,

$$\Upsilon_\gamma(x;t) = \sum_{w\in S_n} d_{w,\gamma}(t)\cdot \mathfrak{S}_w(x;t)$$

where the nonzero coefficients $d_{w,\gamma}(t)$ are specialization of Stanley symmetric functions (up to a sign).

A. Knutson, T. Lam, and D. Speyer. Positroid varieties: Juggling a
Compositio Mathematica, 149:1710–1752, 2013.

T. Lam, S. J. Lee, and M. Shimozono. Back stable Schubert calcul
Compos. Math. 157 (2018), 883–962.

Main Result — Triple Expansion

Applying a stability argument, we get the triple coefficients.

$$\mathfrak{S}_v(x;t)\cdot\mathfrak{S}_u(x;y)=\sum_{w\in S_\infty} c_{v,u}^w(t;y)\cdot\mathfrak{S}_w(x;t).$$

Let v be a p-inverse Grassmannian permutation and u be a q-inverse Grassmannian permutation. Let us pick a (p,m,q)-preclan γ for $m\gg 0$ with $v_\gamma=v$ and $u_\gamma=u$.

Theorem (Chen–Fan–X.–Yao)

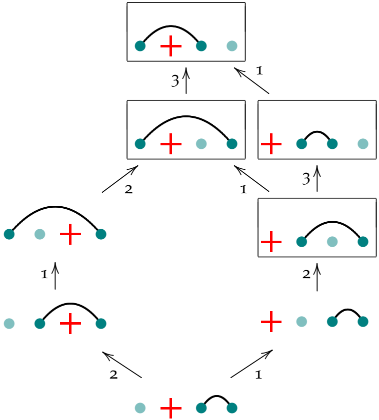
$$c_{v,u}^w(t,y)=\begin{cases} F_{f_{w*\gamma}}(-y_1,\ldots,-y_p;-t_p,\ldots,-t_1,-t_{p+1},\ldots,-t_n), \\ \qquad\qquad \textit{if } w*\gamma \textit{ is separated and } \ell(w*\gamma)=\ell(w)+\ell(\gamma), \\ 0, \qquad \textit{otherwise.}\end{cases}$$

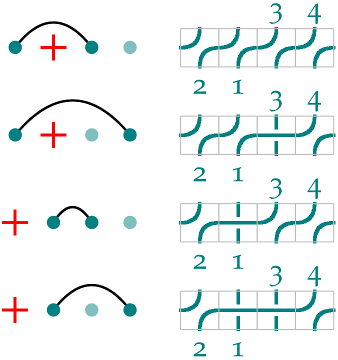
Example

$\gamma =$ 

$v = 3 \quad 1 \quad 2 \quad 4$

$u = 2 \quad 3 \quad 1 \quad 4$





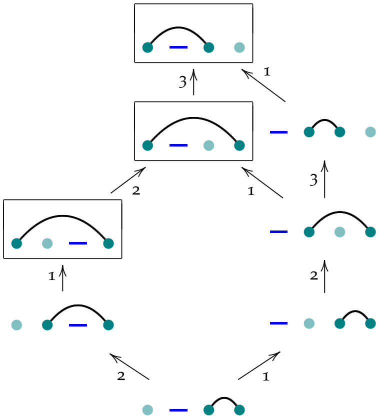
$$\mathfrak{S}_{312}(x;t)\mathfrak{S}_{231}(x;y) = (t_3 - y_1)(t_1 - y_1)\mathfrak{S}_{s_2s_1}(x;t)$$
$$+ (t_3 - y_1)\mathfrak{S}_{s_1s_2s_1}(x;t) + (t_1 - y_1)\mathfrak{S}_{s_3s_2s_1}(x;t) + \mathfrak{S}_{s_3s_1s_2s_1}(x;t)$$

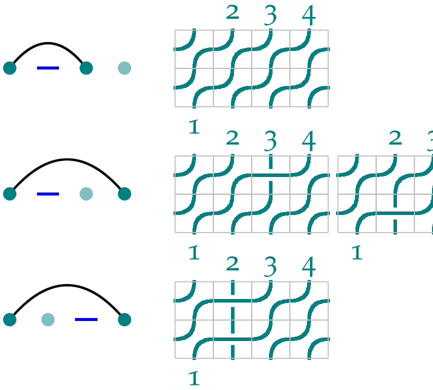
Example

$\gamma =$ 

$v =$ 2 3 1 4

$u =$ 3 1 2 4





$$\mathfrak{S}_{231}(x;t)\mathfrak{S}_{312}(x;y) = (t_2 - y_1)(t_2 - y_2)\mathfrak{S}_{s_1s_2}(x;t)$$
$$+ (t_3 + t_2 - y_1 - y_2)\mathfrak{S}_{s_1s_2s_1}(x;t) + \mathfrak{S}_{s_3s_1s_2s_1}(x;t)$$

A Triple Version of Representatives

From the proof of Theorem, the argument actually implies the existence of a triple version $Y_\gamma(x;t;y)$. They are characterized by the following two properties

$$Y_\gamma(x;t;y) = \mathfrak{S}_{v_\gamma}(x;t)\mathfrak{S}_{u_\gamma}(x;y) \qquad \text{when } \gamma \text{ is Richardson}$$

$$\partial_k Y_\gamma(x;t;y) = \begin{cases} Y_{s_k * \gamma}(x;t;y), & \ell(s_k * \gamma) = \ell(\gamma) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have $Y_\gamma(x;t;\overleftarrow{t}) = Y_\gamma(x;t)$, and the polynomial is stable under adding uncolored nodes $\bullet$. Moreover, the roles of $+$, $-$ are symmetric:

$$\begin{tikzcd} + \arrow[r, bend left] & - \arrow[l, bend left] \end{tikzcd} \iff \begin{tikzcd} t \arrow[r, bend left] & y \arrow[l, bend left] \end{tikzcd}$$

Example

preclan γ	v_γ	u_γ	$\Upsilon_\gamma(x;t;y)$	Richardson?
• + −	213	231	$(x_1 - t_1) \cdot (x_1 - y_1)(x_2 - y_1)$	Yes
• − +	231	213	$(x_1 - t_1)(x_2 - t_1) \cdot (x_1 - y_1)$	Yes
	213	213	$(x_1 - t_1) \cdot (x_1 - y_1)$	Yes
+ • −	123	231	$(x_1 - y_1)(x_2 - y_1)$	Yes
− • +	231	123	$(x_1 - t_1)(x_2 - t_1)$	Yes
+ − •	123	213	$(x_1 - y_1)$	Yes
− + •	213	123	$(x_1 - t_1)$	Yes
	123	123	$x_1 + x_2 - t_1 - y_1$	No
	123	123	1	Yes

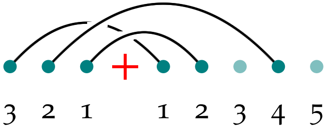
From Triple to Double

Obviously, it suffices to set $y = t$ to get the double coefficients.

$$\mathfrak{S}_v(x;t) \cdot \mathfrak{S}_u(x;t) = \sum_{w \in S_\infty} c_{v,u}^{w}(t) \cdot \mathfrak{S}_w(x;t).$$

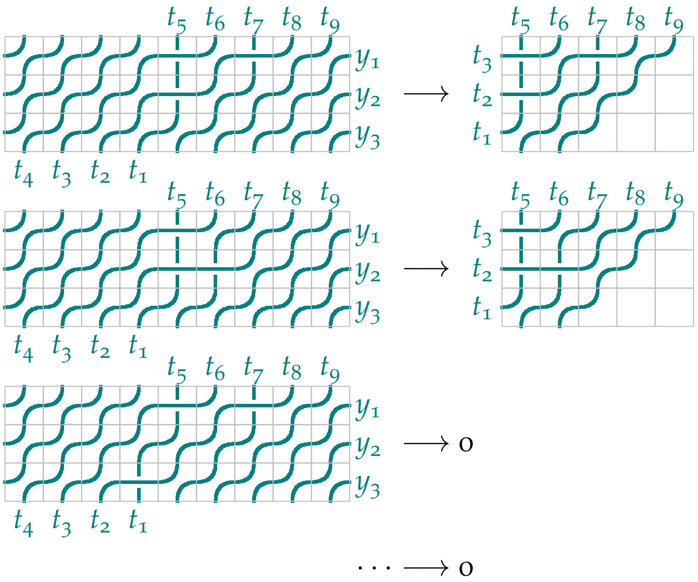
Applying a combinatorial trick (e.g. sign reversed involution), we get the Graham positive double coefficients.

We can define a permutation $S_{m+\min(p,q)}$ for a permutational preclan

$\gamma =$ 

$$w_\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

From affine to finite



Main Result — Double Expansion

Now we can state our main result on double coefficients

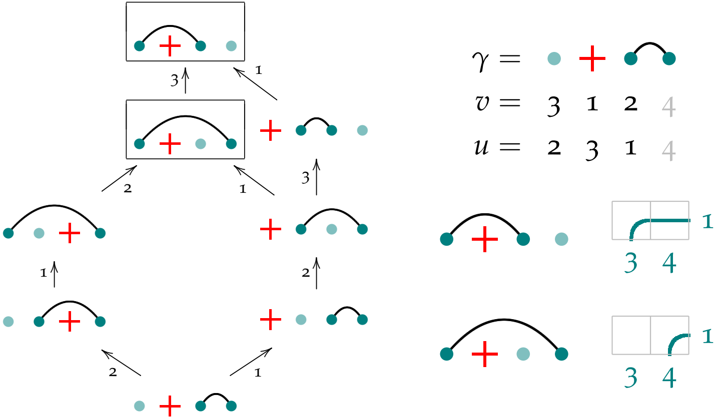
$$\mathfrak{S}_v(x;t) \cdot \mathfrak{S}_u(x;t) = \sum_{w \in S_\infty} c_{v,u}^w(t) \cdot \mathfrak{S}_w(x;t).$$

Let v be a p-inverse Grassmannian permutation and u be a q-inverse Grassmannian permutation. Since now the role of u, v are symmetric, without loss of generality, we can assume $p \geq q$. Let us pick a (p,m,q)-preclan γ for $m \gg 0$ with $v_\gamma = v$ and $u_\gamma = u$.

Theorem (Chen–Fan–X.–Yao)

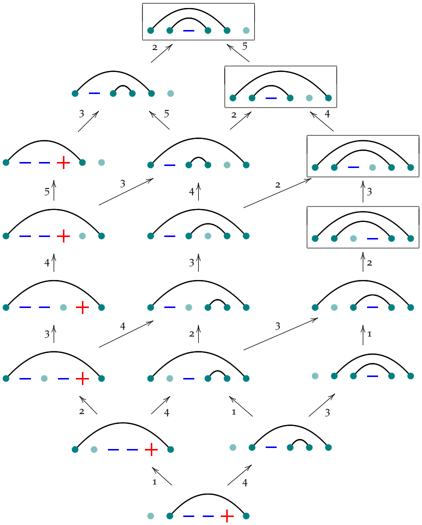
$$c_{v,u}^w(t) = \begin{cases} \mathfrak{S}_{w_\gamma}(-t_q, \ldots, -t_1; -t_{p+1}, -t_{p+2}, \ldots), \\ \qquad \textit{if } w * \gamma \textit{ is permutational and } \ell(w*\gamma) = \ell(w) + \ell(\gamma), \\ 0, \quad \textit{otherwise.} \end{cases}$$

Example



$$\mathfrak{S}_{312}(x;t)\mathfrak{S}_{231}(x;y) = (t_3 - t_1)\mathfrak{S}_{s_1 s_2 s_1}(x;t) + \mathfrak{S}_{s_3 s_1 s_2 s_1}(x;t).$$

Example



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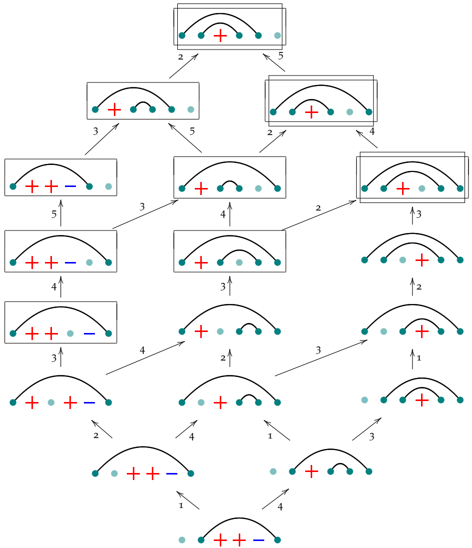

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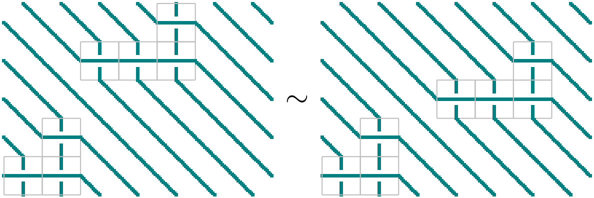
321-avoiding permutations

There is a well-known bijection between

$$\left\{\begin{matrix}\text{321-avoiding} \\ \text{permutations}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{skew shapes uo} \\ \text{to translation}\end{matrix}\right\}.$$

The bijection can be seen from the following diagram



The 321-avoiding permutations include both Grassmannian permutations and inverse Grassmannian permutations.

Main Result — Single Expansion

Now we can state our main result on single coefficients

$$\mathfrak{S}_v(x) \cdot \mathfrak{S}_u(x) = \sum_{w \in S_\infty} c_{v,u}^w \cdot \mathfrak{S}_w(x).$$

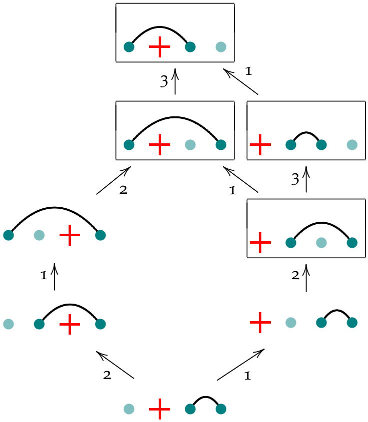
Let v be a p-inverse Grassmannian permutation and u be a 321-avoiding permutation. We can write $u = w_\lambda \tilde{u}$ where w_λ is a q-Grassmannian permutation for a partition $\lambda \subseteq (n-q)^q$ and $\tilde{u}$ is a q-inverse Grassmannian permutation. Let us pick a (p, m, q)-preclan γ for $m \gg 0$ with $v_\gamma = v$ and $u_\gamma = \tilde{u}$.

Theorem (Chen–Fan–X.–Yao)

$$c_{v,u}^w = \begin{cases} EG_{g_\gamma,\lambda}, & \textit{if } w * \gamma \textit{ is separated and } \ell(w*\gamma) = \ell(w) + \ell(\gamma), \\ 0 & \textit{otherwise.} \end{cases}$$

Here EG is the **Edelman–Greene coefficient**.

Example



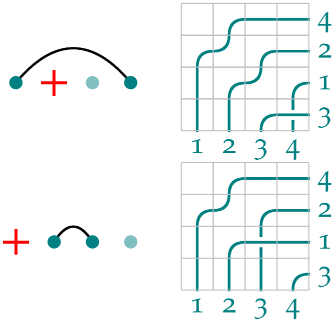
$\gamma =$ 

$$v = 3\ 1\ 2\ 4$$

$$\tilde{u} = 2\ 3\ 1\ 4$$

$$u = 1\ 3\ 2\ 4$$

$$\lambda = \square$$



$$\mathfrak{S}_{312}(x)\mathfrak{S}_{132}(x) = \mathfrak{S}_{s_1 s_2 s_1}(x;t) + \mathfrak{S}_{s_3 s_2 s_1}(x;t).$$

Thank You