Chern classes of Subvarieties over Partial Flag varieties [arXiv:2501.16172]

with Neil J.Y. Fan, Peter L.Guo and Changjian Su

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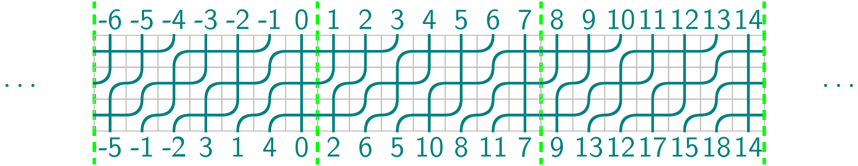


The space of constructible functions

For a variety X, we define the space of **constructible functions**

$$\mathsf{Fun}(X) = \operatorname{span}(\mathbb{1}_W : W \subset X \text{ open}).$$

This forms a functor via

$$\left[Y \xrightarrow{f} X\right] \mapsto \left[\mathsf{Fun}(Y) \xrightarrow{f_*} \mathsf{Fun}(X)\right]$$

$$\text{such that } (f_* \mathbb{1}_Y)(x) = \chi(f^{-1}(x))$$

where χ is the topological Euler characteristic, e.g.

$$\chi(\mathbb{C}) = \chi(\mathsf{pt}) = 1, \qquad \chi(\mathbb{C}^*) = \chi(\varnothing) = 0.$$

Chern classes and Segre classes

There is a natural transformation, generalizing the concept of Chern classes

$$c_{\mathrm{SM}} : \mathsf{Fun}(-) \longrightarrow H_*(-)$$

$$X \text{ smooth} \Rightarrow c_{\mathrm{SM}}(1_X) = c(\mathscr{T}_X)$$

$$X \xrightarrow{f} Y \text{ proper} \Rightarrow (\bigstar) \text{ commutes}$$

$$\begin{array}{ccc} \mathsf{Fun}(X) & \xrightarrow{c_{\mathrm{SM}}} & H_*(X) \\ \downarrow f_* & (\bigstar) & \downarrow f_* \\ \mathsf{Fun}(Y) & \xrightarrow{c_{\mathrm{SM}}} & H_*(Y) \end{array}$$

For a constructible subset $W \subseteq X$, we define the **CSM class**

$$c_{\mathrm{SM}}(W) = c_{\mathrm{SM}}(\mathbb{1}_W) \in H_*(X).$$

When X is smooth, we can identify $H_*(X) \cong H^*(X)$, we define the **SSM class**

$$s_{\mathrm{SM}}(W) = \frac{c_{\mathrm{SM}}(\mathbb{1}_W)}{c(T_X)} = \frac{c_{\mathrm{SM}}(\mathbb{1}_W)}{c_{\mathrm{SM}}(\mathbb{1}_X)} \in H^*(X).$$

Example $\mathbb{P}^1$

We identify $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We know $\mathscr{T}_{\mathbb{P}^1} = \mathscr{O}(2)$, so

$$c_{\text{SM}}(\mathbb{P}^1) = 1 + 2[\text{pt}].$$

It is clear that $c_{\text{SM}}(\text{pt}) = [\text{pt}]$, so

$$c_{\text{SM}}(\mathbb{C}) = c_{\text{SM}}(\mathbb{P}^1) - c_{\text{SM}}(\text{pt}) = 1 + [\text{pt}].$$

If we delete two points

$$c_{\text{SM}}(\mathbb{C}^*) = c_{\text{SM}}(\mathbb{P}^1) - 2c_{\text{SM}}(\text{pt}) = 1.$$

Relation to characteristic cycles

For experts, $\mathsf{Fun}(X)$ is the character group of constructible sheaves. We have a character

$$\chi : D^b_{\mathrm{con}}(X) \longrightarrow \mathsf{Fun}(X), \qquad \mathbb{Q}_W \longmapsto \mathbb{1}_W.$$

Actually, c_{SM}-class (or more precisely s_{SM}-class) can be constructed directly from

$$D^b_{\mathrm{con}}(X) \supseteq \mathbf{Perv}(X) \longrightarrow H^*(X)$$

via the **shadow of characteristic cycles**.

P. Aluffi, L. Mihalcea, J. Schürmann and C. Su, Shadows of characteristic cycles, Verma modules, and positivity of Chern–Schwartz–MacPherson classes of Schubert cells, Duke Math. J.

Flag varieties

Fix a reductive algebraic group

$$G \in \{GL_n, SL_n, PGL_n, SO_n, \ldots\}.$$

Let $\mathcal{B} = G/B$ be its flag variety and $\mathcal{P} = G/P$ be a partial flag variety with natural projection $\pi : \mathcal{B} \to \mathcal{P}$.

For example, when $G = GL_n$, we have an example of $\mathcal{B}$ and $\mathcal{P}$

$$\begin{aligned} \mathrm{Fl}(\mathbb{C}^n) &= \left\{0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n : \dim V_i = i\right\}, \\ \mathrm{Gr}_k(\mathbb{C}^n) &= \left\{V \subset \mathbb{C}^n : \dim V = k\right\}. \end{aligned}$$

The projection is given by $(V_\bullet) \mapsto V_k$.

Stratifications of $\mathcal{B}$

The space $\mathcal{B}$ is stratified by B-orbits (and also B^--orbits)

$$\mathcal{B} = \bigsqcup_{w \in W} \mathring{\Sigma}_w \qquad \mathcal{B} = \bigsqcup_{w \in W} \mathring{\Sigma}^w$$

$$\mathring{\Sigma}_w = \text{Schubert cell} \qquad \mathring{\Sigma}^w = \text{opposite Schubert cell}$$

$$= B\text{-orbit} \qquad = B^-\text{-orbit.}$$

We have a finer stratification

$$\mathcal{B} = \bigsqcup_{u \leq w \in W} \mathring{R}_{u,w}$$

$$\mathring{R}_{u,w} = \text{open Richardson varieties}$$

$$= \mathring{\Sigma}^u \cap \mathring{\Sigma}_w$$

Note that

$$\mathring{\Sigma}^u \cap \mathring{\Sigma}_w \neq \varnothing$$

$$\iff u \leq w.$$

Stratifications of $\mathcal{P}$

Similarly, the space $\mathcal{P}$ is stratified by B-orbits (and also B^--orbit)

$$\mathcal{P} = \bigsqcup_{w \in W^P} \mathring{\Sigma}_w^P \qquad \mathcal{P} = \bigsqcup_{w \in W^P} \mathring{\Sigma}_P^w$$

$$\mathring{\Sigma}_w^P = \text{Schubert cell} \qquad \mathring{\Sigma}_P^w = \text{opposite Schubert cell}$$

$$= B\text{-orbit} \qquad = B^-\text{-orbit.}$$

We can stratify

$$\mathcal{P} = \bigsqcup_{W^P \ni u \le w \in W^P} \mathring{R}_{u,w}^P$$

$$\mathring{R}_{u,w}^P = \text{open Richardson varieties}$$

$$= \mathring{\Sigma}_P^u \cap \mathring{\Sigma}_w^P$$

Note that

$$\mathring{\Sigma}_P^u \cap \mathring{\Sigma}_w^P \neq \varnothing$$

$$\iff u \le w.$$

Projected Richardson varieties

There is finer stratification,

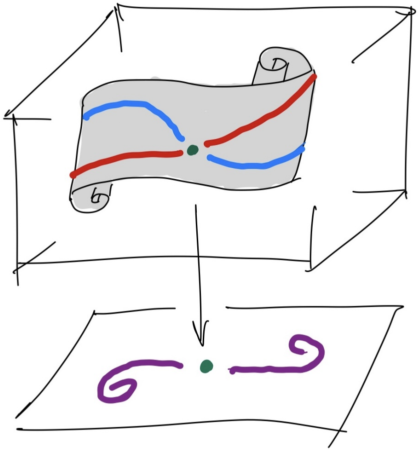
$$\mathcal{P} = \bigsqcup_{u \le w \in W^P} \mathring{\Pi}_{u,w}$$

$$\mathring{\Pi}_{u,w} = \text{projected Richardson}$$
$$= \pi(\mathring{R}_{u,w}).$$

Note that

$$\pi\begin{pmatrix}\text{(opposite)}\\ \text{Schubert}\end{pmatrix} = \begin{matrix}\text{(opposite)}\\ \text{Schubert}\end{matrix}$$

But not the case for Richardson. So in the sense, projected Richardson is a replacement.



Highlights

The projected Richardson varieties were first introduced by Lusztig to study the total positivity.

G. Lusztig, *Total positivity in partial flag manifolds*, Represent. Theory 2 (1998), 70–78.

When $\mathcal{P}$ is cominuscule, projected Richardson varieties represents certain Gromov–Witten invariants.

A.S. Buch, P.-E. Chaput, L.C. Mihalcea and N. Perrin, *Projected Gromov–Witten varieties in cominuscule spaces*, Proc. Amer. Math. Soc. 146 (2018), no. 9, 3647–3660.

In type A, it relates to knot theory and Macdonald theory.

P. Galashin, and T. Lam, *Positroids, knots, and q, t-Catalan numbers*, Duke Math. J. 173(11): 2117-2195 (15 August 2024). DOI: 10.1215/00127094-2023-0049

Computation for Schubert cells over $\mathcal{B}$

Let us explain the computation of **CSM classes of Schubert cells**.

Over $\mathcal{B}$, there is **Demazure–Lusztig operator** for each simple reflection $s_i \in W$

$$T_i : H_T^*(\mathcal{B}) \longrightarrow H_T^*(\mathcal{B}).$$

It coincides with the Springer action (actually not a coincidence).

Theorem (Aluffi–Mihalcea)

$$c_{\mathrm{SM}}(\mathring{\Sigma}_{\mathrm{id}}) = [\mathrm{id}], \qquad c_{\mathrm{SM}}(\mathring{\Sigma}^{w_0}) = [w_0],$$

$$T_i c_{\mathrm{SM}}(\mathring{\Sigma}_w) = c_{\mathrm{SM}}(\mathring{\Sigma}_{ws_i}). \qquad T_i c_{\mathrm{SM}}(\mathring{\Sigma}^w) = c_{\mathrm{SM}}(\mathring{\Sigma}^{ws_i}).$$

P. Aluffi and L. Mihalcea, *Chern–Schwartz–MacPherson classes for Schubert cells in flag manifolds*, Compos. Math. 152 (2016), 2603–2625.

Computation for Schubert cells over $\mathcal{P}$

For $w \in W^P$,

$$\pi(\mathring{\Sigma}_w) = \mathring{\Sigma}^P_w, \quad \Longrightarrow \quad \pi_*(c_{\mathrm{SM}}(\mathring{\Sigma}_w)) = c_{\mathrm{SM}}(\mathring{\Sigma}^P_w).$$

However, there is a more direct way of computing, without passing through $\mathcal{B}$. The technique is the **left operator**

$$T_i^L : H_T^*(\mathcal{P}) \longrightarrow H_T^*(\mathcal{P}).$$

It only acts on the T-equivariant parameters.

Theorem (Mihalcea–Naruse–Su)

$$c_{\mathrm{SM}}(\mathring{\Sigma}^P_{\mathrm{id}}) = [\mathrm{id}], \qquad c_{\mathrm{SM}}(\mathring{\Sigma}_P^{w_0^P}) = [w_0^P],$$

$$T_i^L c_{\mathrm{SM}}(\mathring{\Sigma}^P_w) = c_{\mathrm{SM}}(\mathring{\Sigma}^P_{\overline{s_i w}}), \qquad T_i^{L,\vee} c_{\mathrm{SM}}(\mathring{\Sigma}_P^w) = c_{\mathrm{SM}}(\mathring{\Sigma}_P^{\overline{s_i w}}).$$

L. Mihalcea, H. Naruse and C. Su, Left Demazure–Lusztig operators on equivariant (quantum) cohomology and K-theory, Int. Math. Res. Not. IMRN, 16 (2022):12096–12147.

A close formula for Grassmannian

Over Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$, we can identify

$$W^P = \{\text{partitions inside the rectangle } (n-k)^k\}.$$

There is a symmetric function representative

Theorem (Maulik–Okounkov, Shenfeld)

$$s_{\mathrm{SM}}(\mathring{\Sigma}^\lambda) = \textit{rational analogs of the interpolation Schur functions}$$
$$= \mathrm{Sym}\left(\prod_{i=1}^{k}\prod_{j=1}^{n}\begin{cases}\frac{x_i-y_j}{1+x_i-x_j} & j<\lambda_i+i-k\\ \frac{1}{1+x_i-x_j} & j=\lambda_i+i-k\\ 1 & j>\lambda_i+i-k.\end{cases}\right)$$

- D. Maulik, and A. Okounkov, *Quantum groups and quantum cohomology*, Astérisque, 408, 1–225, 2019.
- D. Shenfeld, Abelianization of stable envelopes in symplectic resolutions, MI, 2013. Thesis (Ph.D.)–Princeton University.

Computation for Richardson varieties

Let us switch to **CSM classes of open Richardson varieties**.

We have a very general theorem on the CSM classes of transversal intersections.

Theorem (Schürmann)

$$c_{\mathrm{SM}}(Z \pitchfork W) = c_{\mathrm{SM}}(Z) \frown s_{\mathrm{SM}}(W)$$

In particular, we have

$$c_{\mathrm{SM}}(\mathring{R}^P_{u,w}) = c_{\mathrm{SM}}(\mathring{\Sigma}^u_P) \frown s_{\mathrm{SM}}(\mathring{\Sigma}^P_w).$$

J. Schürmann, *Chern classes and transversality for singular spaces*, In Singularities in Geometry, Topology, Foliations and Dynamics, Trends in Mathematics, pages 207–231. Birkhäuser, Basel, 2017.

Computation for Richardson varieties

Let us give another perspective.

Theorem (Aluffi–Mihalcea–Schürmann–Su)

$$\int_{\mathcal{P}} c_{\mathrm{SM}}(\mathring{\Sigma}_P^u) s_{\mathrm{SM}}(\mathring{\Sigma}_w^P) = \int_{\mathcal{P}} c_{\mathrm{SM}}(\mathring{R}_{u,w}^P) = \chi(\mathring{R}_{u,w}^P) = \delta_{u,w}.$$

In particular, for any $\gamma \in H_T^*(\mathcal{P})$

$$\begin{aligned}\int_{\mathcal{P}} c_{\mathrm{SM}}(\mathring{R}_{u,w}^P) \cdot \gamma &= \int_{\mathcal{P}} \left(\gamma \cdot c_{\mathrm{SM}}(\mathring{\Sigma}_P^u)\right) \cdot s_{\mathrm{SM}}(\Sigma_P^w) \\ &= \text{coefficient of } c_{\mathrm{SM}}(\mathring{\Sigma}_P^w) \text{ in } \gamma \cdot c_{\mathrm{SM}}(\mathring{\Sigma}_P^u)\end{aligned}$$

It suffices to give the formula for a set of generator of γ.

P. Aluffi, L. Mihalcea, J. Schürmann and C. Su, Shadows of characteristic cycles, Verma modules, and positivity of Chern–Schwartz–MacPherson classes of Schubert cells, Duke Math. J.

A close formula for Grassmannian

For Grassmannian, the Chern classes of dual tautological bundle form a set of generators.

Theorem (Fan, Guo and Xiong)

$$c_r(\mathcal{V}^\vee) c_{\mathrm{SM}}(\mathring{\Sigma}^\lambda) = \sum_\mu c_{\mathrm{SM}}(\mathring{\Sigma}^\mu) \quad \textit{with} \quad \mu = \lambda + r\binom{\textit{decreasing}}{\textit{ribbons}}.$$

It is well-known that the following two basis are dual (aka Cauchy formula)

$$\left\{\begin{array}{c}\text{monomial of}\\ \text{Chern classes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{monomial symmetric}\\ \text{function in Chern roots}\end{array}\right\}.$$

This leads to a ribbon tableaux formula for $c_{\mathrm{SM}}(\mathring{R}_{\lambda,\mu})$.

N. Fan, P. Guo, and R. Xiong, *Pieri and Murnaghan–Nakayama type rules for Chern classes of Schubert cells*, arXiv:2211.06802, 2022.

Projected Richardson varieties

Now, let us switch to the **projected Richardson varieties**.

- Recall
$$\mathring{\Pi}_{u,w} = \pi_*(\mathring{R}_{u,w}) \quad \Longrightarrow \quad c_{\mathrm{SM}}(\mathring{\Pi}_{u,w}) = \pi_*\big(c_{\mathrm{SM}}(\mathring{R}_{u,w})\big).$$
- On the other hand, one can repeat the above argument (with projection formula) to see
$$\int_{\mathcal{P}} c_{\mathrm{SM}}(\mathring{\Pi}_{u,w}) \cdot \gamma = \text{coefficient of } c_{\mathrm{SM}}(\mathring{\Sigma}^w) \text{ in } \pi^*(\gamma) \cdot c_{\mathrm{SM}}(\mathring{\Sigma}^u).$$

This two formulæ provide two aspects of computation. As promised, I will add a new perspective by relating them to affine flag varieties. From now, we will assume the Dynkin diagram of G to be connected for simplicity.

Affine flag varities

Let $\mathrm{Fl}_G = G((z))/(\text{Iwahori})$ be the **affine flag variety**. For an element of (extended) affine Weyl group $w \in W \ltimes X_*(T)$, we have

$$\mathring{\Sigma}_w = \text{Schubert cell} \subset \mathrm{Fl}_G, \qquad c_{\mathrm{SM}}(\mathring{\Sigma}_w) \in H_*^T(\mathrm{Fl}_G).$$

Different from the finite cases, we do not have an opposite Schubert "cell", but we still can **define** algebraically

$$s_{\mathrm{SM}}(\mathring{\Sigma}^w) = \text{dual basis of } c_{\mathrm{SM}}(\mathring{\Sigma}_w) \in \widehat{H_T^*(\mathrm{Fl}_G)}.$$

Theoretically speaking, the class lies in the cohomology of Kashiwara's thick flag variety.

Remark: We believe this class should have a geometric meaning.

Affine Grassmannians

Let $\mathrm{Gr}_G = G(\!(z)\!)/G[[z]]$ be the **affine Grassmannian**. For example, for $G = GL_n$,

$$\mathrm{Gr}_G = \left\{\mathbb{C}[[t]]\text{-lattice } L \subset \mathbb{C}(\!(t)\!)^{\oplus n}\right\}.$$

Let $\lambda \in X_*(T)$ be a dominant coweight such that $W_\lambda = W_P$. We have a torus fixed point

$$z^{-\lambda} G[[z]] \in \mathrm{Gr}_G \, .$$

Then

- the G-orbit of it is isomorphic to $\mathcal{P}$;
- the $G[[z]]$-orbit of it is an affine bundle over the G-orbit.

Comparison

Our main theorem relates two SSM classes.

$$\widehat{H_T^*(\mathcal{P})} \ni s_{\mathrm{SM}}(\mathring{\Pi}_{u,w}) \qquad \textbf{VS} \qquad s_{\mathrm{SM}}(\mathring{\Sigma}^f) \in \widehat{H_T^*(\mathrm{Fl}_G)}.$$

Theorem (FGSX, 2025+)

$$i_{\lambda,*}\left(s_{\mathrm{SM}}(\mathring{\Pi}_{u,w}) \cdot c^T(\mathcal{N})\right) = (j_\lambda^* \circ r^*)\left(s_{\mathrm{SM}}(\mathring{\Sigma}^f)\right) \in H_T^*(\mathrm{Gr}_\lambda)_{\mathrm{loc}},$$

where

- $u \leq w \in W^P$ and $f = ut_\lambda w^{-1} \in Wt_\lambda W \subset \widehat{W}$;
- $\mathcal{N}$ is the normal bundle of G-orbit $\mathcal{P}$ in $G[[z]]$-orbit Gr_λ;
- $i_\lambda : \mathcal{P} \hookrightarrow \mathrm{Gr}_\lambda$ the inclusion and $j_\lambda : \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}_G$ the inclusion;
- $r : \mathrm{Gr}_G \cong \Omega K \subseteq LK \to LK/(T \cap K) \cong \mathrm{Fl}_G$ the "wrong way map".

Question

Let us explain the evidence that the two classes are related.

Let us start from the answer of the following geometric question. Recall $\pi : \mathcal{B} \to \mathcal{P}$ is the natural projection.

Question

How to characterize the pair (u, w) *such that the* $\pi_*(\mathbb{1}_{\mathring{R}_{u,w}})$ *is non-zero?*

Note that $\pi_*(\mathbb{1}_{\mathring{R}_{u,w}}) = 0 \not\Longrightarrow \pi(\mathring{R}_{u,w}) = \varnothing$. For example, when $G = SL_2$, $\mathcal{B} = \mathbb{P}^1$ and $\mathcal{P} = \mathrm{pt}$, we have

$$\mathring{R}_{\mathrm{id},s} = \mathbb{P}^1 \setminus \{0, \infty\}, \qquad \pi_*(\mathbb{1}_{\mathring{R}_{\mathrm{id},s}}) = \chi(\mathring{R}_{\mathrm{id},s}) = 0.$$

The answer turns out to be very combinatorial.

Answer

Let us define the **extended P-Bruhat order**

$$u \leq_P w \iff \text{there exists a chain } u \xrightarrow{P} u_1 \xrightarrow{P} \cdots \xrightarrow{P} u_{k-1} \xrightarrow{P} w$$

$$u \xrightarrow{P} w \iff w = ut > u \text{ for some reflection } t \text{ such that } wW_P \neq uW_P.$$

This definition is motivated by the Chevalley formula of CSM classes of Schubert cells.

Theorem (FGSX, 2025+)

The following statements are equivalent

- $u \leq_P w$;
- $f \leq t_\mu$ *for some* $\mu \in W\lambda$;
- $\pi_*(\mathbb{1}_{\mathring{R}_{u,w}}) \neq 0$;

where $u, w \in W$ *and* $f = ut_\lambda w^{-1}$.

Wrong way map

Recall the "wrong way map"

$$r : \mathrm{Gr}_G \cong \Omega K \subseteq LK \to LK/(T \cap K) \cong \mathrm{Fl}_G \,.$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
W\lambda & = & W\lambda & \overset{\subset}{\longrightarrow} & X_*(T) & \overset{\subset}{\longrightarrow} & \widehat{W} \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \downarrow\cap \\
\mathcal{P} & \underset{i_\lambda}{\overset{\subset}{\longrightarrow}} & \mathrm{Gr}_\lambda & \underset{j_\lambda}{\overset{\subset}{\longrightarrow}} & \mathrm{Gr}_G & \underset{r}{\longrightarrow} & \mathrm{Fl}_G
\end{array}$$

in particular,

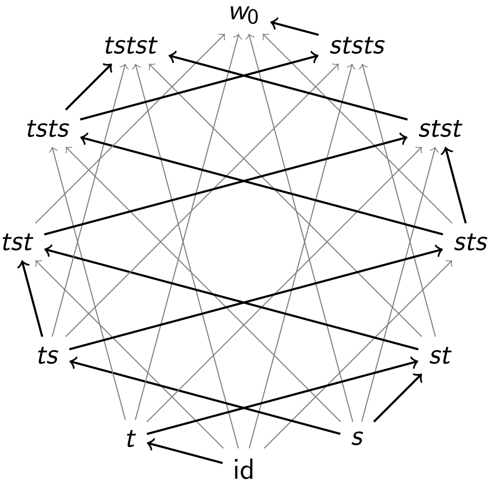
for $\gamma \in H_T^*(\mathrm{Fl}_G)$,

$$(j_\lambda^* \circ r^*)(\gamma)|_{t_\lambda} = \gamma|_{t_\lambda}.$$

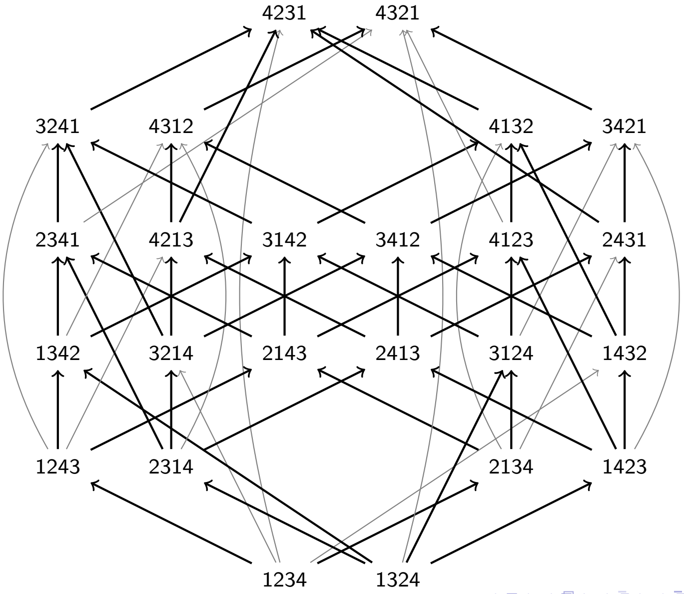
From above, we roughly have

$$\pi_*(c_{\mathrm{SM}}(\mathring{R}_{u,w})) \neq 0 \iff u \leq_P w \iff (j_\lambda^* \circ r^*)(s_{\mathrm{SM}}(\mathring{\Sigma}^f)) \neq 0.$$

Example ($W = G_2$, $W_P = \{1, s\}$)



Example ($W = S_4$, $W_P = S_1 \times S_2 \times S_1$)



Example

Theorem (Fan–Guo–Xiong)

When $W = S_n$ and $W_P = S_k \times S_{n-k}$ (i.e. $\mathcal{P}$ is a Grassmannian)

$$u \leq_P w \iff \begin{cases} u(1) \leq w(1) \\ \cdots \\ u(k) \leq w(k) \end{cases} \text{and} \begin{cases} u(k+1) \geq w(k+1) \\ \cdots \\ u(n) \geq w(n) \end{cases}$$

We have a similar description in $W = BC_n$ and $W_P = S_n$ (i.e. $\mathcal{P}$ is a maximal isotropic Grassmannian or maximal Lagrangian Grassmannian).

N. Fan, P. Guo, and R. Xiong, *Pieri and Murnaghan–Nakayama type rules for Chern classes of Schubert cells*, arXiv:2211.06802, 2022.

Grassmannians

Now let us restrict ourselves to Grassmannian. The (open) projected Richardson varieties are known as (open) **positroid varieties**, after Postnikov. We choose the fundamental coweight $\lambda = \varpi_k^\vee$.

Theorem (Knutson–Lam–Speyer)

$$\left\{ u t_\lambda w^{-1} : u \leq w \in W^P \right\} = \left\{ \begin{matrix} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijective} \end{matrix} : \begin{matrix} f(i+n) = f(i) + n \\ \frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k \\ i \leq f(i) \leq i + n \end{matrix} \right\}.$$

The right-hand side is called the **bounded affine permutations**, obviously bijective to **decorated permutations** by Postnikov.

A. Knutson, T. Lam, and D. Speyer, *Positroid varieties: juggling and geometry*, Compos. Math. 149 (2013), no. 10, 1710–1752.

Symmetric function representative

We take $G = GL_n$. Let us identify

$$H_T^*(\mathrm{pt}) = \mathbb{Q}[y_1, \ldots, y_n]$$
$$H_T^*(\mathrm{Gr}_k(\mathbb{C}^n)) = \text{a quotient algebra of } H_T^*(\mathrm{pt})[x_1, \ldots, x_k]^{S_k}.$$

Let us denote $\mathring{\Pi}_f = \mathring{\Pi}_{u,w}$ for $f = ut_\lambda w^{-1}$. Then we have

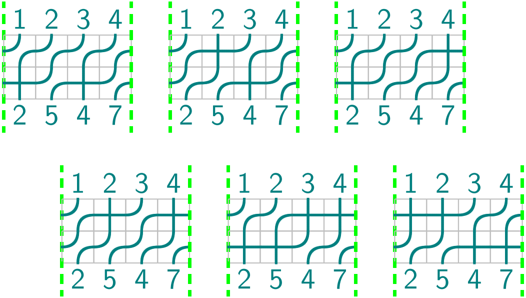
Theorem (FGSX, 2025)

$$s_{\mathrm{SM}}(\mathring{\Pi}_f) = \begin{array}{c}\textit{weighted sum of certain}\\ \textit{“periodic pipe dreams”}\end{array} \in H_T^*(\mathrm{Gr}_k(\mathbb{C}^n)).$$

Comparing with Shimozono–Zhang, the lowest degree component of the weighted sum is a **(double) affine Stanley polynomial**.

Example

Gr(2, 4)
$u = 1324$
$w = 2413$
$f = 2547$



Thus

$$s_{\mathrm{SM}}(\mathring{\Pi}_f) = \frac{1}{\prod_{i=1}^{2}\prod_{j=1}^{4}(1+x_i-y_j)} \begin{pmatrix} (x_2-y_1)(x_2-y_3)+(x_1-y_2)(x_2-y_3) \\ +(x_1-y_4)(x_2-y_1)+(x_1-y_2)(x_1-y_4) \\ +(x_1-y_3)(x_1-y_4)(x_2-y_1)(x_2-y_2) \\ +(x_1-y_1)(x_1-y_2)(x_2-y_3)(x_2-y_4) \end{pmatrix}$$

R-matrices

The proof uses a diagrammatic calculation of the classical R-matrices.

The R-matrices we are using is the classical one, i.e.

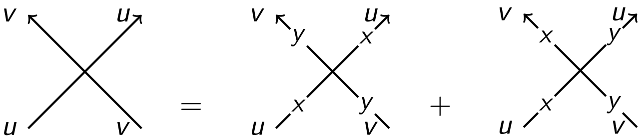
$$R(x) : \frac{x}{1+x} s_i + \frac{x}{1+x} \in \text{Group ring of } \tilde{S}_n$$

which is from the Yangian $Y_\hbar(\mathfrak{gl}_n)$ on $V = \mathbb{C}^n$. These R-matrices were used to compute the SSM classes of type A flag varieties. An explanation is, in type A,

$$T^*\mathcal{P} = \mathfrak{M}\left(\begin{array}{l} \boxed{n} \\ \ | \\ \bigcirc - \cdots - \bigcirc \end{array}\right)$$

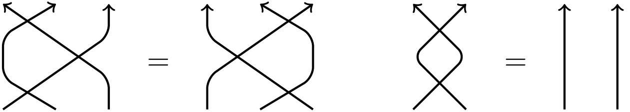
is a Nakajima quiver varieties.

We represent it diagrammatically by a cross



$$R(u-v) \qquad \frac{u-v}{1+u-v} \qquad \frac{1}{1+u-v}$$

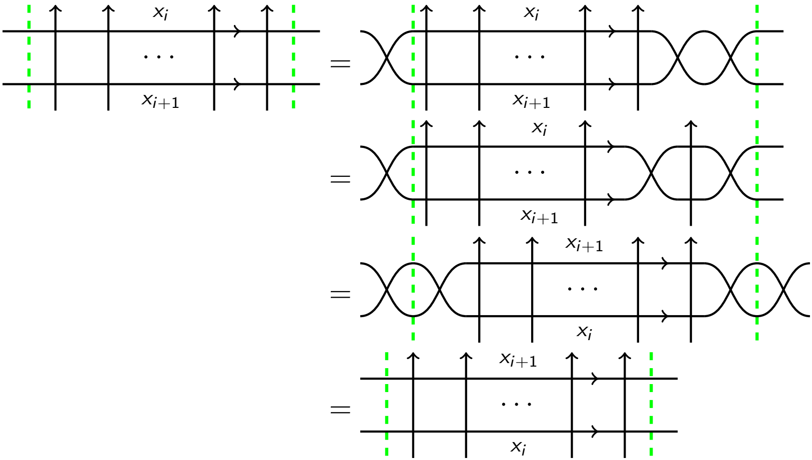
Then the **Yang–Baxter equation** and the **unitary equation** can be drawn as the invariance of two local moves



Yang–Baxter equation (**YBE**) unitary equation (**UE**)

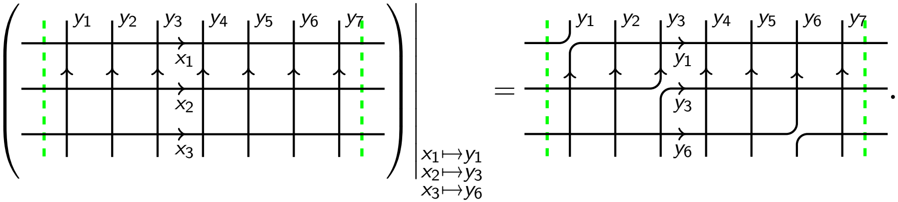
The proof

The generating function is symmetric in $x_1, \ldots, x_k$ by the following old argument due to Baxter (commutativity of transfer matrices).



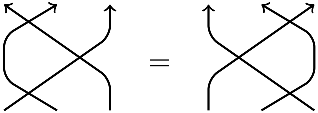
The proof

It rests to prove the localization agrees. This can also be done by a diagram calculus.



The resulting diagram computes the localization of SSM of affine Schubert cells (say, a Billey type formula). By our theorem, it is also the SSM of the open positroid varieties. So the proof is complete.

Thank You



Yang–Baxter equation (**YBE**)