Note on Quiver Representations

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Contents

1 Associative Algebras

1.1. We fix a field k. All the algebras and the modules are assumed to be finite-dimensional. Let R be an algebra. Denote R-mod the category of finite-dimensional modules over R.

1.2. Radical Let R be an algebra. Denote $\operatorname{rad} R$ be the **Jacobson radical**. It can be described as follows.

$$u \in \operatorname{rad} R \iff^{\forall x \in R,} 1 - xu \in \operatorname{unit} R \iff^{\forall x \in R,} 1 - ux \in \operatorname{unit} R.$$

Then it is the intersection of all maximal left (right) ideals, thus an ideal. It is also a nilpotent ideal. `Hint: It is a classic ring theory excises that` $1-ab$ `is invertible if and only if` $1-ba$ `is invertible. Say` $1+b(1-ab)^{-1}a$. `If` $1-xu \in \mathfrak{M}$ `for some maximal ideal and` $x \in R$, `then` u `cannot in` $\mathfrak{M}$. `Conversely, if` $x \notin \mathfrak{M}$, `then` $1 \in Rx+\mathfrak{M}$. `Note that` $\operatorname{rad} R^n = \operatorname{rad} R \cdot \operatorname{rad} R^n$ `for some` n, `then by Nakayama lemma, it is zero.`

1.3. Nakayama Lemma For any module $M \in R$-mod, denote the **radical of M** by $\operatorname{rad} M = \operatorname{rad} R \cdot M$. We call $M/\operatorname{rad} M$ the **top** of M. For a module morphism $M \xrightarrow{f} N$, it induces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{rad} M & \longrightarrow & M & \longrightarrow & M/\operatorname{rad} M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & \searrow f' & \downarrow f'' & & \\
0 & \longrightarrow & \operatorname{rad} N & \longrightarrow & N & \longrightarrow & N/\operatorname{rad} N & \longrightarrow & 0
\end{array}$$

The **Nakayama lemma** claims that

$$f \text{ is surjective} \iff f' \text{ is surjective} \iff f'' \text{ is surjective.}$$

`Hint:` $f(M) + \operatorname{rad} R \cdot N = N$, `thus` $\operatorname{rad} R \cdot N/f(M) = 0$. `So` $N = f(M)$, `this is the usage of classic Nakayama lemma. Pick a set of minimal generators etc.`

1.4. Semisimple For any module $M \in R$-mod, $\operatorname{rad} M$ is the intersection of all maximal proper submodule, and the top of $M/\operatorname{rad} M$ is semisimple (direct sum of simple modules). Thus the algebra $R/\operatorname{rad} R$ is semisimple. `Hint: Consider the inclusion of` $M' \to M$ `for any maximal proper submodule.`

We see $\operatorname{rad} R{\cdot}M{+}M'$ must be M'. For any maximal submodule, we can construct $M/M' \hookrightarrow \bigoplus M/M' \hookrightarrow M/\operatorname{rad} M \twoheadrightarrow M/M'$. Then we can split a simple summand, etc.

1.5. Projective modules For any projective module $P \in \operatorname{proj} R\text{-}\mathsf{mod}$, the natural ring homomorphism $\operatorname{End}(P) \to \operatorname{End}(P/\operatorname{rad} P)$ is surjective has kernel $\operatorname{rad}\operatorname{End}(P)$. In particular,

$$\left\{\begin{array}{l}\text{indecomposable}\\ \text{projective modules}\end{array}\right\}/\cong \quad \underset{\text{taking projective cover}}{\overset{\text{taking top}}{\rightleftarrows}} \quad \left\{\text{simple modules}\right\}/\cong$$

gives a bijection of finite sets. Hint: It follows from the fact that the map is surjective and $\operatorname{End}(P/\operatorname{rad} P)$ is already semi-simple, thus the kernel must be the radical. To see it is a bijection, it suffices to show each simple module has a projective cover. Actually, we can lift the idempotent elements by a nilpotent ideal. Say $e^2 \equiv e \bmod J$, then $(3e^2 - 2e^3)^2 \equiv 3e^2 - 2e^3 \bmod J^2$. Pick any simple module S, and any projective $P \twoheadrightarrow S$. The projection to S in $P/\operatorname{rad} P$ can be lift to a projection in P.

Warning For general indecomposable module M, $M/\operatorname{rad} M$ is not necessarily simple.

1.6. Projective cover For any module M, the lift of the projective cover for $M/\operatorname{rad} M$ is a projective cover for M. In particular, in the category R-mod, every object has a projective cover.

1.7. Basic Algebra By decompose $R/\operatorname{rad} R$, we see that any indecoposable projective module is a direct summand of left regular module R. Consider the algebra $R' = \operatorname{End}_R(P)^{\mathrm{op}}$ where P is the direct sum of all indecomposable projective modules (with multiplicity 1). Then R-mod and R'-mod are Morita equivalent. Then each indecomposable module has multiplicity one in R'. This is called **basic algebra**.

1.8. Duality Denote $\operatorname{Hom}_k(-, k)$ the **duality functor**. It defines an anti-equivalence of category R-mod and R^{op}-mod. In particular, for any module $M \in R$-mod, the sum of all simple module is called **socle**. Dually,

$$\left\{\begin{array}{l}\text{indecomposable}\\ \text{injective modules}\end{array}\right\}/\cong \quad \underset{\text{taking injective hull}}{\overset{\text{taking socle}}{\rightleftarrows}} \quad \left\{\text{simple modules}\right\}/\cong$$

gives a bijection of finite sets. For any module M, the extension of the injecitve hull for $\operatorname{soc} M$ is an injective hull for M. In particular, the injective hull of each object in R-mod lies in R-mod.

Exercises

1.9. Let $\{P_1, \ldots, P_n\}$ be the set of indecomposable projective modules over R. Let $S_i = P_i / \operatorname{rad} P_i$ the corresponding simple modules. Show that

$$\dim_k \operatorname{Hom}(P_i, P_j) = \text{multiplicity of } S_i \text{ in composition series of } P_j.$$

Hint: Since $\operatorname{Hom}(P, -)$ is exact for projective P, and $\operatorname{Hom}(M, S) = \operatorname{Hom}(M/\operatorname{rad} M, S)$ for simple S.

1.10. Show that M is indecomposable if and only if $\operatorname{End}(M)$ is a local ring.
Hint: A local ring has no nontrivial idempotent. $\operatorname{End}(M)$ is a local ring follows from Fitting lemma.

Warning When M is simple, $\operatorname{End}(M)$ is a division algebra (Schur lemma), but the converse is not true in general.

2 Quivers

2.1. For a pseudo-abelian category $\mathcal{C}$ over k (that is, an k-categroy with each idempotent morphism realized as a projection to some object). Assume this category has Krull–Schmidt property.

2.2. Radical To do this, for indecomposable objects M and N, denote the **radical** $\operatorname{rad}_{\mathcal{C}}(M,N) \subseteq \operatorname{Hom}_{\mathcal{C}}(M,N)$ the space of non-isomorphism maps. That is, it is just $\operatorname{Hom}(M,N)$ when $M \not\cong N$, and it is the radical of the local ring $\operatorname{End}(M)$.

Denote $\operatorname{rad}^2_{\mathcal{C}}(M,N)$ the space spanned by $f \circ g$ with $g \in \operatorname{rad}_{\mathcal{C}}(M,L)$ and $f \in \operatorname{rad}_{\mathcal{C}}(L,N)$ for an indecomposable L. Equivalently, $M \xrightarrow{g} K \xrightarrow{f} N$ with $K \in \mathcal{C}$ and g not a split injection, f not a split surjection.

Similarly, we denote $\operatorname{rad}^n_{\mathcal{C}}(M,N)$ the space spanned by $f_1 \circ \cdots \circ f_n$ with each $f_i \in \operatorname{rad}_{\mathcal{C}}(\bullet,\bullet)$. We take the convention that $\operatorname{rad}^0_{\mathcal{C}} = \operatorname{Hom}_{\mathcal{C}}(M,N)$.

2.3. Quiver Category A quiver is a directed graph. For a quiver Q, we define the category

$$k\langle Q\rangle = \begin{cases} \textbf{Obj}: & i \quad \text{vertex of } Q \\ \textbf{Mor}: & \begin{array}{c} i \\ \downarrow \\ j \end{array} \in \bigoplus k\cdot(\text{paths from } i \to j) \end{cases}$$

The path algebra is defined by

$$kQ = \bigoplus_{i,j \text{ vertices}} \mathbb{1}_j \operatorname{Hom}(i,j)\mathbb{1}_i = \bigoplus k\cdot(\text{all paths}).$$

with $\{\mathbb{1}_i\}$ the formal orthogonal idempotents. Here we take empty path into consideration (the identity morphism).

2.4. Quiver of a category The **quiver** $Q=(I,H)$ of the category $\mathcal{C}$ is a directed graph with I the equivalence classes of indecomposable objects and

$$\#\{M \to N\} = \dim_k \operatorname{rad}_{\mathcal{C}}(M,N)/\operatorname{rad}^2_{\mathcal{C}}(M,N).$$

If we pick a lift of a choice of basis in $\operatorname{rad}_{\mathcal{C}}(M,N)$. Then each arrow corresponds to a morphism in $\mathcal{C}$. It defines a functor $k\langle Q\rangle \to \mathcal{C}$ which is full (surjecive in Hom). Moreover, it is compatible with the radical filtration. That is,

$$\operatorname{rad}^n_{\mathcal{C}}(M,N) = \operatorname{span}(\text{paths } M \to N \text{ of length} \geq n).$$

Thus, the quiver of the category is an approximation (of degree 1) of the category.

2.5. Quiver of an algebra For an algebra R, assume that $R/\operatorname{rad} R$ is a product of copies of k's (for example k is algebraically closed). The quiver of R is the quiver of $\operatorname{proj} R\text{-}\mathsf{mod}$. Then there is a surjective algebra homomorphism $kQ^{\mathrm{op}} \to R$ with kernel in $\operatorname{rad}^2 kQ$.

2.6. Auslander–Reiten Quiver of an algebra For an algebra R, the Auslander–Reiten Quiver (AR quiver) is the quiver of $R\text{-}\mathsf{mod}$.

2.7. Quiver Reprsentation Denote the category of **quiver representation** to be $Q\text{-}\mathsf{rep} = \operatorname{Fun}_k(k\langle Q\rangle, k\text{-}\mathsf{mod})$. Equivalently, $Q\text{-}\mathsf{rep} = kQ\text{-}\mathsf{mod}$. Note that, to give a quiver representation is to give a vector space for each vertex and to give a linear map for each arrow.

Warning My notation of kQ is converse to most of books (where they use right modules mostly).

2.8. Consider the quiver

$$Q : \underset{1}{\bullet} \xrightarrow{f} \underset{2}{\bullet}$$

Then $kQ = ke_1 \oplus ke_2 \oplus kf$ with product

$\downarrow \cdot \rightarrow$	e_1	f	e_2
e_1	e_1	0	0
f	f	0	0
e_2	0	f	e_2

It is isomorphic to the algebra $\begin{pmatrix} k & k \\ & k \end{pmatrix}$. A Q-representation is just to give $V_1 \xrightarrow{f} V_2$. Thus it is just classified by $\dim V_1$, $\dim V_2$ and $\operatorname{rank} f$. There are three indecomposable objects in $Q\text{-}\mathsf{rep}$.

$$S(1) = [k \to 0] \qquad S(2) = P(2) = [0 \to k] \qquad P(1) = [k \xrightarrow{\mathrm{id}} k].$$

Note that $S(i)$ is simple for $i = 1, 2$. For $V = [V_1 \xrightarrow{f} V_2]$, we have

$$\operatorname{Hom}(S(1), V) = \ker f \qquad \operatorname{Hom}(S(2), V) = V_2 \qquad \operatorname{Hom}(P(1), V) = V_1.$$

Thus $P(i)$ is projective cover of $S(i)$ for $i = 1, 2$. The following is the Auslander–Reiten quiver of kQ

$$\begin{array}{ccccc} & & P(1) & & \\ & \nearrow & & \searrow & \\ P(2) = S(2) & & & & S(1) \end{array}$$

Exercise

2.9. Let $\{P_1, \dots, P_n\}$ be the set of indecomposable projective modules over R. Show that

$$\operatorname{rad}^n(P_i, P_j) = \operatorname{Hom}(P_i, \operatorname{rad}^n P_j).$$

Hint: Any $f \in \operatorname{Hom}(P_i, P_j)$, it is an isomorphism if and only if the induced map $S_i \to S_j$ is an isomorphism. The general case follows from definition.

2.10. Under the same notation, assume $S_i = P_i / \operatorname{rad} P_i$. Show that

$$\dim \operatorname{Ext}^1(S_j, S_i) \cong \dim \operatorname{rad}(P_i, P_j)/\operatorname{rad}^2(P_i, P_j).$$

Hint: We have $\operatorname{rad} P_j \to P_j \to S_j$. Then we get

$$0 \to \operatorname{Hom}(S_j, S_i) \xrightarrow{\cong} \operatorname{Hom}(P_j, S_i) \to \underbrace{\operatorname{Hom}_R(\operatorname{rad} P_j, S_i)}_{\operatorname{Hom}_{R/\operatorname{rad} R}(\operatorname{rad} P_j/\operatorname{rad}^2 P_j, S_i)} \to \operatorname{Ext}(S_j, S_i) \to 0$$

So $\operatorname{Ext}(S_j, S_i) = \operatorname{Hom}_{R/\operatorname{rad} R}(\operatorname{rad} P_j / \operatorname{rad}^2 P_j, S_i)$. Since $R/\operatorname{rad} R$ is semisimple, it has the same dimension as $\operatorname{Hom}_{R/\operatorname{rad} R}(S_i, \operatorname{rad} P_j/\operatorname{rad}^2 P_j) = \operatorname{Hom}_R(P_i, \operatorname{rad} P_j /$ Since $\operatorname{Hom}_R(P_i, -)$ is additive, this proves the assertion.

2.11. Consider the quiver

$$Q : \underset{1}{\bullet} \xrightarrow{g} \underset{2}{\bullet} \xrightarrow{f} \underset{3}{\bullet}$$

Then $kQ = ke_1 \oplus ke_2 \oplus kf \oplus kg \oplus k(fg)$. Show that it is isomorphic to the algebra $\begin{pmatrix} k & k & k \\ & k & k \\ & & k \end{pmatrix}$.

3 Functor Category

3.1. Functor Category Denote $\mathrm{Fun}(R)$ the category of additive functor from R-mod to k-mod. Note that it is an abelian category with kernel and cokernel object-wise. We have the **Yoneda embedding**, a contravariant functor $R\text{-mod} \longrightarrow \mathrm{Fun}(R)$ sending M to $\tilde{M} := \mathrm{Hom}_R(M, -)$. Then

$$\mathrm{Hom}_{\mathrm{Fun}(R)}(\tilde{M}, F) = F(M), \qquad \widetilde{\operatorname{cok} f} = \ker \tilde{f}.$$

In particular, the Yoneda embedding is fully faithful (isomorphic in Hom).

3.2. Finitely Generated Functor Note that $\tilde{M}$ is projective in $\mathrm{Fun}(R)$ by definition. We say $F \in \mathrm{Fun}(R)$ **finitely generated** if it is a quotient of $\tilde{M}$. By Yoneda embedding,

$$\left\{ \begin{array}{c} \text{indecomposable} \\ \text{modules in } R\text{-mod} \end{array} \right\} / \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{indecomposable finitely} \\ \text{generated projective objects} \end{array} \right\} / \cong$$

gives a bijection of sets (not necessarily finite). `Hint: If` $\tilde{M} \twoheadrightarrow F$ `with` F `projective, then the splitting can be realized as an idempotent over` $\tilde{M}$`, then over` M`, so` F `is representable.`

3.3. Simple Functor For any simple object S in $\mathrm{Fun}(R)$, it has a finitely generated projective cover say $\tilde{M}$ for some indecomposable $M \in R$-mod. Moreover, $S(N) \cong \mathrm{Hom}(M, N)/\operatorname{rad}(M, N)$ for any indecomposable N. `Hint: Actually,` M `is the indecomposable module such that` $S(M) = \mathrm{Hom}(\tilde{M}, S) \neq$ `The nonzero element in` $\mathrm{Hom}(\tilde{M}, S)$ `must be surjective. Note that` $S(M)$ `has to be a` $\mathrm{End}(M)$`-module, the maximal choice of the kernel is` $\operatorname{rad}(M, M$ $\operatorname{rad}(M)$`.`

3.4. For an indecomposable module M, $\operatorname{rad}(M, L) = \bigoplus \operatorname{rad}(M, L_i)$ where $L = \bigoplus L_i$ the decomposition of indecomposable modules. Equivalently, it is given by the space of $M \to L$ which is not a split injection.

3.5. Almost split We say $M \xrightarrow{f} N$ is **left almost split** if

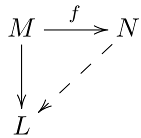
$$\tilde{N} \to \tilde{M} \to \tilde{M}/\operatorname{rad}\tilde{M} \to 0$$

is a resolution. That is, it is surjective for any L

$$\mathrm{Hom}(N, L) \twoheadrightarrow \operatorname{rad}(M, L).$$

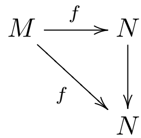
Equivalently,

Any map $M \xrightarrow{g} L$ which is not a split injection factor through N.



Moreover, we say f is **minimal** if the above resolution is minimal. It is equivalent to,

Any endmorphism of N commuting with f is invertible.

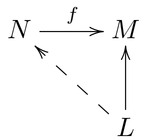


Hint: By a modification of the proof of Fitting lemma.

3.6. Combinatorially, assume $[M \xrightarrow{f} N] = \bigoplus [M \xrightarrow{f_i} N_i]$ with N_i all indecomposable (possibly with repeatition). The condition is equivalent to say, for any path $M \to L$ of length ≥ 1, it has to go through the sum of $M \xrightarrow{f_i} N_i$. So it is not difficult to see that when f_i corresponds to the arrow of AR quiver from M, f is left almost split and minimal.

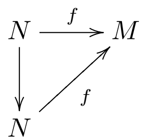
3.7. Duality Similarly, we can consider $\mathrm{Fun}^\vee(R)$ the category of additive contravariant functor from R-mod to k-mod. We can similarly define **Yoneda embedding** $M \mapsto \mathrm{Hom}_R(-, M)$, **radical** $\mathrm{rad}(-, M)$ the space of map to M which is not a split surjection, **right almost split**

Any map $L \to M$ which is not a split surjection factor through N.



and minimal

Any endmorphism of N commuting with f is invertible.



map. Then we see that when f_i corresponds to the arrow of AR quiver to M, f is right almost split and minimal.

3.8. Auslander–Reiten translation For $M \in R\text{-mod}$, define the **transpose** $\operatorname{Tr} M \in R^{\mathrm{op}}\text{-mod}$ by sequence

$$0 \to \operatorname{Hom}(M, R) \to \operatorname{Hom}(P_0, R) \to \operatorname{Hom}(P_1, R) \to \operatorname{Tr} M \to 0$$

where $P_1 \to P_0 \to M \to 0$ is the minimal resolution of M. Define the **Auslander–Reiten translation** $\tau M = D \operatorname{Tr} M \in R\text{-mod}$. It maps non-projective indecomposable modules to non-injective indecomposable modules.

3.9. Stable Hom For two modules $M, N \in R\text{-mod}$, define the **stable hom**

$$\underline{\operatorname{Hom}}_R(M, N) = \operatorname{Hom}_R(M, N) \Big/ \{M \to P \to N : \text{for some projective } P\}.$$

We have **Auslander–Reiten formula**

$$\underline{\operatorname{Hom}}_R(M, N) = \operatorname{Tor}(\operatorname{Tr} M, N) = D \operatorname{Ext}_{R^{\mathrm{op}}}(\operatorname{Tr} M, DN) = D \operatorname{Ext}_R(N, \tau M)$$

where $D = \operatorname{Hom}_k(-, k)$ the duality functor. `Hint: One can apply the universal coefficient theorem on the complex` $P_1 \to P_0$ `(spectral sequence).`

3.10. Denote $\underline{\mathrm{Fun}}(R)$ the subcategory of $\mathrm{Fun}(R)$ vanishing on projective modules. Denote $\underline{\mathrm{Fun}}^\vee(R)$ the subcategory of $\mathrm{Fun}^\vee(R)$ vanishing on projective modules. Then the Auslander–Reiten formula can be reformulated as

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{M \mapsto \underline{\operatorname{Hom}}_R(M,-)} & \underline{\mathrm{Fun}}(R) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle D} \\ R\text{-mod} & \xrightarrow[M \mapsto \operatorname{Ext}_R(-,M)]{} & \underline{\mathrm{Fun}}^\vee(R) \end{array}$$

Exercises

3.11. For a projective resolution $P \twoheadrightarrow N$, show that

$$\begin{aligned} \underline{\operatorname{Hom}}_R(M, N) &= \operatorname{cok}\Big[\operatorname{Hom}_R(M, P) \to \operatorname{Hom}_R(M, N)\Big] \\ &= \operatorname{cok}\Big[\operatorname{Hom}_R(M, R) \otimes_R N \to \operatorname{Hom}_R(M, N)\Big]. \end{aligned}$$

4 Auslander–Reiten Theory

4.1. Almost split sequence For a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

Then

$$\begin{matrix} f \text{ is left almost split and minimal} \\ \text{(in particular } M \text{ is indecomposable)} \end{matrix} \iff \begin{matrix} g \text{ is right almost split and minimal} \\ \text{(in particular } N \text{ is indecomposable)} \end{matrix}$$

$$\iff \begin{matrix} f \text{ is left almost split} \\ g \text{ is right almost split} \end{matrix}$$

In this case we say this sequence is **almost split**. `Hint: The trick is standard --- taking pull back to construct splitting/factorization etc.`

4.2. Auslander–Reiten Theorem When M is indecomposable but not projective,

$$0 \longrightarrow \tau M \longrightarrow L \longrightarrow M \longrightarrow 0$$

is an almost split sequence.

Proof Denote $\tilde{M} = \mathrm{Hom}(-, M)$, and $S_M = \tilde{M}/\operatorname{rad}\tilde{M}$. Note that $\tilde{M} \to S_M$ fact through $\underline{\mathrm{Hom}}(-, M)$. Then we get

$$DS_M \hookrightarrow D\,\underline{\mathrm{Hom}}(-, M) = \mathrm{Ext}_R(M, \tau -).$$

Take into M, the nonzero element $* \in DS_M(M)$ corresponds to an extension L. We will show it is an almost split sequence. For any $N \to M$ which is not a split surjection, that is, in $\operatorname{rad}(M, N)$,

$$\begin{array}{ccccccc} * & \in & DS_M(M) & \hookrightarrow & D\,\underline{\mathrm{Hom}}(M, M) & = & \mathrm{Ext}_R(M, \tau M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \in & DS_N(M) & \hookrightarrow & D\,\underline{\mathrm{Hom}}(M, N) & = & \mathrm{Ext}_R(N, \tau M) \end{array}$$

Thus the pull back splits, we see $N \to M$ factor through L. Similarly, we do it dually on τM. Q.E.D.

4.3. As a result, when M is indecomposable non-projective. in $\operatorname{Fun}^{\vee}(R)$, we have minimal resolution

$$0 \longrightarrow \widetilde{\tau M} \longrightarrow \tilde{L} \longrightarrow \tilde{M} \longrightarrow \tilde{M}/\operatorname{rad}\tilde{M} \longrightarrow 0.$$

In $\operatorname{Fun}(R)$, we have minimal resolution

$$0 \longrightarrow \widetilde{\tau^{-1}N} \longrightarrow \tilde{L} \longrightarrow \tilde{N} \longrightarrow \tilde{N}/\operatorname{rad}\tilde{N} \longrightarrow 0,$$

where $N = \tau M$ is indecomposable non-injective.

4.4. When P is indecomposable projective,

$$\operatorname{rad}(-,P) = \operatorname{Hom}(-,\operatorname{rad}P)$$

is representable, thus we have minimal resolution in $\operatorname{Fun}^{\vee}(R)$

$$0 \longrightarrow \widetilde{\operatorname{rad}P} \longrightarrow \tilde{P} \longrightarrow \tilde{P}/\operatorname{rad}\tilde{P} \longrightarrow 0.$$

Dually, when I is indecomposable injective, we have minimal resolution

$$0 \longrightarrow \widetilde{I/\operatorname{soc}I} \longrightarrow \tilde{I} \longrightarrow \tilde{I}/\operatorname{rad}\tilde{I} \longrightarrow 0.$$

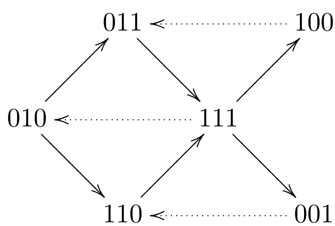
4.5. Consider the example

$$Q : \bullet \longrightarrow \bullet \longleftarrow \bullet$$

It has 6 indecomposable representations

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}, \quad \underbrace{\begin{bmatrix}0 & 1 & 0\end{bmatrix}}_{\text{projective}}, \quad \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \quad \underbrace{\begin{bmatrix}1 \overset{\text{id}}{\to} 1 & 0\end{bmatrix}}_{\text{projective}}, \quad \underbrace{\begin{bmatrix}0 & 1 \overset{\text{id}}{\leftarrow} 1\end{bmatrix}}_{\text{projective}}, \quad \begin{bmatrix}1 \overset{\text{id}}{\to} 1 \overset{\text{id}}{\leftarrow} 1\end{bmatrix}.$$

Then we can draw the AR quiver



4.6. Consider $k[x]/(x^3)$

$Q:$  $\quad x^3 = 0$

It has 3 indecomosable representations

$$1 : x = (0), \qquad 2 : x = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, \qquad 3 : x = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}.$$

The AR quiver

Exercises

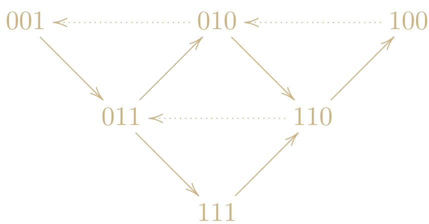
4.7. Draw the AR quiver of Q-rep with Q

$$Q : \bullet \longrightarrow \bullet \longrightarrow \bullet$$

`Hint: It has 6 indecomposable representations`

$$[1 \quad 0 \quad 0], \quad [0 \quad 1 \quad 0], \quad \underbrace{[0 \quad 0 \quad 1]}_{\texttt{projective}}, \quad [1 \xrightarrow{\text{id}} 1 \quad 0], \quad \underbrace{[0 \quad 1 \xrightarrow{\text{id}} 1]}_{\texttt{projective}}, \quad \underbrace{[1 \xrightarrow{\text{id}} 1 \xrightarrow{\text{id}} 1]}_{\texttt{projective}}.$$

`The the AR quiver`



5 Quiver Representations

5.1. Quiver Representations Assume Q is a quiver without oriented loop. For any $M \in Q\text{-rep}$, the radical is the sum of images of arrows, the socle is the intersection of kernel of arrows.

§ For each vertex i, denote $S(i)$ the one-dimensional representation supported over i, that is

$$S(i): \text{ the path at } i, \quad \text{i.e.} \quad S(i)_j = \delta_{ij} \cdot k.$$

Denote $P(i)$ the representation such that

$$P(i): \text{ paths from } i, \quad \text{i.e.} \quad P(i)_j = \bigoplus k(\text{path from } i \text{ to } j)$$

the arrows is the tautological one. Denote $I(i)$ the representation such that

$$I(i): \text{dual of paths to } i, \quad \text{i.e.} \quad I(i)_j^* = \bigoplus k(\text{path from } j \text{ to } i)$$

the arrows is the tautological one.

§ Then, the set of simple representation is $\{S(i)\}$. The projective cover of $S(i)$ is $P(i)$. More precisely, we have the following exact sequence

$$0 \longrightarrow \bigoplus_{\text{arrow } i \to j} P(j) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0.$$

The injective hull of $S(i)$ is $I(i)$. More precisely, we have the following exact sequence

$$0 \longrightarrow S(i) \longrightarrow I(i) \longrightarrow \bigoplus_{\text{arrow } j \to i} I(j) \longrightarrow 0.$$

In particular, the category $Q\text{-rep}$ has projective dimension 1 (for nontrivial Q).

§ We see that the quiver for $\text{proj}(Q\text{-rep})$ is Q^{op}.

5.2. Dimension vector Denote the vertices set of Q to be I. For a $V \in Q\text{-rep}$, define the **dimension vector** $\mathbf{dim}\, V = (\dim V_i)_{i \in I}$. Then this defines an isomorphism of Grothendieck group

$$G_0(Q\text{-rep}) \xrightarrow{\mathbf{dim}} \mathbb{Z}^I,$$

where

$$G_0(Q\text{-rep}) = \frac{\bigoplus_{V\in Q\text{-rep}} \mathbb{Z}\cdot[V]}{[V_2]=[V_1]+[V_3] : \begin{array}{c}\text{short exact sequence}\\ 0\to V_1\to V_2\to V_3\to 0\end{array}}$$

5.3. Euler form For representation V, W, define the **Euler form**

$$\langle V, W\rangle = \dim \mathrm{Hom}_{Q\text{-rep}}(V, W) - \dim \mathrm{Ext}_{Q\text{-rep}}(V, W).$$

Note that this is bi-additive, thus factor though $G_0(Q\text{-rep})$. Say, for $\mathbf{v}, \mathbf{w}$ two dimension vectors, define

$$\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{\text{vertex } i} v_i w_i - \sum_{\text{arrow } i\to j} v_i w_j$$

Then $\langle V, W\rangle = \langle \mathbf{dim}\, V, \mathbf{dim}\, W\rangle$. `Hint: Since we computed Ext of simple modules.`

5.4. Moduli of Q-rep Let $\mathbf{v} = (v_i)_{i\in I} \in \mathbb{N}^I$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Denote

$$E(\mathbf{v}) = \prod_{\text{arrow } i\to j} \mathrm{Hom}_k(k^{v_i}, k^{v_j}), \quad G(\mathbf{v}) = \prod_{\text{vertex } i} \mathrm{GL}(k^{v_i}).$$

The group $G(\mathbf{v})$ acts on $E(\mathbf{v})$ by conjugation. Then tautologically, we have

$$\left\{\begin{array}{c}\text{representations of}\\ \text{dimenison } \mathbf{v} \text{ in } Q\text{-rep}\end{array}\right\}/\cong \ \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c} G(\mathbf{v}) \text{ orbits}\\ \text{of } E(\mathbf{v})\end{array}\right\}$$

So the space $E(\mathbf{v})$ with the group action $G(\mathbf{v})$ is said to be the **moduli of Q-rep**. One can also think it as the quotient stack. Let us compute the dimension of $E(\mathbf{v})$ and $G(\mathbf{v})$.

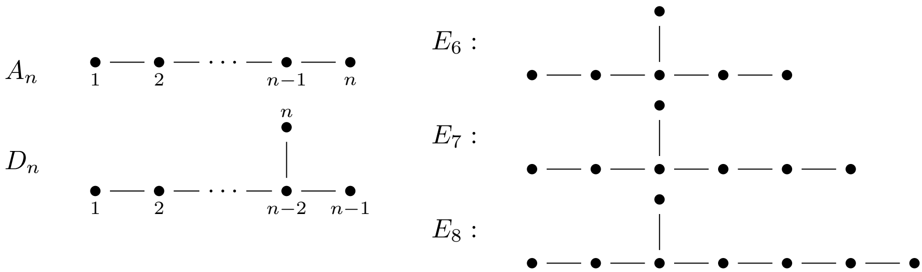
$$\dim E(\mathbf{v}) = \sum_{\text{arrow } i\to j} v_i v_j, \qquad \dim G(\mathbf{v}) = \sum_{\text{vertex } i} v_i^2.$$

In particular,

$$\dim E(\mathbf{v}) - \dim G(\mathbf{v}) = -\langle \mathbf{v}, \mathbf{v}\rangle.$$

5.5. We see that if the $G(\mathbf{v})$-orbits are finite, then we need to require that $\dim E(\mathbf{v}) - \dim G(\mathbf{v}) < 0$ (since the action of $k^\times \in G(\mathbf{v})$ is trivial, we need to quotient by $\mathbb{P}G(\mathbf{v})$). That is, the Euler form is positive-definite. This only happens when the underlying graph is a disjoint union of **simply-laced Dynkin diagrams**, say **ADE type**.

5.6. Simply-laced Dynkin Diagrams



5.7. Gabriel Theorem A quiver Q with finite many indecomposable representations if and only if the underlying graph is disjoint union of simply-laced Dynkin diagrams.

We have proved the "only if" part. The "if" part will be done in the next section.

Exercises

5.8. Show that

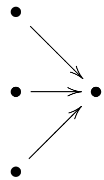
$$\mathrm{Hom}_Q(P(i), V) = V_i, \qquad \mathrm{Hom}_Q(S(i), V) = \ker \left[V_i \to \bigoplus_{\text{arrow } i \to j} V_j \right].$$

Dually,

$$D\,\mathrm{Hom}_Q(V, I(i)) = V_i, \qquad D\,\mathrm{Hom}_Q(V, S(i)) = \mathrm{cok} \left[\bigoplus_{\text{arrow } j \to i} V_j \to V_i \right].$$

6 Reflection Functors

6.1. Let us start from an example. Consider the quiver

$Q:$ 

By consider "three different lines passing through 0 in the plane",

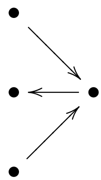
$$\begin{array}{ccc} & \ell_1 & \\ & \searrow & \\ \ell_2 & \longrightarrow & k^2 \\ & \nearrow & \\ & \ell_3 & \end{array}$$

we have a representation of dimension $\begin{smallmatrix} 1 \\ 1\,2 \\ 1 \end{smallmatrix}$. It is easy to see this representation is indecomposable.

If we take the cokernel of one arrow, say

$$\begin{array}{ccc} & \ell_1 & \\ & \searrow & \\ k^2/\ell_2 & \longleftarrow & k^2 \\ & \nearrow & \\ & \ell_3 & \end{array}$$

, we get an indecomposable representation for Q'.

$Q':$ 

6.2. For a quiver Q, and a **sink** i (no arrow from i). $S(i) = P(i)$ is projective. If $V \in Q\text{-rep}$ is indecomposable but not $S(i)$, then the map

$$\bigoplus_{\text{arrow } i \to j} V_j \longrightarrow V_i$$

is surjective. Dually, for a **sourse** i (no arrow to i), $S(i) = I(i)$ is injective. If $V \in Q\text{-rep}$ is indecomposable but not $S(i)$, then the map

$$V_i \longrightarrow \bigoplus_{\text{arrow } i \to j} V_j$$

is injective. `Hint: The kernel can be thought as a coply of` $S(i)$ `which is injective.`

6.3. In general, for a quiver Q and a vertex i, we define a new quiver S_iQ by reverse all arrows incident (to or from) i. Define the **reflection functor** when i is a sink

$$S_i^+ : \quad Q\text{-rep} \longrightarrow S_iQ\text{-rep}$$

sending V to S_i^+V by replacing the $V_i \to V_j$ by $V_j \to (S_i^+V)_i$ such that the following sequence

$$0 \longrightarrow (S_i^+V)_i \longrightarrow \bigoplus_{\text{arrow } i \to j} V_j \longrightarrow V_i$$

exact. In particular, $S_i^+S(i) = 0$.

Define the **reflection functor** when i is a source

$$S_i^- : \quad Q\text{-rep} \longrightarrow S_iQ\text{-rep}$$

sending V to S_i^-V by replacing the $V_j \to V_i$ by $(S_i^+V)_i \to V_j$ such that the following sequence

$$V_i \longrightarrow \bigoplus_{\text{arrow } i \to j} V_j \longrightarrow (S_i^-V)_i \longrightarrow 0$$

exact. In particular, $S_i^-S(i) = 0$.

6.4. From the discussion above, we see that for a sink (or source) i, the functor $S_i^{\mp}S_i^{\pm} : Q\text{-rep} \to Q\text{-rep}$ is natural isomorphic to the projection to the direct summand without $S(i)$ (this is a functor since $\mathrm{Hom}(-, S(i)) = 0$ for this summand).

6.5. Denote the quadratic form $B(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$ and $\alpha_i = \mathbf{e}_i$ the standard basis. Define $s_i\mathbf{v} = \mathbf{v} - \frac{2B(\mathbf{v},\alpha_i)}{B(\alpha_i,\alpha_i)}\alpha_i = \mathbf{v} - B(\mathbf{v}, \alpha_i)\alpha_i$. Note that

$$\mathbf{dim}\, S_i^{\pm}V = \max(s_i\, \mathbf{dim}\, V, \mathbf{0})$$

for indecomposable representation V.

6.6. For any simply-laced Dynkin diagram Q (actually, any quiver whose underlying graph is a tree), we can find an order $I = \{1, \ldots, n\}$ such that

$$Q_0\text{-rep} \xrightarrow{S_1^+} Q_1\text{-rep} \xrightarrow{S_2^+} \cdots \longrightarrow Q_n\text{-rep} \xrightarrow{S_n^+} Q_0\text{-rep}.$$

is well-defined (i.e. i is a sink for Q_i). We call the above functor the **Coxeter functor**. `Hint: Actually, we can assign certain height of each vertex such that all arrows go down, then reflection of a sink is to pick the lowest vertices up.`

6.7. The standard theory of Coxeter group tells us the group generated by s_i (Weyl group) has a Coxeter representataion

$$\begin{aligned} s_i^2 &= 1 && \text{for all vertex } i \\ s_i s_j s_i &= s_j s_i s_j && i \text{ --- } j \text{ exists an edge} \\ s_i s_j &= s_j s_i && i \quad\ \ j \text{ no edge} \end{aligned}$$

Denote the product of s_i in any order is said to be a **Coxeter element**. There is no nonzero element $\mathbb{R}^I_{\geq 0}$ fixed by Coxeter element c. Moreover, every nonzero element of $\mathbb{R}^I_{\geq 0}$ will be send out by some power of c. `Hint: Say` $c = s_n \cdots s_1$. `Then the first component of` $s_1 \mathbf{v}$ `will never be changed after the end, so it is zero, so` $v_1 = 0$ `and` $s_1\mathbf{v} = \mathbf{v}$. `The by induction we see` $\mathbf{v} = 0$. `For any` $\mathbf{v} \in \mathbb{R}^I_{\geq 0}$, `the element` $\mathbf{v} + c\mathbf{v} + \cdots c^{h-1}\mathbf{v}$ `is fixed by` c, `where` h `is the order of` c, `the Coxeter number.`

6.8. By the discussion above, every indecomposable module is killed by iterated Coxeter functor. That is, it is a simple representation in some Q_i-rep. By reflecting back, we see it has finite many indecomposable representations. This finishes the proof of Gabriel theorem.

6.9. We see that

$$\left\{ \begin{array}{c} \text{indecomposable} \\ \text{representations in } Q\text{-rep} \end{array} \right\} / \cong \xrightarrow{\mathbf{dim}} \left\{ \begin{array}{c} \text{positive roots} \\ \text{of the root system} \end{array} \right\}$$

is a bijection. `Hint: When it vanishes depends only on the dimension vector, thus it is injective. Any root can be reflected to a simple root, then we can just reflect back to get an indecomposable representat`

Exercises

6.10. Show that the Coxeter elements are all conjugate. `Hint:` $s_1 s_2 \cdots \sim s_2 s_1 s_2 \cdots s_2 = s_1 s_2 s_1 \cdots s_2 \sim s_1 \cdots s_2 s_1 s_2 = s_1 \cdots s_1 s_2 s_1 \sim s_2 s_1 \cdots$.

7 More Examples

7.1. **Type A** The type A quiver

$$A_n : \underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n-1}{\bullet} - \underset{n}{\bullet}$$

The indecomposable representations are given by connected subgraphs.

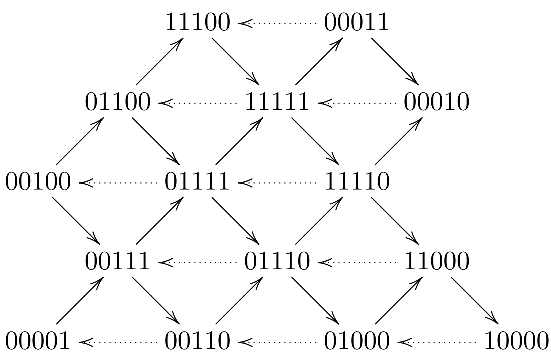
$$0 \cdots 0\, 1 \cdots 1\, 0 \cdots 0.$$

The maps between 1's are all identities.

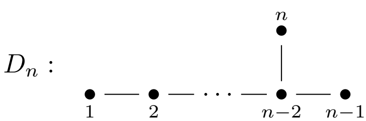
7.2. Consider

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

The AR quiver is



7.3. **Type D** The type D quiver



We have indecomposable representations given by connected subgraphs.

$$0\cdots01\cdots10\cdots\overset{0}{0}0, \quad 0\cdots01\cdots\cdots\overset{1}{1}0, \quad 0\cdots01\cdots\cdots\overset{0}{1}1, \quad 0\cdots01\cdots\cdots\overset{1}{1}1.$$

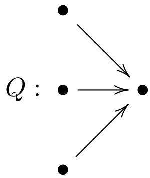
We also has "3-star representation"

$$0\cdots01\cdots12\cdots\overset{1}{2}1,$$

with identities between 2's and 1's respectively, and the three maps between 1 and 2 are given as D_4.

Consider three different lines passing through 0 in the plane. This gives the case of one of orientations (reflection functor). For other orientations, just replacing the inclusion by its cokernel.

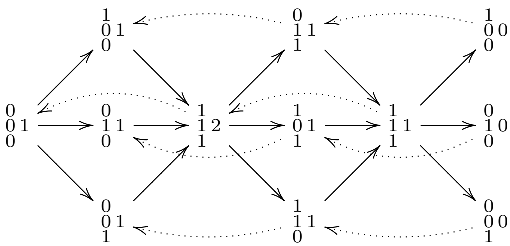
7.4. Let us consider



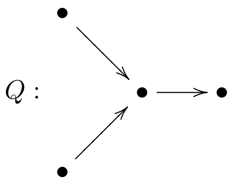
It has 12 indecomposable modules

$$\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix},\ \begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix},\ \begin{smallmatrix}0&\\0&0\\1&\end{smallmatrix},\ \underbrace{\begin{smallmatrix}0&\\0&1\\0&\end{smallmatrix}},\ \underbrace{\begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix}},\ \underbrace{\begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix}},\ \underbrace{\begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix}},\ \begin{smallmatrix}1&\\1&1\\0&\end{smallmatrix},\ \begin{smallmatrix}0&\\1&1\\1&\end{smallmatrix},\ \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix},\ \begin{smallmatrix}1&\\1&1\\1&\end{smallmatrix},\ \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}.$$

The representation $\begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}$ is given by three different lines going through 0 in a plane. The AR quiver



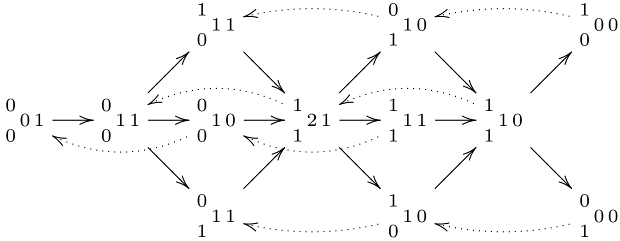
7.5. Let us consider



It has 12 indecomposable modules

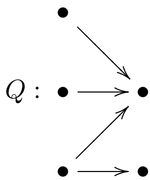
$$\begin{smallmatrix}1&&\\&0&0\\0&&\end{smallmatrix},\ \underbrace{\begin{smallmatrix}0&&\\&0&1\\0&&\end{smallmatrix}},\ \begin{smallmatrix}0&&\\&0&0\\1&&\end{smallmatrix},\ \begin{smallmatrix}0&&\\&1&0\\0&&\end{smallmatrix},\ \begin{smallmatrix}1&&\\&1&0\\0&&\end{smallmatrix},\ \underbrace{\begin{smallmatrix}0&&\\&1&1\\0&&\end{smallmatrix}},\ \begin{smallmatrix}0&&\\&1&0\\1&&\end{smallmatrix},\ \underbrace{\begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix}},\ \underbrace{\begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix}},\ \begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix},\ \begin{smallmatrix}1&&\\&1&1\\1&&\end{smallmatrix},\ \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}.$$

The AR quiver

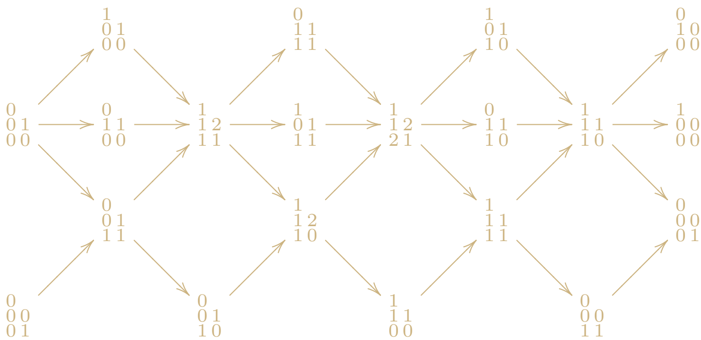


Exercises

7.6. Compute for the quiver



Hint:



The τ is omitted.

8 Tilting Modules

8.1. Morita Theory For a **projective generator** $P \in B\text{-mod}$ (any $M \in B\text{-mod}$ is a quotient of P^n). Denote $A = \mathrm{End}_B(P)^{\mathrm{op}}$. Then the adjoint functor gives

$$A\text{-mod} \underset{\mathrm{Hom}_B(P,-)}{\overset{P\otimes_A -}{\rightleftarrows}} B\text{-mod}$$

an equivalence. Since the natural transform

$$\mathrm{id} \longrightarrow \mathrm{Hom}_B(P, P \otimes_A -)$$

is an isomorphism on any finitely generated projective module, and both sides are right exact, thus it is an isomorphism; the natural transform

$$P \otimes_A \mathrm{Hom}_B(P, -) \longrightarrow \mathrm{id}$$

is an isomorphism on P, and both sides are right exact, thus it is an isomorphism.

8.2. Actually, if we denote $Q = \mathrm{Hom}_B(P, B)$. Then

$$Q \otimes_B - = \mathrm{Hom}_B(P, B) \otimes_B - = \mathrm{Hom}_B(P, -).$$

So $Q \otimes_B P \cong A$ and $P \otimes_A Q \cong B$.

8.3. General Tilting Module Now we want to do the same work on derived category. Assume the algebra R is of finite homological dimension. Denote $D(R)$ the derived category of bounded complexes over R. We say a module $M \in R\text{-mod}$ is a **(general) tilting module** if

$$\mathrm{Ext}^{\geq 1}(T, T) = 0$$

and the minimal triangulated category containing T is $D(R)$. Then the adjoint functor gives

$$D(S) \underset{\mathbf{R}\,\mathrm{Hom}_R(T,-)}{\overset{T\otimes^{\mathbf{L}}_S -}{\rightleftarrows}} D(R)$$

where $S = \mathrm{End}(T)^{\mathrm{op}}$.

8.4. **Tilting Module** We say a module $M \in R\text{-mod}$ is a **tilting module** if

$$\mathrm{Ext}^{\geq 2}(T,-) = 0, \qquad \mathrm{Ext}^1(T,T) = 0$$

and there is a short exact sequence

$$0 \longrightarrow R \longrightarrow T_1 \longrightarrow T_2 \longrightarrow 0$$

with $T_1, T_2 \in \mathrm{add}\, T$ the set of direct sums of summands of T.

Two stronger conditions ensure that T is of projective dimension ≤ 1 as R-mod and also as mod-S. `Hint: The functor` $\mathrm{Hom}_R(-,T)$ `is exact on` $0 \to R \to T_1 \to T_2 \to 0$`, so we get a resolution of` T `as` S`-module of length` ≤ 1.

8.5. Denote the T-torsion part and T-torsionfree part

$$\mathcal{T}(T) = \ker[\mathrm{Ext}^1_R(T,-)], \qquad \mathcal{F}(T) = \ker[\mathrm{Hom}_R(T,-)].$$

Denote

$$\mathcal{X}(T) = \ker[T \otimes_S -], \qquad \mathcal{Y}(T) = \ker[\mathrm{Tor}_S(T,-)].$$

The **Brenner and Butler theorem** tells us for any $M \in R\text{-mod}$, there is a functorial exact sequence

$$0 \longrightarrow \underbrace{T \otimes_S \mathrm{Hom}_R(T,M)}_{\in \mathcal{T}(T)} \longrightarrow M \longrightarrow \underbrace{\mathrm{Tor}_S(T, \mathrm{Ext}^1_R(T,M))}_{\in \mathcal{F}(T)} \longrightarrow 0$$

and for any $N \in S\text{-mod}$, there is a functorial exact sequence

$$0 \longrightarrow \underbrace{\mathrm{Ext}^1_R(T, \mathrm{Tor}_S(T,N))}_{\in \mathcal{X}(T)} \longrightarrow N \longrightarrow \underbrace{\mathrm{Hom}_R(T, T \otimes_S N)}_{\in \mathcal{Y}(T)} \longrightarrow 0$$

`Hint: This is a standard usage of Grothendieck spectral sequence (derived version).`

8.6. Moreover, we conclude that

$$\mathcal{T}(T) = \mathrm{im}[T \otimes_S -)], \qquad \mathcal{F}(T) = \mathrm{im}[\mathrm{Tor}_S(T,-)].$$
$$\mathcal{X}(T) = \mathrm{im}[\mathrm{Ext}^1_R(T,-)], \qquad \mathcal{Y}(T) = \mathrm{im}[\mathrm{Hom}_R(T,-)].$$

The functors

$$\mathcal{Y}(T) \underset{\mathrm{Hom}_R(T,-)}{\overset{T \otimes_S -}{\rightleftarrows}} \mathcal{T}(T), \qquad \mathcal{X}(T) \underset{\mathrm{Ext}^1_R(T,-)}{\overset{\mathrm{Tor}_S(T,-)}{\rightleftarrows}} \mathcal{F}(T)$$

are equivalences.

$$\mathrm{Hom}_R(\mathcal{T}(T), \mathcal{F}(T)) = 0, \qquad \mathrm{Hom}_S(\mathcal{X}(T), \mathcal{Y}(T)) = 0.$$

Exercises

8.7. Let $\operatorname{Gen} T$ be the category of modules can be written as a quotient of T. Let T be a tilting module. Show that $\operatorname{Gen} T = \mathcal{T}(T)$. Hint: Show that $\operatorname{im}[T \otimes_S -] \subseteq \operatorname{Gen} T \subseteq \ker[\operatorname{Ext}^1_R(T, -)]$

8.8. Denote $\overline{\operatorname{Hom}}_R(M, N) = \operatorname{Hom}_R(M, N) / \{M \to I \to N : \text{from some injective } I\}$. Show the dual **Auslander–Reiten formula**

$$\overline{\operatorname{Hom}}_R(M, N) = D \operatorname{Ext}_R(\tau^{-1} N, M).$$

Or

$$\operatorname{Ext}_R(M, N) = D \underline{\operatorname{Hom}}_R(\tau^{-1} N, M) = D\overline{\operatorname{Hom}}_R(N, \tau M).$$

8.9. Assume $\operatorname{Ext}^{\geq 2}(-, \tau M) = 0$, then

$$\operatorname{Ext}^1(N, \tau M) = D \operatorname{Hom}(M, N).$$

Dually, if $\operatorname{Ext}^{\geq 2}(N, -) = 0$, then

$$\operatorname{Ext}^1(N, M) = D \operatorname{Hom}(M, \tau N).$$

Hint: The condition tells $\operatorname{Hom}_R(M, R) = 0$. Thus $\underline{\operatorname{Hom}}(M, N) = \operatorname{Hom}(M, N)$

9 Reflection Functors Again

9.1. Let Q be a quiver. Let i be an vertex. Let us compute $\tau S(i)$. The following resolution is minimal

$$0 \longrightarrow \bigoplus_{\text{arrow } i\to j} P(j) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0.$$

Thus by a careful computation, we get

$$0 \longrightarrow \tau S(i) \longrightarrow \bigoplus_{\text{arrow } i\to j} I(j) \longrightarrow I(i) \longrightarrow 0.$$

When i is a sink, $S(i) = I(i)$, the above sequence is an almost split sequence. Dually,

$$0 \longrightarrow P(i) \longrightarrow \bigoplus_{\text{arrow } j\to i} P(j) \longrightarrow \tau^{-1}S(i) \longrightarrow 0 \qquad (*)$$

When i is a source, $S(i) = P(i)$, the above sequence is an almost split sequence.

9.2. Let i be a sink. Let $V \in Q\text{-}\mathsf{rep}$. We see $\mathrm{Hom}_{Q\text{-}\mathsf{rep}}(P(i), V) = V_i$, thus

$$\mathrm{Hom}_{Q\text{-}\mathsf{rep}}(\tau^{-1}S(i), V) = \ker\left[\bigoplus_{\text{arrow } j\to i} V_j \longrightarrow V_i\right].$$

Consider

$$T = \tau^{-1}S(i) \oplus \bigoplus_{j\neq i} P(j).$$

Then we see directly from the AR quiver that $\mathrm{End}_R(T)^{\mathrm{op}} = S_iQ$. Moreover,

$$\mathrm{Hom}_R(T, V) = S_i^+ V.$$

We will show T is a tiling module. It is extension-free by Auslander–Reiten formula

$$\mathrm{Ext}^1(\tau^{-1}S(i), P(j)) = D\overline{\mathrm{Hom}}(P(j), S(i)) = 0.$$

The sequence $(*)$ shows the last condition of definition of tilting module. Moreover, for $V \in S_iQ\text{-}\mathsf{rep}$, one can check that

$$T \otimes_{S_iQ} V = S_i^- V.$$

9.3. Assume i is a sink, then the functorial exact sequence is

$$0 \longrightarrow S_i^- S_i^+ V \longrightarrow V \longrightarrow V/S_i^- S_i^+ V \longrightarrow 0$$

Note that $V/S_i^- S_i^+ V$ is a copy of $S(i) = P(i)$ thus above sequence splits.

9.4. Then one can consider the derived version of reflection functor $\mathbf{R}\,\mathrm{Hom}_R(T, -) = \mathbf{R}S_i^+$. Actually, it send $S(i)$ to $\mathrm{Ext}^1(T, S(i)) = \mathrm{Ext}^1(\tau^{-1}S(i), S(i)) = S(i)[-1]$ (from the $(*)$ above).

For a complex in derived category of Q-rep, define

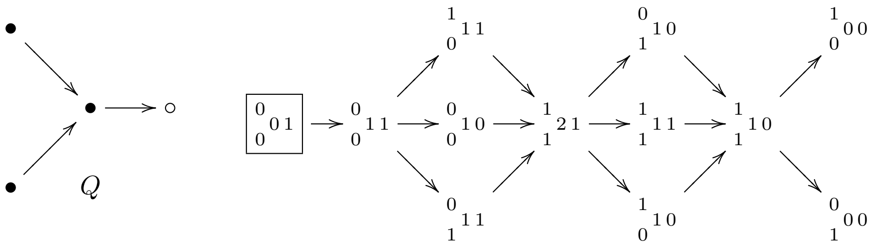
$$\mathbf{dim}\, V^\bullet = \sum (-1)^i\, \mathbf{dim}\, V^i \in \mathbb{Z}^I.$$

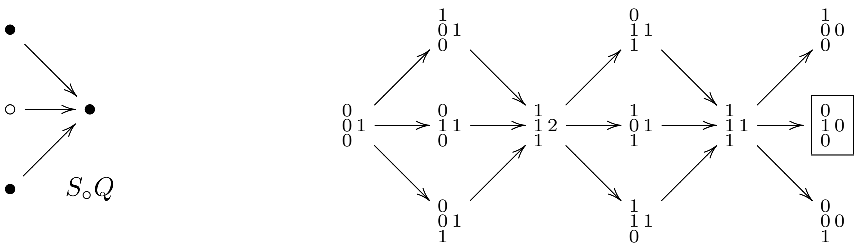
Then

$$\mathbf{dim}\, \mathbf{R}S_i^+(V^\bullet) = s_i\, \mathbf{dim}\, V^\bullet.$$

Similar result holds for $T \otimes^{\mathbf{L}}_{S_iQ} - = \mathbf{L}S_i^-$.

9.5. Consider the example.





Note that

$$\left\{ \begin{array}{c} \text{representations of} \\ s_\circ Q \text{ in box} \end{array} \right\} \underset{\mathrm{Ext}_Q(T,-)}{\overset{\mathrm{Tor}_{S_\circ Q}(T,-)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{representations of} \\ Q \text{ in box} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{representations of} \\ s_\circ Q \text{ out of box} \end{array} \right\} \underset{\operatorname{Hom}_Q(T,-)=S_\circ^+}{\overset{T\otimes_{s_\circ Q}-=S_\circ^-}{\rightleftarrows}} \left\{ \begin{array}{c} \text{representations of} \\ Q \text{ out of box} \end{array} \right\}$$

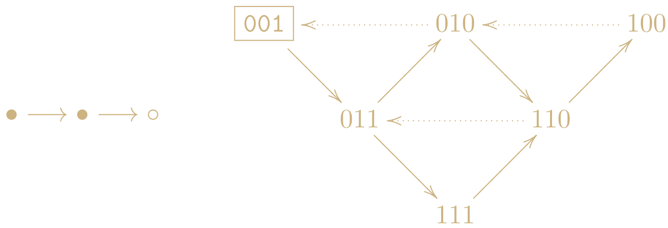
are both equivalences.

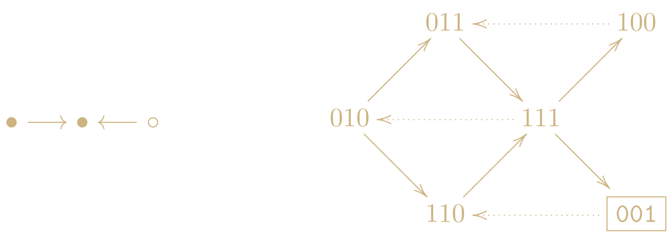
Exercises

9.6. Analyse

$$\bullet \longrightarrow \bullet \longrightarrow \circ \qquad \xrightarrow{S_\circ} \qquad \bullet \longrightarrow \bullet \longleftarrow \circ$$

`Hint:`





10 Hall–Ringel Algebras

10.1. Let $\mathbf{q}$ be a power of a prime number, and $q = \mathbf{q}^{1/2}$ a square root. Assume $k = \mathbb{F}_{q^2}$ the finite field. We are going to do counting over k-rep. This relates the Lie algebra corresponding to the Dynkin diagrams.

10.2. Denote

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]!_q = [1]_q \cdots [n]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

Note that

$$q^{-n} \cdot \#\mathbb{P}^n = [n+1]_q, \qquad q^{-n(n-1)/2} \cdot \# \mathcal{F}\ell(n) = [n]!_q,, \qquad q^{-k(n-k)} \cdot \# \mathcal{G}r(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

10.3. Denote the **Hall algebra of Q**

$$\mathcal{H}(Q) = \bigoplus_{V \in Q\text{-rep}/\cong} \mathbb{Z} \cdot [V]$$

the formal direct sum of isomorphic classes of Q-rep. It is equipped with the **Hall convolution**

$$[U] * [V] = \sum \# \left\{ W' \subseteq W : \begin{array}{c} W' \cong V \\ W/W' \cong U \end{array} \right\} \cdot [W].$$

Denote the **virtual Hall convolution**

$$[U] \cdot [V] = q^{\langle U, V \rangle} [U] * [V].$$

It is not difficult to see $(\mathcal{H}(Q), *)$ or $(\mathcal{H}(Q), \cdot)$ forms an associative algebra graded by **dim**.

10.4. We are going to compute the case

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}.$$

Denote

$$\theta_1 = S(1) = [1 \quad 0], \qquad \theta_2 = S(2) = P(2) = [0 \quad 1].$$

§ Consider the degree $\mathbf{dim} = 2\,0$. The only module is $[2 \quad 0]$. Thus $\theta_1 * \theta_1 = \#\mathbb{P}^1 \cdot [2 \quad 0]$. As a result,

$$\theta_1^2 = q^1 \cdot \#\mathbb{P}^1 \cdot [2 \quad 0] = q^2 [2]_q \cdot [2 \quad 0].$$

§ Similarly, for the degree **dim** $= 0\,2$,

$$\theta_2^2 = q^2[2]_q \cdot [0 \quad 2].$$

§ Consider the degree **dim** $= 1\,1$. There are two candidates $P(1) = [1 \xrightarrow{\text{id}} 1]$ and $S(1) \oplus S(2) = [1 \quad 1]$.

	$[1 \to 1]$	$[1 \quad 1]$
$S(2) \subseteq ?$	unique	unique
$S(1) \subseteq ?$	never	unique

Thus

$$\theta_1 \cdot \theta_2 = q^{-1}[1 \to 1] + q^{-1}[1 \quad 1], \qquad \theta_2 \cdot \theta_1 = [1 \quad 1].$$

Note that $\langle \theta_1, \theta_2 \rangle = -1$, and $\langle \theta_2, \theta_1 \rangle = 0$.

§ Consider the degree **dim** $= 2\,1$. There are two candidates $S(1) \oplus P(1) = [2 \to 1]$ and $S(1)^{\oplus 2} \oplus S(2) = [2 \quad 1]$.

	$[2 \to 1]$	$[2 \quad 1]$
$S(2) \subseteq ?$	unique	unique
$S(1) \subseteq ?$	unique, the kernel	$\mathbb{P}^1$-many choices

Thus

$$\begin{aligned} [2 \quad 0] \cdot \theta_2 &= q^{-2} \cdot [2 \to 1] + q^{-2} \cdot [2 \quad 1] \\ [1 \to 1] \cdot \theta_1 &= \quad q \cdot [2 \to 1] \\ [1 \quad 1] \cdot \theta_1 &= \qquad\qquad q^2 \cdot [2]_q \cdot [2 \quad 1] \end{aligned}$$

In particular,

$$\begin{aligned} \theta_1^2\theta_2 &= [2]_q \cdot [2 \to 1] + [2]_q \cdot [2 \quad 1] \\ \theta_1\theta_2\theta_1 &= \quad [2 \to 1] + q[2]_q \cdot [2 \quad 1] \\ \theta_2\theta_1^2 &= \qquad q^2[2]_q \cdot [2 \quad 1] \end{aligned}$$

So we get the **Serre relation**

$$\theta_1^2\theta_2 - [2]_q\theta_1\theta_2\theta_1 + \theta_2\theta_1^2 = 0.$$

§ Consider the degree **dim** $= 1\,2$. There are two candidates $S(2) \oplus P(1) = [1 \to 2]$ and $S(1) \oplus S(2)^{\oplus 2} = [1 \quad 2]$.

	$[1 \to 2]$	$[1 \quad 2]$
$S(2) \subseteq ?$	$\mathbb{P}^1$-many choices *	$\mathbb{P}^1$-many choices
$S(1) \subseteq ?$	never	unique

*: the quotient is $[1 \quad 1]$ if it coincides with image, and $[1 \to 1]$ otherwise.

Thus

$$\begin{aligned}
[1 \to 1] \cdot \theta_2 &= q^2 \cdot [1 \to 2] \\
[1 \quad 1] \cdot \theta_2 &= \ 1 \cdot [1 \to 2] + q[2]_q \cdot [1 \quad 2] \\
[0 \quad 2] \cdot \theta_1 &= \qquad\qquad\qquad\qquad [1 \quad 2]
\end{aligned}$$

In particular,

$$\begin{aligned}
\theta_1 \theta_2^2 &= (q + q^{-1})[1 \to 2] + \ [2]_q \cdot [1 \quad 2] \\
\theta_2 \theta_1 \theta_2 &= \qquad\qquad [1 \to 2] + q[2]_q \cdot [1 \quad 2] \\
\theta_2^2 \theta_1 &= \qquad\qquad\qquad\qquad q^2 [2]_q \cdot [1 \quad 2]
\end{aligned}$$

So we get the **Serre relation**

$$\theta_1^2 \theta_2 - [2]_q \theta_1 \theta_2 \theta_1 + \theta_2 \theta_1^2 = 0.$$

10.5. Assume the underlying graph of Q is a simply-laced Dynkin diagram. The computation is local thus Serre relation also holds. Thus we get the **Ringel map**

$$U_v(\mathfrak{g})^+ \big|_{v=q} \longrightarrow \mathcal{H}(Q)$$

where $U_q(\mathfrak{g})^+$ is the negative part of quantum group of the corresponding Lie algebra. This is an isomorphism by PBW basis theorem for $U_v(\mathfrak{g})^+$ and the classification of representations of Q.

Exercises

10.6. Show that for the quiver $\bullet \longrightarrow \bullet$

$$[m \quad 0] * [0 \quad n] = [m \quad n], \qquad [m \quad 0] \cdot [0 \quad n] = q^{mn} [m \quad n].$$

10.7. Show that for the quiver $\bullet \longrightarrow \bullet$

$$[m \quad 0] * [n \quad 0] = q^{mn} \begin{bmatrix} m+n \\ m \end{bmatrix} [m+n \quad 0], \qquad [m \quad 0] \cdot [n \quad 0] = q^{2mn} \begin{bmatrix} m+n \\ m \end{bmatrix} [m+n \quad 0].$$

In particular,

$$\theta_1^n = q^{n(n-1)} [n]!_q \cdot [n \quad 0].$$