# NOTES ON QUANTUM COHOMOLOGY

RUI XIONG

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# 1. QUANTUM PRODUCT

## 1.1. **The moduli space of stable maps.**

1.1.1. *Stable maps.* A quasi-stable curve with $n$-marked point is

$$(C, p_1, \dots, p_n)$$

where $C$ is a projective, connected, reduced, (at worst) nodal curve of arithmetic genus $0$, $p_1, \dots, p_n \in C$ are distinct regular points on $C$. We call

$$\{\text{special points}\} = \{\text{marked points}\} \cup \{\text{nodal points}\}.$$

For a variety $X$, $\beta \in \mathrm{Eff}(X)$, we define the moduli space of stable maps

$$\overline{\mathcal{M}}_n(X, \beta) = \left\{ (f, C, p_1, \dots, p_n) : \begin{array}{c} (C, p_1, \dots, p_n) \text{ is quasi-stable} \\ f : C \to X \text{ with } f_*[C] = \beta, \\ \text{and the stability condition} \end{array} \right\} /\text{re-parametrization.}$$

Here the stability condition is

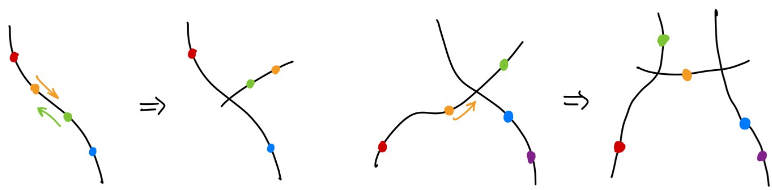
$$\boxed{\begin{array}{c} \text{If } f \text{ is constant over an irreducible component of } C, \\ \text{then there must be at least 3 special points on it.} \end{array}}$$

Equivalently, the automorphism group $\mathrm{Aut}(f, C, p_1, \dots, p_n)$ is finite. We denote

$$\overline{\mathcal{M}}_n(X) = \bigcup_{\beta} \overline{\mathcal{M}}_n(X, \beta), \qquad \overline{\mathcal{M}}_n = \overline{\mathcal{M}}_n(\mathrm{pt}).$$

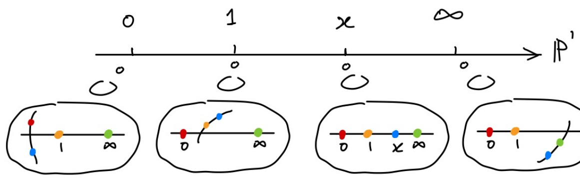
1.1.2. *Compactification.* It turns out $\overline{\mathcal{M}}_n(X, \beta)$ is a compactification of

$$\left\{ (f, \mathbb{P}^1, p_1, \dots, p_n) : \begin{array}{c} f : \mathbb{P}^1 \to X \text{ with } f_*[\mathbb{P}^1] = \beta \\ p_1, \dots, p_n \in \mathbb{P}^1 \text{ distinct} \end{array} \right\} /\text{re-parametrization.}$$



When $n = 3$, as any three points can be moved to $(0, 1, \infty)$ by a re-parametrization $\mathrm{Aut}(\mathbb{P}^1)$, the moduli space $\overline{\mathcal{M}}_3(X)$ is a compactification of $\mathrm{Mor}(\mathbb{P}^1, X)$.

1.1.3. *Example.* We have
$$\overline{\mathcal{M}}_3 = \mathrm{pt}, \qquad \overline{\mathcal{M}}_4 = \mathbb{P}^1.$$




1.1.4. *Example.* We have
$$\overline{\mathcal{M}}_3(\mathbb{P}^1, 1) = \mathrm{pt}, \qquad \overline{\mathcal{M}}_3(X, 0) = X.$$

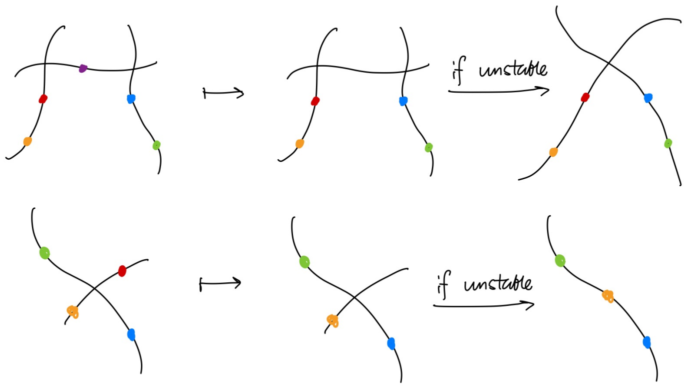
1.1.5. *Expected dimension.* At the point $(f, C, p_1, \dots, p_n)$, the tangenet space is the difference of the following
$$\begin{aligned}
(\text{deforming } f) &= \text{tangent fields of } X \text{ along } C \\
&= H^0(C, f^*\mathcal{T}_X). \\
(\text{infinitesimal automorphisms}) &= (\text{infinitesimal reparametrization}) \\
&= \text{tangent fields of } C \text{ vanishing at } p_1, \dots, p_n \\
&= H^0(C, \mathcal{T}_C(-p_1 - \cdots - p_n)) \\
&= \mathrm{Ext}^0(\omega_C(p_1 + \cdots + p_n), \mathcal{O}_C).
\end{aligned}$$
By Riemann–Roch
$$\chi(C, f^*\mathcal{T}_X) = \dim X + \langle \beta, c_1(\mathcal{T}_X)\rangle$$
$$\chi(C, \mathcal{T}_C(-p_1 - \cdots - p_n)) = -n + 3.$$
So the expected dimension of $\overline{\mathcal{M}}_n(X, \beta)$ is
$$\dim X + \langle \beta, c_1(\mathcal{T}_X)\rangle + n - 3.$$

## 1.2. **Gromov–Witten invariants.**

1.2.1. *Morphisms.* We have a morphism called *evaluation*
$$\mathrm{ev} : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow X \times \cdots \times X : \qquad (f, C, p_1, \dots, p_n) \longmapsto (f(p_1), \dots, f(p_n)).$$
We denote $\mathrm{ev}_i$ the $i$-th component. We have a forgetful morphism $\mathrm{ft}$
$$\mathrm{ft}_i : \overline{\mathcal{M}}_{n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(X, \beta)$$
by forgetting the $i$-th marked point and collapsing branches if necessary to get a stable map. Note that this map is not defined for $\beta = 0$ and $n = 2$, as $\overline{\mathcal{M}}_2(X, 0) = \varnothing$. Similarly for $f : X \to Y$, we have
$$f_* : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(Y, f_*\beta).$$

In particular, we have

$$\mathrm{ft}_X : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow \overline{\mathcal{M}}_n.$$




1.2.2. *Gromov–Witten invariants.* For $\gamma_1, \gamma_2, \gamma_3 \in H^*(X)$, we define

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta := \int_{\overline{\mathcal{M}}_n(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3).$$

Note that $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta = 0$ unless

$$(\deg \gamma_1 + \deg \gamma_2 + \deg \gamma_3) = \dim X + \langle \beta, c_1(\mathcal{T}_X) \rangle.$$

Here $\deg \gamma = k$ if $\gamma \in H^{2k}(X)$.

1.2.3. *Meaning.* Assume $\gamma_i = [Z_i]$ for subvariety $Z_i \subset X$. Then the meaning of Gromov–Witten invariant can be understood as

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta = \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{array}{c} f_*[\mathbb{P}^1] = \beta, \ f(0) \in Z_1, \\ f(1) \in Z_2, \ f(\infty) \in Z_3 \end{array} \right\}.$$

Note that now

$$\text{reparametrization} = \mathrm{Aut}(\mathbb{P}^1, 0, 1, \infty) = \text{trivial group}.$$

1.2.4. *Novikov Ring.* Denote *Novikov ring*

$$\mathbb{Q}[\![\mathrm{Eff}(X)]\!] = \mathbb{Q}[\![q^\beta]\!]_{\beta \in \mathrm{Eff}(X)} / \langle q^0 = 1, \ q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2} \rangle.$$

We will equip the degree

$$\deg q^\beta = \langle \beta, c_1(\mathcal{T}) \rangle.$$

## 1.3. **Quantum cohomology.**

1.3.1. *Quantum cohomology.* We define

$$QH^*(X) = H^*(X, \mathbb{Q})[\![\mathrm{Eff}(X)]\!]$$

with the quantum product $*$ uniquely determined by

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \sum_{\beta \in \mathrm{Eff}(X)} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta,$$

where $\langle\ ,\ \rangle$ is the Poincaré pairing. As $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_0 = \langle \gamma_1\gamma_2, \gamma_3 \rangle$, quantum product is a q-deformation of classical product

$$\gamma_1 * \gamma_2 = \gamma_1\gamma_2 + (\mathsf{quantum\ correction})$$

with

$$(\mathsf{quantum\ correction}) \in \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta H^*(X)$$

which tends to $0$ under the limit $\lim_{q \to 0} : H^*(X, \mathbb{Q})[\![\mathrm{Eff}(X)]\!] \to H^*(X, \mathbb{Q})$.

1.3.2. *Commutativity.* Note that this expression is symmetric under any permutation of $\gamma_1, \gamma_2, \gamma_3$, so quantum product is commutative

$$\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$$

and satisfies the *Frobenius property*

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \langle \gamma, \gamma_2 * \gamma_3 \rangle.$$

1.3.3. *Associativity.* Let us consider

$$\mathrm{ft}_X : \overline{\mathcal{M}}_4(X) \longrightarrow \overline{\mathcal{M}}_4 = \mathbb{P}^1.$$

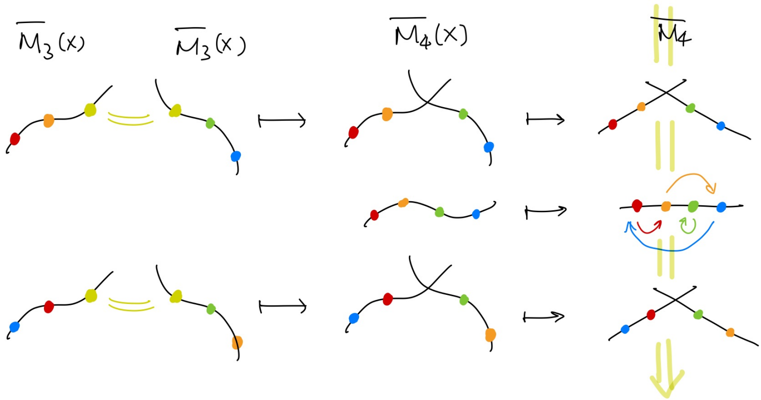
For the nodal curve $C$ on $\overline{\mathcal{M}}_4$, we have

$$\mathrm{ft}_X^{-1}(\{C\}) = \bigcup_{\beta_1 + \beta_2 = \beta} \overline{\mathcal{M}}_3(X, \beta_1) \times_X \overline{\mathcal{M}}_3(X, \beta_2).$$

Here

$$\begin{tikzcd}
\overline{\mathcal{M}}_3(X, \beta_1) \times_X \overline{\mathcal{M}}_3(X, \beta_2) \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\text{fibre product}"] & \overline{\mathcal{M}}_3(X, \beta_1) \arrow[d, "\mathrm{ev}_3"] & \overline{\mathcal{M}}_4(X, \beta) \\
\overline{\mathcal{M}}_3(X, \beta_2) \arrow[r, "\mathrm{ev}_3"] & X. &
\end{tikzcd}$$

The map is given by gluing the last marked points.




Let us compute

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_4(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3 \boxtimes \gamma_4)\, \mathrm{ft}^*([\mathrm{pt}]) \\
&= \sum_{\beta_1+\beta_2=\beta} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times_X \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes \mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1 \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes 1) \\
&= \sum_{\beta_1+\beta_2=\beta} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes \mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1 \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes 1)(\mathrm{ev}_3 \boxtimes \mathrm{ev}_3)^*(\Delta_X) \\
&= \sum_{\beta_1+\beta_2=\beta} \sum_w \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes \mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \sigma_w \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes \sigma^w) \\
&= \sum_{\beta_1+\beta_2=\beta} \sum_w \langle \gamma_1, \gamma_2, \sigma_w \rangle_{\beta_1} \langle \gamma_3, \gamma_4, \sigma^w \rangle_{\beta_2},
\end{aligned}$$

where $\{\sigma_w\} \subset \mathrm{H}^*(X)$ is a basis and $\{\sigma^w\}$ is its dual basis under Poincaré duality. Note that

$$\Delta_X = \sum_w \sigma_w \otimes \sigma^w \in= \mathrm{H}^*(X) \otimes \mathrm{H}^*(X) = \mathrm{H}^*(X \times X).$$

As a result, we have

$$\begin{aligned}
&\sum_{\beta\in \mathrm{Eff}(X)} q^{\beta}\int_{\overline{\mathcal{M}}_4(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3\boxtimes\gamma_4)\,\mathrm{ft}^*([\mathrm{pt}])\\
&=\sum_{\beta_1+\beta_2}\sum_{w} q^{\beta_1}\langle\gamma_1,\gamma_2,\sigma_w\rangle_{\beta_1} q^{\beta_2}\langle\gamma_3,\gamma_4,\sigma^w\rangle_{\beta_2}\\
&=\sum_{w}\langle\gamma_1*\gamma_2,\sigma_w\rangle\langle\gamma_3*\gamma_4,\sigma^w\rangle\\
&=\langle\gamma_1*\gamma_2,\gamma_3*\gamma_4\rangle=\langle(\gamma_1*\gamma_2)*\gamma_3,\gamma_4\rangle.
\end{aligned}$$

Note that this is invariant under any permutation of $\gamma_1,\gamma_2,\gamma_3,\gamma_4$. In particular, we have associativity

$$(\gamma_1*\gamma_2)*\gamma_3=(\gamma_2*\gamma_3)*\gamma_1=\gamma_1*(\gamma_2*\gamma_3).$$

1.3.4. *Remark.* When $\gamma_i=[Z_i]$ for subvariety $Z_i\subset X$. This also tells

$$\langle\gamma_1*\gamma_2,\gamma_3*\gamma_4\rangle=\sum_{\beta} q^{\beta}\#\left\{\mathbb{P}^1\xrightarrow{f}X : f_*[\mathbb{P}^1]=\beta, f(c_i)\in Z_i\right\}$$

for any given four points $c_1,\dots,c_4\in\mathbb{P}^1$.

1.3.5. *Identity.* Let $\beta>0$. Let us consider

$$\mathrm{ft}_3:\overline{\mathcal{M}}_3(X,\beta)\longrightarrow\overline{\mathcal{M}}_2(X,\beta).$$

Then

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)}\mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes 1)\\
&=\int_{\overline{\mathcal{M}}_3(X,\beta)}\mathrm{ft}_3^*(\mathrm{ev}^*(\gamma_1\boxtimes\gamma_2))\\
&=\int_{\overline{\mathcal{M}}_2(X,\beta)}\mathrm{ev}^*(\gamma_1\boxtimes\gamma_2)\,\mathrm{ft}_{3*}(1)=0
\end{aligned}$$

Here $\mathrm{ft}_{3*}(1)=0$ by degree reason. When $\beta=0$,

$$\int_{\overline{\mathcal{M}}_3(X,0)}\mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes 1)=\int_X\gamma_1\gamma_2=\langle\gamma_1,\gamma_2\rangle.$$

This proves

$$\langle\gamma_1*1,\gamma_2\rangle=\langle\gamma_1,\gamma_2\rangle.$$

So $1\in H^*(X)\subset QH^*(X)$ is the identity

$$\gamma_1*1=\gamma_1.$$

## 2. Properties and Examples

### 2.1. **Divisor equation.**

2.1.1. *Divisor.* Let $\lambda$ be a divisor. When $\beta > 0$,

$$\begin{aligned}\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \lambda) &= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ft}_3^*(\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2))\,\mathrm{ev}_3^*(\lambda) \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)\,\mathrm{ft}_{3*}(\mathrm{ev}_3^*(\lambda)).\end{aligned}$$

By degree reason, $\mathrm{ft}_{3*}(\mathrm{ev}_3^*(\lambda))$ is a number. So it equals to the intersecting number of the generic fibre and $\mathrm{ev}_3 *(\lambda)$. For a generic stable map $(f, \mathbb{P}^1, p_1, p_2)$, the fibre along $\mathrm{ft}_3$ is $\mathbb{P}^1$ itself, and $\mathrm{ev}_3$ is identified with $f$. So the intersecting number is $\langle \beta, \lambda \rangle$. We conclude that

$$\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \lambda) = \langle \lambda, \beta \rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2).$$

In other word,

$$\langle \gamma_1 * \lambda, \gamma_2 \rangle = \langle \gamma_1 \lambda, \gamma_2 \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} \langle \lambda, \beta \rangle q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2).$$

2.2. **Remark.** This can be understood as follows. Assume $\lambda = [D]$ for a codimension 1 subvariety $D \subset X$.

$$\langle \gamma_1, \gamma_2, \lambda \rangle_\beta = \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{array}{c} f_*[\mathbb{P}^1] = \beta,\ f(0) \in Z_1, \\ f(1) \in D,\ f(\infty) \in Z_2 \end{array} \right\}.$$

Note $D$ intersects any $\mathbb{P}^1 \to X$ by $\langle \beta, \lambda \rangle$ points. Thus

$$\langle \gamma_1, \gamma_2, \lambda \rangle_\beta = \langle \beta, \lambda \rangle \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{array}{c} f_*[\mathbb{P}^1] = \beta, \\ f(0) \in Z_1,\ f(\infty) \in Z_2 \end{array} \right\} / \mathbb{C}^\times.$$

Note that now

$$\mathsf{reparametrization} = \mathrm{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{C}^\times.$$

### 2.3. **Product.**

2.3.1. *Product.* Let $X$ and $Y$ be two varieties. We have a birational

$$\overline{\mathcal{M}}_3(X \times Y, (\beta, \beta')) \longrightarrow \overline{\mathcal{M}}_3(X, \beta) \times \overline{\mathcal{M}}_3(Y, \beta')$$

induced by two projections. Note that, this is birational only for $n = 3$ in which case $\overline{\mathcal{M}}_3(X)$ is a compactification of $\mathrm{Mor}(\mathbb{P}^1, X)$. We can conclude

$$QH^*(X \times Y) \longrightarrow QH^*(X) \otimes QH^*(Y)$$

is an algebra isomorphism.

2.3.2. *Corollory.* When $\beta_1, \beta_2 > 0$

$$\int_{\overline{\mathcal{M}}_2(X\times Y,(\beta_1,\beta_2))} \mathrm{ev}^* \big((\gamma_1 \otimes \gamma_1') \boxtimes (\gamma_2 \otimes \gamma_2')\big) = 0.$$

This can be proved using divisor equation. For any ample divisor $\lambda \in H^2(X)$,

$$\begin{aligned}
&\langle(\gamma_1 \otimes \gamma_1') * (\lambda \otimes 1), \gamma_2 \otimes \gamma_2'\rangle \\
&= \langle \gamma_1\lambda, \gamma_1'\rangle + \sum_{\beta_1,\beta_2} \langle \lambda, \beta_1\rangle q^{\beta_1} q^{\beta_2} \int_{\overline{\mathcal{M}}_2} \mathrm{ev}^*((\gamma_1 \otimes \gamma_1') \boxtimes (\gamma_2 \otimes \gamma_2')).
\end{aligned}$$

Note that $\langle \lambda, \beta_1\rangle > 0$. On the other hand,

$$\langle(\gamma_1 \otimes \gamma_1') * (D \otimes 1), \gamma_2 \otimes \gamma_2'\rangle = \langle \gamma_1 * \lambda, \gamma_2\rangle\langle \gamma_2, \gamma_2'\rangle$$

having no $q^{\beta_2}$-term.

2.3.3. *Remark.* Let us give a direct proof of this fact. When $\beta_1, \beta_2 > 0$, we have the following diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_2(X \times Y, (\beta, \beta')) \arrow[r, "(*)"] \arrow[d, "\mathrm{ev}"] & \overline{\mathcal{M}}_2(X, \beta) \times \overline{\mathcal{M}}_2(Y, \beta) \arrow[d, "\mathrm{ev} \boxtimes \mathrm{ev}"] \\
X \times Y \times X \times Y \arrow[r] & X \times X \times Y \times Y
\end{tikzcd}$$

Note that

$$\dim \text{left-hand side of } (*) - \dim \text{right-hand side of } (*) = 1.$$

By degree reason, the Gromov–Witten invariant vanishes.

## 2.4. **Projective spaces.**

2.4.1. *Example.* We have

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times.$$

We know

$$\begin{aligned}
H^2(\mathbb{P}^n) &= \mathbb{Z} \cdot H, \qquad H = [\text{a hyperplane}] = c_1(\mathcal{O}(1)) \\
H^2(\mathbb{P}^n) &= \mathbb{Z} \cdot \ell, \qquad \ell = [\text{a straight line}].
\end{aligned}$$

Recall that

$$H^*(\mathbb{P}^n) = \mathbb{Z}[H]/\langle H^n\rangle, \qquad \langle H^a, H^b\rangle = \delta_{a+b=n}.$$

Since the tangent bundle $\mathcal{T}_X$ can be put into the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}(1)^{N+1} \longrightarrow \mathcal{T}_X \longrightarrow 0,$$

we have $c_1(\mathcal{T}_X) = (n+1)H$. As a result, $q := q^\ell$ has degree $n+1$.

2.4.2. *Approach A.* Let us compute when $a+b=n+1$

$$H^a * H^b = (??)q.$$

That is,

$$\langle H^a * H^b, H^n\rangle = (??).$$

Note that $H^k$ is represented by a codimension $k$-plane, and in particular, $H^n$ is represented by a point. By the geometric meaning,

$$(??) = \#\left\{\begin{array}{c}\text{straight lines going through a point } P\\ \text{a } (n-a)\text{-plane } A \text{ and a } (n-b)\text{-plane } B\end{array}\right\}$$

Note that the affine span of $P$ and $A$ intersects a unique point $Q$ with $B$. Then $PQ$ is the straight line going through $P$, $A$ and $B$. So $(??)=1$. Thus when $a+b=n+1$, we have

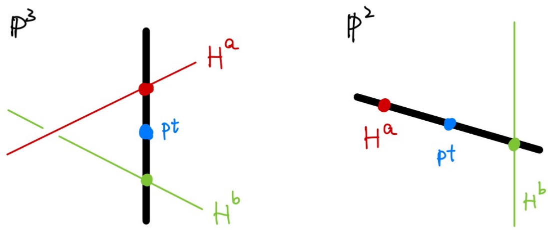
$$H^a * H^b = q.$$

By degree reason, we can conclude that, for $0 \le a, b \le n$,

$$H^a * H^b = \begin{cases} H^{a+b}, & a+b \le n\\ qH^{a+b-n-1}, & a+b > n.\end{cases}$$

So we have the following presentation of quantum cohomology

$$QH^*(\mathbb{P}^n) = \mathbb{Q}[H,q]/\langle H^{n+1} = q\rangle.$$




2.4.3. *Approach B.* There is anther approach of doing this. Let us compute

$$\underbrace{H * \cdots * H}_{n+1} = (??)q.$$

Recall that

$$\begin{aligned}\mathrm{Mor}_{\deg=1}(\mathbb{P}^1,\mathbb{P}^N) &= \{\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^N : f_*[\mathbb{P}^1] = \ell\}\\ &= \left\{(s_0,\dots,s_n) : \begin{array}{l} s_i \in H^0(\mathbb{P}^1,\mathcal{O}(1))\\ s_0\cdots s_n \text{ vanishes nowhere}\end{array}\right\}/\mathbb{C}^\times.\end{aligned}$$

Actually, for any $f:\mathbb{P}^1 \to \mathbb{P}^n$ of degree 1, the corresponding $(s_0,\dots,s_n)$ is given by

$$s_i = f^*(x_i), \qquad \text{the } i\text{-th coordinate } x_i \in H^0(\mathbb{P}^1,\mathcal{O}(1)) = \mathbb{C}^{n+1}.$$

Conversely, $f$ is defined by

$$f(x) = [s_0(x) : \cdots : s_n(x)] \in \mathbb{P}^n, \qquad x \in \mathbb{P}^1.$$

Let $H_i = \{x_i = 0\} \subset \mathbb{P}^n$ be the coordinate hyperplane. Let $c_0, \ldots, c_n \in \mathbb{P}^1$ be given points. Then

$$\left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^N : \begin{array}{l} f_*[\mathbb{P}^1] = \ell \\ f(c_i) \in H_i \end{array} \right\} = \left\{ (s_0, \ldots, s_n) : \begin{array}{l} s_i \in H^0(\mathbb{P}^1, \mathcal{O}(1)) \\ s_0 \cdots s_n \text{ vanishes nowhere} \\ s_i(c_i) = 0 \end{array} \right\} / \mathbb{C}^\times.$$

Note that

$$s_i(c_i) = 0 \iff s_i \in \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(c_i), \mathcal{O}(1)) \cong \mathbb{C}.$$

For a given generic $x \in \mathbb{P}^1$, we see that

$$\left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^n : \begin{array}{l} f_*[\mathbb{P}^1] = \ell \\ f(c_i) \in H_i \end{array} \right\} \xrightarrow{\mathrm{ev}_x} \mathbb{P}^N$$

is an isomorphism. Thus

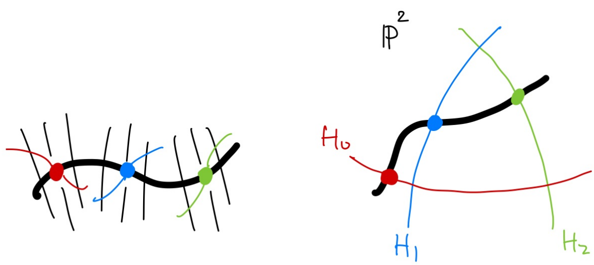
$$\# \left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^n : \begin{array}{l} f_*[\mathbb{P}^1] = \ell,\ f(c_i) \in H_i \\ f(x) = \text{a given point} \end{array} \right\} = 1.$$

This proves

$$\langle H * \cdots * H, [\mathrm{pt}] \rangle = q.$$

That is,

$$\underbrace{H * \cdots * H}_{n+1} = q.$$




### 2.5. **Full flag variety in $\mathbb{C}^3$.**

2.5.1. *Example.* Let us consider the full flag variety

$$X = \mathcal{F}\ell_2 = \left\{ 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 \right\}.$$

We have a tautological flag bundle

$$0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{O}_X^3.$$

Let us denote

$$x_1 = -c_1(\mathcal{V}_1), \qquad x_2 = -c_1(\mathcal{V}_2/\mathcal{V}_1), \qquad x_3 = -c_1(\mathcal{O}_X^3/\mathcal{V}_2).$$

The usual cohomology is given by

$$H^*(\mathcal{F}\ell_2) = \mathbb{Z}[x_1, x_2, x_3] \Big/ \left\langle \begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_1x_2 + x_1x_3 + x_1x_2 = 0 \\ x_1x_2x_3 = 0 \end{array} \right\rangle.$$

We have the following dual basis

$$1 \leftrightarrow x_1^2x_2, \qquad x_1 \leftrightarrow x_1x_2, \qquad x_1 + x_2 \leftrightarrow x_1^2.$$

Let us consider

$$X_1 = \mathbb{P}^2 = \{0 \subset V_1 \subset \mathbb{C}^3\}, \qquad X_2 = (\mathbb{P}^2)^\vee = \{0 \subset V_2 \subset \mathbb{C}^3\}.$$

We have forgetful map $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$. Denote

$$\beta_1 = \text{fibre of } \pi_1, \quad q_1 = q^{\beta_1}, \qquad \beta_2 = \text{fibre of } \pi_2, \quad q_2 = q^{\beta_2}.$$

The intersection form is

| $\langle\, ,\, \rangle$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $\beta_1$ | $1$ | $-1$ | $0$ |
| $\beta_2$ | $0$ | $1$ | $-1$ |

Since

$$c_1(\mathcal{T}_X) = (x_1 - x_2) + (x_2 - x_3) + (x_1 - x_3) = 2x_1 - 2x_3.$$

We have

$$\deg q_1 = \deg q_2 = 2.$$

By degree reason,

$$\lambda_1 * \lambda_2 = \lambda_1\lambda_2 + (\text{a number})q_1 + (\text{a number})q_2.$$
$$\lambda_1 * \lambda_2 * \lambda_3 = \lambda_1\lambda_2\lambda_3 + (\text{a divisor})q_1 + (\text{a divisor})q_2.$$

2.5.2. *Relation A.* We can get the quadratic relation as follows. For two divisors $\lambda_1, \lambda_2$, by using the divisor equation twice, we have

$$\begin{aligned} \langle \lambda_1 * \lambda_2, \gamma \rangle &= \langle \lambda_1\lambda_2, \gamma \rangle + \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \langle \lambda_1, \beta \rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\lambda_1 \boxtimes \gamma) \\ &= \langle \lambda_1\lambda_2, \gamma \rangle + \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \langle \lambda_1, \beta \rangle \langle \lambda_2, \beta \rangle \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma). \end{aligned}$$

The key observation is, we can identify

$$\begin{tikzcd} \overline{\mathcal{M}}_1(X, \beta_1) \arrow[r, equal, "\mathrm{ev}"] \arrow[d, "\mathrm{ft}_1"'] & X \arrow[d, "\pi_1"] \\ \overline{\mathcal{M}}_0(X, \beta_1) \arrow[r, equal] & X_1 \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}_1(X, \beta_2) \arrow[r, equal, "\mathrm{ev}"] \arrow[d, "\mathrm{ft}_2"'] & X \arrow[d, "\pi_2"] \\ \overline{\mathcal{M}}_0(X, \beta_2) \arrow[r, equal] & X_2 \end{tikzcd}$$

By taking $\gamma = [\mathrm{pt}]$, we get

$$\lambda_1 * \lambda_2 = \lambda_1\lambda + \langle \lambda_1, \beta_1\rangle\langle \lambda_2, \beta_1\rangle q_1 + \langle \lambda_1, \beta_2\rangle\langle \lambda_2, \beta_2\rangle q_2.$$

We can now compute

| $*$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | $x_1^2 + q_1$ | $x_1x_2 - q_1$ | $x_1x_3$ |
| $x_2$ | $x_1x_2 - q_1$ | $x_2^2 + q_1 + q_2$ | $x_2x_3 - q_2$ |
| $x_3$ | $x_1x_3$ | $x_2x_3 - q_2$ | $x_3^2 + q_2$ |

So we can conclude that

$$x_1x_2 + x_2x_3 + x_3x_1 + q_1 + q_2 = 0.$$

2.5.3. *Relation B.* We further have

$$\overline{\mathcal{M}}_2(X, \beta_1) = X \times_{X_1} X, \qquad \overline{\mathcal{M}}_2(X, \beta_2) = X \times_{X_2} X.$$

We have

$$\begin{tikzcd}
\overline{\mathcal{M}}_2(X,\beta_1) \arrow[rrr, "\mathrm{ft}_2"] \arrow[dr, equal] \arrow[drr, bend left, "\mathrm{ev}_1"] \arrow[ddr, bend right, "\mathrm{ev}_2"] \arrow[ddd, "\mathrm{ft}_1"'] & & & \overline{\mathcal{M}}_1(X,\beta_1) \arrow[dl, equal, "\mathrm{ev}"'] \arrow[ddd, "\mathrm{ft}"] \\
& X \times_{X_1} X \arrow[r] \arrow[d] & X \arrow[d, "\pi_1"] & \\
& X \arrow[r, "\pi_1"] \arrow[dl, equal, "\mathrm{ev}"'] & X_1 \arrow[dr, equal] & \\
\overline{\mathcal{M}}_1(X,\beta_1) \arrow[rrr, "\mathrm{ft}"] & & & \overline{\mathcal{M}}_0(X,\beta_1)
\end{tikzcd}$$

It is well-known that the composition

$$\begin{aligned}
&[H^*(X) \xrightarrow{\text{pull}} H^*(X \times_{X_i} X) \xrightarrow{\text{push}} H^{*-2}(X)] \\
&= [H^*(X) \xrightarrow{\text{push}} H^{*-2}(X_i) \xrightarrow{\text{pull}} H^*(X)] \\
&= \partial_i \text{ the BGG Demazure operator.}
\end{aligned}$$

The BGG Demazure operator acts as

$$\partial_1 f = \frac{f - f|_{x_1 \leftrightarrow x_2}}{x_1 - x_2}, \qquad \partial_2 f = \frac{f - f|_{x_2 \leftrightarrow x_3}}{x_2 - x_3}.$$

For a divisor $\lambda$, by divisor relation,

$$\lambda * \gamma = \lambda\gamma + q_1\langle \lambda, \beta_1\rangle \partial_1(\gamma) + q_2\langle \lambda, \beta_2\rangle \partial_2(\gamma) + (\text{other quantum terms}).$$

But by degree reason, there will be no other quantum terms. As a result,

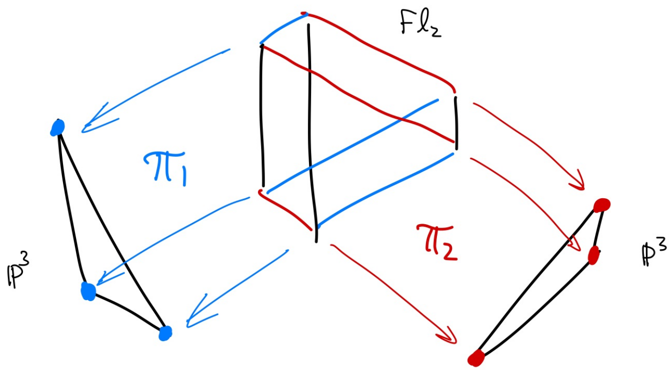
$$\begin{aligned} x_1 * (x_2 * x_3) = x_2 * (x_1 * x_3) &= x_2 * (x_1 x_3) \\ &= x_1 x_2 x_3 + q_1 \langle x_2, \beta_1 \rangle \partial_1 (x_1 x_3) + q_2 \langle x_2, \beta_2 \rangle \partial_2 (x_1 x_3) \\ &= 0 - q_1 x_3 - q_2 x_1. \end{aligned}$$

This proves

$$x_1 * x_2 * x_3 + q_1 x_3 + q_2 x_1 = 0.$$

In summary, the relations are given by the coefficients of characteristic polynomial of

$$\begin{bmatrix} x_1 & q_1 & \\ -1 & x_2 & q_2 \\ & -1 & x_3 \end{bmatrix}.$$




2.6. **Grassmannian in $\mathbb{C}^4$.**

2.6.1. *Example.* Let us consider

$$X = \mathrm{Gr}(2,4) = \{V \subset \mathbb{C}^4 : \dim V = 2\}.$$

We have a tautological exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X^4 \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Let us denote

$$D = e_1 = h_1 = -c_1(\mathcal{V}) = c_1(\mathcal{Q}), \qquad e_2 = c_2(\mathcal{V}), \qquad h_2 = c_2(\mathcal{Q}).$$

The relation is

$$(1 - e_1 y + e_2 y^2)(1 + h_1 y + h_2 y^2) = 1 \quad \text{(as a polynomial in } y).$$

We have $\mathcal{T}_X = \mathcal{H}om(\mathcal{V}, \mathcal{Q})$, so $c_1(\mathcal{T}_X) = nD$. Let $\ell$ be the primitive generator of $\mathrm{Eff}(X)$, we denote $q = q^\ell$. We have $\deg q = n$. Now let us consider

$$\begin{tikzcd}
\mathcal{F}\ell_4 = \{0 \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4\} \arrow[r, equal] & \overline{\mathcal{M}}_1(X,\ell) \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{ft}"] & \\
\mathrm{Gr}(2,4) & & \overline{\mathcal{M}}_0(X,\ell) = \{0 \subset V_1 \subset V_3 \subset \mathbb{C}^4\}
\end{tikzcd}$$

We can identify

$$Y = \overline{\mathcal{M}}_0(X,\ell), \qquad \mathcal{F}\ell_4 = \overline{\mathcal{M}}_1(X,\ell).$$

2.6.2. *Relation.* By degree reason, we have

$$e_2 * h_2 = e_2 h_2 + (\text{a number})q.$$

Note that

$$\text{the number} = \int_{\overline{\mathcal{M}}_3(X,\ell)} \mathrm{ev}^*(e_2 \boxtimes h_2 \boxtimes [\mathsf{pt}]).$$

We can identify

$$\overline{\mathcal{M}}_3(X,\ell) = \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4\,.$$

We have

$$\begin{aligned}
H^*(\overline{\mathcal{M}}_3(X,\ell)) &= H^*(\mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4) \\
&= H^*(\mathcal{F}\ell_4) \otimes_{H^*(Y)} H^*(\mathcal{F}\ell_4) \otimes_{H^*(Y)} H^*(\mathcal{F}\ell_4) \\
H^*(Y) &= \text{invariant algebra of } H^*(\mathcal{F}\ell_4) \text{ under } x_2 \leftrightarrow x_3.
\end{aligned}$$

Let us denote

$$x_i = x_i \otimes 1 \otimes 1, \qquad y_i = 1 \otimes x_i \otimes 1, \qquad z_i = 1 \otimes 1 \otimes x_i.$$

Note that

$$x_1 = y_1 = z_1, \qquad x_4 = y_4 = z_4.$$

We can represent

$$e_2 = x_1 x_2, \qquad h_2 = x_1^2 + x_1 x_2 + x_2^2, \qquad [\mathsf{pt}] = x_1^2 x_2^2.$$

As a result,

$$\mathrm{ev}^*(\cdots) = (x_1 x_2)(x_1^2 + x_1 y_2 + y_2^2)(x_1^2 z_2^2).$$

The pushforward is given by

$$\partial_2^x \partial_2^y \partial_2^z, \qquad \partial_2^f = \frac{f - f|_{x_2 \leftrightarrow x_3}}{x_2 - x_3}, \text{ etc.}$$

So

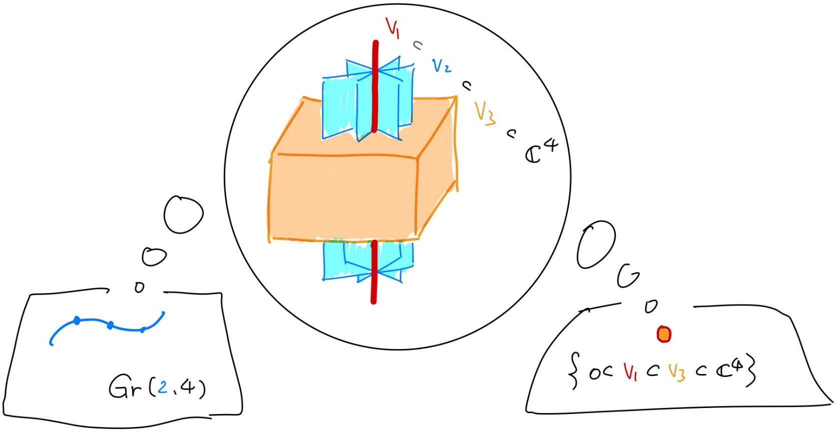
$$\begin{aligned}
\mathrm{ft}_*(\mathrm{ev}^*(\cdots)) &= (x_1)(x_1 + y_2 + y_3)(x_1^2(z_2 + z_3)) \\
&= (x_1)(x_1 + x_2 + x_3)(x_1^2(x_2 + x_3)) = [\mathsf{pt}].
\end{aligned}$$

As a result,

$$e_2 * h_2 = q.$$

So the relation is

$$(1 - e_1 y + e_2 y^2)(1 + h_1 y + h_2 y^2) = 1 + q.$$



## 3. Fundamental solution

The purpose of this section is to establish the theory of fundamental solution of quantum differential equations.

### 3.1. **Psi class.**

3.1.1. *Universal curve.* We could view the forgetful morphism

$$\mathrm{ft}_{n+1} : \overline{\mathcal{M}}_{n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(X, \beta)$$

the universal curve. That is, the fibre of a stable map $(f, C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_n(X, \beta)$ is $C$ itself. We also have universal sections $\sigma_i$ $(1 \leq i \leq n)$

$$\sigma_i : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow \overline{\mathcal{M}}_{n+1}(X, \beta)$$

by attaching a

$$\mathbb{P}^1 \ni p_{n+1}, \ (\text{new } p_i), \ (\text{attaching point})$$

on the $i$-th marked point.

3.1.2. *Universal cotangent line.* We define the *universal cotangent line* to be

$$\mathbb{L}_i = \sigma_i^*(\text{relative dualizing sheaf of } \mathrm{ft}_{n+1})$$

a line bundle over $\overline{\mathcal{M}}_n(X, \beta)$. In particular, at each point $(f, C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_n(X, \beta)$, the fibre of $\mathbb{L}_i$ is the cotangent line at $p_i \in C$. The *psi-class* is defined to be

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_n(X, \beta), \mathbb{Q}).$$

3.1.3. *Local computation.* The following computation is very important in the computation of psi-classes. Consider the family of curves with 1 marked point

$$(1, h) \in C_h = \{(x, y) : xy = h\} \subset \mathbb{C}^2, \qquad h \in \mathbb{C}.$$

Then we have

$$\begin{aligned} &\upsilon : \mathbb{C}^2 \longrightarrow \mathbb{C}, && (x, y) \longmapsto xy; && \text{(universal family)} \\ &\sigma : \mathbb{C} \longrightarrow \mathbb{C}^2, && h \longmapsto (1, h). && \text{(universal section)} \end{aligned}$$

We denote $\mathbb{L}$ the universal cotangent line. Note that the 2-nd projection defines a morphism $\mathbb{L}^* \longrightarrow \mathcal{T}_{\mathbb{C}}$, i.e.

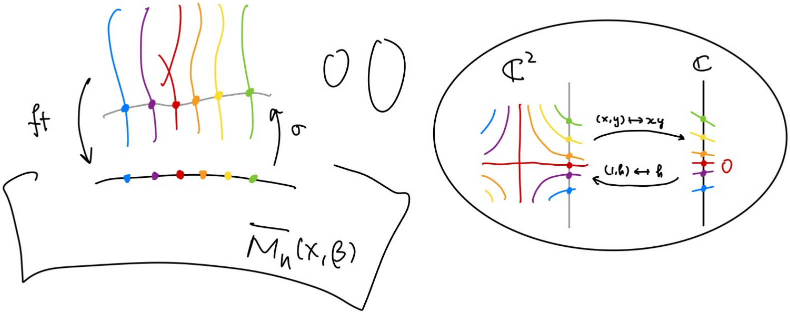
$$\text{tangent line of } C_h \text{ at } (1, h) \xrightarrow{\mathrm{pr}_2} \text{tangent line of } \mathbb{C} \text{ at } h.$$

Note that this morphism has a zero at $h = 0$. So we have

$$\mathbb{L} \otimes \mathcal{T}_{\mathbb{C}} \simeq \mathcal{O}(\{0\}), \qquad \text{i.e.} \qquad \mathbb{L} \simeq \Omega_{\mathbb{C}}(\{0\}).$$

The principle is

$$\boxed{\psi_i - \mathrm{ft}^* \psi_i = \left[ \begin{array}{c} \text{locus of curves collapsing on} \\ \text{the branch of the } i\text{-th marked point} \end{array} \right]}$$




3.1.4. *Example.* Let $C = (f, \mathbb{P}^1, p_1, \dots, p_n)$ be a generic stable map on $\overline{\mathcal{M}}_n(X, \beta)$. We know $\mathbb{P}^1 \simeq \mathrm{ft}_{n+1}^{-1}(C)$. Let us compute the restriction of $\mathbb{L}_i$ to $\mathbb{P}^1$. The first guess is

$$\mathbb{L}_i|_{\mathbb{P}^1} \quad \text{“="} \quad \Omega_{\mathbb{P}^1} = \mathcal{O}(-2).$$

But this is not true. At the point $p_i \in \mathbb{P}^1$, the corresponding curve is $\sigma_i(C) \in \mathrm{ft}_{n+1}^{-1}(C)$, whose $i$-th marked point is not $p_i$. From the local computation above, we actually have

$$\mathbb{L}_i|_{\mathbb{P}^1} = \Omega_{\mathbb{P}^1}(p_1 + \cdots + p_n) = \mathcal{O}(n-2).$$

3.1.5. *Example.* Recall the forgetful map

$$\mathrm{ft}_{n+1} : \overline{\mathcal{M}}_{n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(X, \beta).$$

We shall compare psi classes for different number of marked points. The first guess is

$$\mathrm{ft}_{n+1}^* \psi_i \quad \text{“="} \quad \psi_i.$$

But this is not true. When forgetting the $(n+1)$-th marked point, we might need collapsion to get a stable map. The local computation shows

$$\psi_i - \mathrm{ft}_{n+1}^* \psi_i = \left[\text{image of } \sigma_i : \overline{\mathcal{M}}_n(X, \beta) \to \overline{\mathcal{M}}_{n+1}(X, \beta)\right]$$

3.1.6. *Example.* Consider the forgetful map

$$\mathrm{ft}_X : \overline{\mathcal{M}}_3(X, \beta) \longrightarrow \overline{\mathcal{M}}_3.$$

We shall compare psi classes between them. The first guess is

$$\mathrm{ft}_X^* \psi_i \quad \text{“="} \quad \psi_i = 0.$$

But this is not true. When forgetting the underlying space $X$, we might need collapsion to get a stable map. The local computation shows

$$\psi_3 = \psi_3 - \mathrm{ft}_X^* \psi_3 = \sum_{\beta = \beta_1 + \beta_2} \left[\overline{\mathcal{M}}_3(X, \beta_1) \underset{\Delta_X}{\times} \overline{\mathcal{M}}_2(X, \beta_2)\right].$$

Here

$$\begin{tikzcd}
\overline{\mathcal{M}}_3(X,\beta_1) \underset{\Delta_X}{\times} \overline{\mathcal{M}}_2(X,\beta_2) \arrow[r] \arrow[d] \arrow[rr, hook, bend left, "\text{glue}"] \arrow[dr, phantom, "\text{fibre product}"] & \overline{\mathcal{M}}_3(X,\beta_1) \arrow[d, "\mathrm{ev}_3"] & \overline{\mathcal{M}}_3(X,\beta_1+\beta_2) \\
\overline{\mathcal{M}}_2(X,\beta_2) \arrow[r, "\mathrm{ev}_2"'] & X &
\end{tikzcd}$$

## 3.2. **Fundamental solution.**

3.2.1. *GW invariant twisted by psi class.* For $\gamma_1, \gamma_2, \gamma_3 \in H^*(X)$, let us consider a *gravitational correlator*

$$\langle \gamma_1, \gamma_2, \tau_a \gamma_3 \rangle_\beta := \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)\psi_3^a.$$

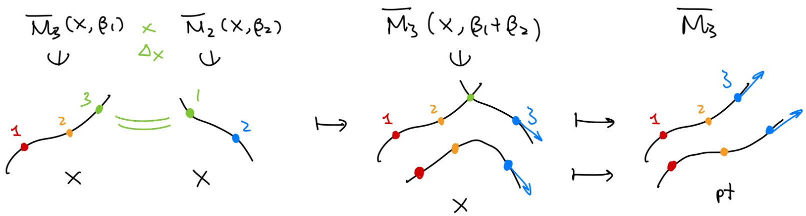
Let us pick a basis $\{\sigma_w\} \subset H^*(X)$ with dual basis $\{\sigma^w\}$.

3.2.2. *Appraoch A.* Let us apply Example 3.1.6. When $a \geq 1$,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)\psi_3^a \\
&= \sum_{\beta=\beta_1+\beta_2} \int_{\overline{\mathcal{M}}_3(X,\beta)} [\overline{\mathcal{M}}_3(X,\beta_1) \times_{\Delta_X} \overline{\mathcal{M}}_2(X,\beta_2)] \cdot \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)\psi_3^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times_{\Delta_X}\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)(1 \boxtimes \psi_2)^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \Delta_X \boxtimes \gamma_3)(1 \boxtimes \psi_2)^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2} \sum_w \int_{\overline{\mathcal{M}}_3(X)\times\overline{\mathcal{M}}_2(X)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \sigma_w \boxtimes \sigma^w \boxtimes \gamma_3)\psi_3^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2} \sum_w \int_{\overline{\mathcal{M}}_3(X,\beta_1)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \sigma_w) \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w \boxtimes \gamma_3)\psi_2^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2} \sum_w \langle \gamma_1, \gamma_2, \sigma_w \rangle_{\beta_1} \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w \boxtimes \gamma_3)\psi_2^{a-1}
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\beta\in \mathrm{Eff}(X)} q^{\beta}\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)\psi_3^a\\
&=\sum_w q^{\beta_1}\langle\gamma_1,\gamma_2,\sigma_w\rangle_{\beta_1}\sum_{\beta_2} q^{\beta_2}\int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w\boxtimes\gamma_3)\psi_2^{a-1}\\
&=\sum_\beta q^{\beta}\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1*\gamma_2\boxtimes\gamma_3)\psi_2^{a-1}
\end{aligned}$$




3.2.3. *Approach B.* Let us apply Example 3.1.5. Let us denote

$$D=\left[\text{image of }\sigma_2:\overline{\mathcal{M}}_2(X,\beta)\to\overline{\mathcal{M}}_3(X,\beta)\right].$$

Note that $\sigma_2^*\mathbb{L}_2$ is trivial, i.e. $D\cdot\psi_2=0$. When $a\geq 1$,

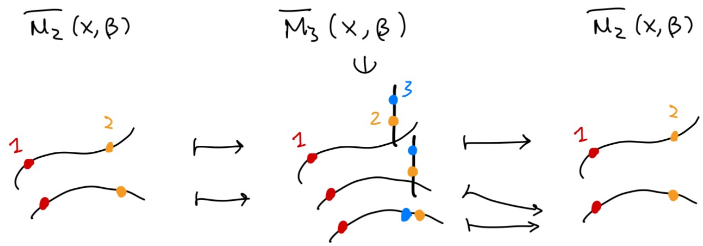
$$\psi_2^a=(\mathrm{ft}_3^*\psi_2+D)\psi_2^{a-1}=\mathrm{ft}_3^*\psi_2\cdot\psi_2^{a-1}=\cdots=\mathrm{ft}_3^*\psi_2^a+D\cdot\mathrm{ft}_3^*\psi_2^{a-1}.$$

Let us assume $\gamma_2=\lambda$ is a divisor. When $\beta>0$,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)\psi_3^a\\
&=\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\boxtimes\lambda)\psi_2^a\\
&=\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\boxtimes\lambda)\left(\mathrm{ft}_3^*\psi_2^a+D\cdot\mathrm{ft}_3^*\psi_2^{a-1}\right)\\
&=\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\boxtimes\lambda)\,\mathrm{ft}_3^*\psi_2^a+\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\boxtimes\lambda)D\cdot\mathrm{ft}_3^*\psi_2^{a-1}\\
&=\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ft}_3^*\left(\mathrm{ev}(\gamma_1\boxtimes\gamma_3)\psi_2^a\right)\mathrm{ev}_3^*(\lambda)+\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3)\sigma_2^*(\mathrm{ev}_3^*\lambda)\psi_2^{a-1}\\
&=\langle\lambda,\beta\rangle\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3)\psi_2^a+\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\cdot\lambda)\psi_2^{a-1}\\
&=\int_{\overline{\mathcal{M}}_2(X,\beta)} \langle\lambda,\beta\rangle\,\mathrm{ev}^*(\gamma_1\boxtimes\gamma_3)\psi_2^a+\mathrm{ev}^*(\gamma_1\boxtimes\gamma_3\cdot\lambda)\psi_2^{a-1}
\end{aligned}$$

Here we use the facts

$$\mathrm{ft}_{3*}\,\mathrm{ev}_3(\lambda) = \langle \lambda, \beta \rangle, \qquad \mathrm{ev}_3 \circ \sigma_3 = \mathrm{ev}_2, \qquad \mathrm{ft}_3\,\sigma_2 = \mathrm{id}\,.$$




3.2.4. *Summary.* By equalizing the results by two approaches, we get ($a \geq 1$)

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3)\psi_3^a \\
= &\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 * \lambda \boxtimes \gamma_3)\psi_2^{a-1} \\
= &\sum_{\beta \in \mathrm{Eff}(X)\backslash\{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \langle \lambda, \beta \rangle\, \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3)\psi_2^a + \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \cdot \lambda)\psi_2^{a-1}.
\end{aligned}$$

When $\beta = 0$, $\overline{\mathcal{M}}_2(X, \beta) = \varnothing$, so the integral is understood as $0$. Recall

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3) \\
&= \langle \gamma_1, \lambda \cdot \gamma_3 \rangle + \sum_{\beta \in \mathrm{Eff}(X)\backslash 0} q^\beta \langle \lambda, \beta \rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3) \\
&= \langle \gamma_1 * \lambda, \gamma_3 \rangle
\end{aligned}$$

For any polynomial (or a power series) $T(\psi)$, we denote $T^{\downarrow}(\psi) = \frac{T(\psi) - T(0)}{\psi}$. We have

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3) T(\psi_3) \\
= &\langle \gamma_1 * \lambda, \gamma_3 \rangle T(0) + \sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 * \lambda \boxtimes \gamma_3) T^{\downarrow}(\psi_2) \\
= &\langle \gamma_1, \lambda \cdot \gamma_3 \rangle T(0) + \sum_{\beta \in \mathrm{Eff}(X)\backslash 0} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \langle \lambda, \beta \rangle\, \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3) T(\psi_2) + \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \cdot \lambda) T^{\downarrow}(\psi
\end{aligned}$$

3.2.5. *Notations.* Let us introduce more notations

- Let us take a formal variable $z$. Now let us consider
$$\mathsf{T}(\psi) = \frac{1}{z-\psi} = \frac{1/z}{1-\psi/z} = \frac{1}{z} + \frac{\psi}{z^2} + \frac{\psi^2}{z^3} + \cdots .$$
Then
$$\mathsf{T}^{\downarrow}(\psi) = \frac{1}{\psi}\left(\frac{1}{z-\psi} - \frac{1}{z}\right) = \frac{1}{z(z-\psi)} = \frac{1}{z}\mathsf{T}(\psi).$$
- For any divisor $\lambda$ denote $\partial_\lambda$ the differential operator on $QH^*(X)$ with
$$\partial_\lambda q^\beta = \langle \lambda, \beta \rangle q^\beta.$$
Here, a differential operator is an $H^*(X)$-linear operators with Leibniz rule.
- Let us denote $p \ln q$ the unique function with
$$\partial_\lambda(p \ln q) = \lambda.$$
It can be constructed by $p \ln q = \sum p_i \ln q^{\beta_i}$ for $\{\beta_i\} \subset \mathrm{Eff}(X) \subset H_2(X)$ a basis with $\{p_i\} \subset H^2(X)$ its dual basis. In particular,
$$\partial_\lambda(e^{p \ln q/z}) = \frac{1}{z} e^{p \ln q/z} \lambda.$$

3.2.6. *Fundamental solution.* Let us denote a functional $S$ as follows. For $\gamma, \gamma' \in H^*(X)$,
$$S(\gamma, \gamma') = \langle \gamma, e^{p \ln q/z} \gamma' \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes e^{p \ln q/z} \gamma') \frac{1}{z - \psi_2}.$$
Then we can write down the equation
$$\frac{1}{z} S(\gamma * \lambda, \gamma') = \partial_\lambda S(\gamma, \gamma').$$
In particular, let us denote an operator $S$ such that
$$\langle \gamma, S(\gamma') \rangle = S(\gamma, \gamma') \qquad \text{i.e.} \quad S(\gamma') = \sum_w \sigma_w \cdot S(\sigma^w, \gamma').$$
In particular,
$$\begin{aligned} S(\gamma * \lambda, \gamma') &= \langle \gamma * \lambda, S(\gamma') \rangle = \langle \gamma, \lambda * S(\gamma') \rangle \\ \partial_\lambda S(\gamma, \gamma') &= \partial_\lambda \langle \gamma, S(\gamma') \rangle = \langle \gamma, \partial_\lambda S(\gamma') \rangle. \end{aligned}$$
Thus for any $\gamma' \in H^*(X)$, we have
$$\partial_\lambda S(\gamma') - \frac{1}{z} \lambda * S(\gamma') = 0.$$
In particular, $S(\gamma')$ solves the quantum differential equation (discussed later). We call the operator $S$ the *fundamental solution*.

3.2.7. *Remark.* Since

$$\lim_{z\to\infty} S(\gamma') = \gamma'$$

the operator $S$ is nondegenerate.

## 3.3. **J-function.**

3.3.1. *J-function.* Let us define $J$ to be the unique class such that

$$\langle J, \gamma' \rangle = S(1, \gamma') = \langle 1, S(\gamma') \rangle, \qquad \text{i.e.} \qquad J = \sum_w \sigma_w \cdot S(1, \sigma^w).$$

If we think $S$ as a matrix, then each column of $S$ is a solution of quantum differential equation. The J-function is by definition the row of $S$ corresponding to $1 \in H^*(X)$.

3.3.2. *Simplification.* By definition

$$\begin{aligned} J &= \sum_w \sigma_w \cdot S(1, \sigma^w) \\ &= \sum_w \sigma_w \left( \langle 1, e^{p \ln q/z} \sigma^w \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(1 \boxtimes e^{p \ln q/z} \sigma^w) \frac{1}{z - \psi_2} \right). \end{aligned}$$

More general, for $\beta > 0$, let us denote

$$D = [\text{image of } s_1 : \overline{\mathcal{M}}_1(X, \beta) \to \overline{\mathcal{M}}_2(X, \beta)].$$

Similar as what we did in Approach B 3.2.3, we have

$$\begin{aligned} &\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(1 \boxtimes \gamma) \psi_2^a \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes 1) \psi_1^a \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes 1)(\mathrm{ft}_2^* \psi_1^a + D \cdot \mathrm{ft}_2^* \psi_1^{a-1}) \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ft}_2^*(\mathrm{ev}^*(\gamma) \psi_1^a) + \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma) \psi_1^{a-1} \\ &= 0 + \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma) \psi_1^{a-1}. \end{aligned}$$

Let us denote $\psi = \psi_1 \in H^2(\overline{\mathcal{M}}_1(X,\beta))$. So

$$\begin{aligned}
J &= \sum_w \sigma_w \langle 1, e^{p\ln q/z}\sigma^w\rangle + \sum_{\beta\in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \sum_w \sigma_w \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(e^{p\ln q}\sigma^w)\frac{1}{z(z-\psi)} \\
&= e^{p\ln q/z} + e^{p\ln q/z} \sum_{\beta\in \mathrm{Eff}(X)\setminus\{0\}} q^\beta\, \mathrm{ev}_* \frac{1}{z(z-\psi)} \\
&= e^{p\ln q/z}\left(1 + \sum_{\beta\in \mathrm{Eff}(X)\setminus\{0\}} q^\beta\, \mathrm{ev}_* \frac{1}{z(z-\psi)}\right).
\end{aligned}$$

3.4. **Relations.** Let $D = f(z\partial_\lambda, q)$ be a differential operator with $f$ a noncommutative polynomial. If

$$DJ = 0$$

then $\lim_{z\to 0} f(\lambda, q) = 0$ in $QH^*(X)$.

*Proof.* Note that

$$z\partial_\lambda S(\gamma') = \lambda * S(\gamma').$$

When $f$ takes form of

$$\sum (\text{a function in } q)\cdot(\text{differential operators}),$$

we have

$$DS(\gamma') = f(\lambda *, q)S(\gamma').$$

Thus

$$\begin{aligned}
0 = \langle DJ, \gamma'\rangle &= D\langle J, S(\gamma')\rangle = D\langle 1, S(\gamma')\rangle \\
&= \langle 1, DS(\gamma')\rangle = \langle 1, f(\lambda *, q)S(\gamma')\rangle = \langle f(\lambda *, q), S(\gamma')\rangle.
\end{aligned}$$

Since $S(\gamma')$ is non-degenerate, $f(\lambda, q) = 0$ in $QH^*(X)$.

The general case follows from the fact that

$$[z\partial_\lambda, \text{multiplcation by } q^\beta] = z\cdot \text{multiplication by } \partial_\lambda q^\beta,$$

which is killed by $\lim_{z\to 0}$. $\square$

## 4. Quasi maps

### 4.1. **Normal bundle in terms of Psi class.**

4.1.1. *Local computation.* Recall the family of curves

$$C_h = \{(x, y) : xy = h\} \subset \mathbb{C}^2, \qquad h \in \mathbb{C}.$$

The ideal for $C_0 = (x\text{-axis}) \cup (y\text{-axis})$ is

$$\mathfrak{m} = \langle xy \rangle \subset R := \mathbb{C}[x, y].$$

So the normal bundle of $C_0$ is

$$\mathfrak{m}/\mathfrak{m}^2 = xyR/\mathfrak{m} = \mathcal{O}_{C_0}(x) \otimes \mathcal{O}_{C_0}(y).$$

Thus we can naturally identify the normal bundle of the singleton $C_0 \in \{C_h\}$ with

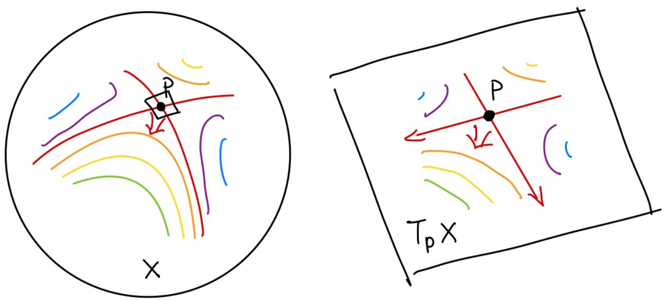
$$(\text{tangent line of } 0 \text{ along } x\text{-axis}) \otimes (\text{tangent line of } 0 \text{ along } y\text{-axis}).$$

Say, by the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[d] & \times & \mathbb{C} \arrow[d] \arrow[r] & \mathbb{C}^2 \arrow[d] & & \\
\{x\text{-axis}\} & \times & \{y\text{-axis}\} \arrow[r] & \{C_h\} & \simeq & \mathbb{C}.
\end{tikzcd}$$

The principle is

$$\boxed{\text{smoothing of the nodal point} = \text{tensor product of two tangent directions}}$$




4.1.2. *Example.* Let us consider the morphism

$$\overline{\mathcal{M}}_{n+1}(X, \beta_1) \times_X \overline{\mathcal{M}}_{m+1}(X, \beta_2) \longrightarrow \overline{\mathcal{M}}_{m+n}(X, \beta_1 + \beta_2)$$

by gluing the first marked points. Then the normal bundle of this morphism is the restriction of $(\mathbb{L}_1 \boxtimes \mathbb{L}_1)^*$.

4.1.3. *Example.* Let us consider the morphism

$$\overline{\mathcal{M}}_{n+1}(X,\beta_1)\times\overline{\mathcal{M}}_{m+1}(Y,\beta_2)\longrightarrow\overline{\mathcal{M}}_{m+n}(X\times Y,(\beta_1,\beta_2))$$

by gluing the first marked points. Then the normal bundle is the restriction of $(\mathbb{L}_1\boxtimes\mathbb{L}_1)^*$.

4.1.4. *Example.* Let us consider

$$\overline{\mathcal{M}}_n(X,\beta)\times\mathbb{P}^1\longrightarrow\overline{\mathcal{M}}_{n-1}(X\times\mathbb{P}^1,(\beta,1))$$

by sending $(C,x)$ to the curve obtained by first putting $C$ vertically at the point $x\in\mathbb{P}^1$ and then gluing a $\mathbb{P}^1$ horizontally at the first marked point. Then the normal bundle is $\mathbb{L}_1^*\boxtimes\mathcal{T}_{\mathbb{P}^1}$.

4.2. **Quasi-maps.**

4.2.1. *Remark.* Let $\mathcal{L}$ and $\mathcal{V}$ be two vector bundles. For a sheaf morphism $s:\mathcal{L}\to\mathcal{V}$, we have (by Nakayama lemma)

$$s\text{ is surjective}\iff s\text{ is fibrewise surjective.}$$

While we only have

$$s\text{ is injective}\Leftarrow s\text{ is fibrewise injective.}$$

Actually, when $\mathcal{L}$ is a line bundle,

$$s\text{ is injective}\iff s\text{ is nonzero (on each connected component).}$$

4.2.2. *Quasi maps for projective space.* Recall that

$$\operatorname{Mor}(C,\mathbb{P}^N)=\bigcup_{\mathcal{L}\in\operatorname{Pic}(C)}\operatorname{Surj}(\mathcal{O}_C^{N+1}\to\mathcal{L})/\mathbb{C}^*.$$

By taking dual,

$$\operatorname{Surj}(\mathcal{O}_C^{N+1}\to\mathcal{L})/\mathbb{C}^*\hookrightarrow\operatorname{Inj}(\mathcal{L}^\vee\to\mathcal{O}_C^{N+1})/\mathbb{C}^*=\mathbb{P}(H^0(C,\mathcal{L})^{N+1}).$$

We define quasi-map by

$$\operatorname{QM}(C,\mathbb{P}^N)=\bigcup_{\mathcal{L}}\mathbb{P}(H^0(C,\mathcal{L})^{N+1}).$$

When $C=\mathbb{P}^1$, we define

$$\operatorname{QM}(\mathbb{P}^N)=\bigcup_{d\geq 0}\operatorname{QM}(\mathbb{P}^N,d)=\bigcup_{d\geq 0}\mathbb{P}(\mathbb{C}[x]^{N+1}_{\deg\leq d}).$$

It is a compactification of the space of $\mathbb{P}^1\to\mathbb{P}^N$ of degree d.

4.2.3. *Quasi maps for general* X. Assume we can embed

$$X \longrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$$

using primitive nef divisors $D_1, \ldots, D_m$. For $\beta \in \mathrm{Eff}(X)$, denote

$$\beta_1 = \langle D_1, \beta \rangle, \ldots, \beta_m = \langle D_m, \beta \rangle.$$

We can view

$$\begin{aligned}
\mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X) &\subset \mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, \mathbb{P}^{N_1} \times \cdots \mathbb{P}^{N_m}) \\
&= \mathrm{Mor}_{\deg=\beta_1}(\mathbb{P}^1, \mathbb{P}^{N_1}) \times \cdots \times \mathrm{Mor}_{\deg=\beta_m}(\mathbb{P}^1, \mathbb{P}^{N_m}) \\
&\subset \mathrm{QM}(\mathbb{P}^{N_1}, \beta_1) \times \cdots \times \mathrm{QM}(\mathbb{P}^{N_m}, \beta_m).
\end{aligned}$$

We define

$$\mathrm{QM}(X, \beta) = \text{closure of } \mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X) \text{ in } \mathrm{QM}(\mathbb{P}^{N_1}, \beta_1) \times \cdots \times \mathrm{QM}(\mathbb{P}^{N_m}, \beta_m)$$

and $\mathrm{QM}(X) = \bigcup_{\beta \in \mathrm{Eff}(X)} \mathrm{QM}(X, \beta)$.

4.2.4. *Remark.* We can think as follows. For sections $s_0, \ldots, s_N \in H^0(C, \mathcal{L})$, we define a rational map

$$C \longrightarrow \mathbb{P}^N, \qquad x \mapsto [s_0(x) : \cdots : s_N(x)].$$

This defines a morphism when $s_0, \ldots, s_N$ has no common zeros. In general, the closure of C defines a morphism $C \to \mathbb{P}^N$ but with class

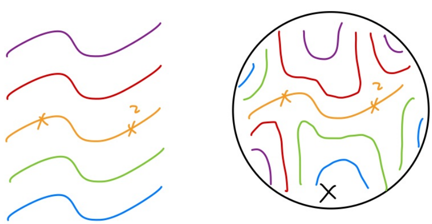
$$\mathcal{L}(-\text{common zeros}).$$

We call those common zeros by *marked points* (with multiplicity). So we have

$$\mathrm{QM}(\mathbb{P}^n, d) = \bigsqcup_{0 \le d' \le d} \mathrm{Mor}_{\deg=d'}(\mathbb{P}^1, \mathbb{P}^n) \times \mathrm{Sym}^{d-d'} C.$$

A quasi map can be uniquely recorded as a morphism $C \to \mathbb{P}^N$ and marked zeros. Generally, a quasi map over X can be uniquely recorded as a morphism $\mathbb{P}^1 \to X$ with colored marked point. That is,

$$\mathrm{QM}(X, \beta) = \bigsqcup_{0 \le \beta' \le \beta} \mathrm{Mor}_{\deg=\beta}(C, \mathbb{P}^n) \times \prod_{i=1}^{m} \mathrm{Sym}^{\langle \beta - \beta', D_i \rangle} \mathbb{P}^1.$$



4.2.5. *Fixed locus.* There is $\mathbb{C}^\times$-action on $\mathrm{QM}(X)$ induced from $\mathbb{P}^1$. Firstly, let us look at

$$\mathrm{QM}(\mathbb{P}^N, d) = \mathbb{P}(\mathbb{C}[x]^{N+1}_{\deg \le d}).$$

We have

$$\mathrm{QM}(\mathbb{P}^N, d)^{\mathbb{C}^\times} = \bigcup_{0 \le d' \le d} x^{d'} \mathbb{P}(\mathbb{C}^{N+1}) = \bigcup_{0 \le d' \le d} \mathbb{P}^N.$$

That is, it is set of constant quasi-map with $d$ marked point at $0$ and $d - d'$ marked point at $\infty$. More generally, we have

$$\mathrm{QM}(X, \beta)^{\mathbb{C}^\times} = \bigcup_{0 \le \beta' \le \beta} x^{\beta'} \cdot X.$$

4.2.6. *Pseudo evaluation.* Recall we have a morphism

$$\epsilon\upsilon^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\mathrm{QM}(X, \beta))$$

such that the restricting to any fixed component

$$\mathrm{Pic}(\mathrm{QM}(X, \beta)) \longrightarrow \mathrm{Pic}(x^{\beta'} X) \simeq \mathrm{Pic}(X)$$

is identity. For any polynomial $f(x_1, \ldots, x_m)$, we want to compute

$$\int_{\mathrm{QM}(X,\beta)} f(\epsilon\upsilon^* D_1, \ldots, \epsilon\upsilon^* D_m).$$

## 4.3. **Graph Space.**

4.3.1. *Graph Space.* Let us consider the graph space

$$G_0(X, \beta) = \overline{\mathcal{M}}_0(\mathbb{P}^1 \times X, (1, \beta)).$$

Note that $G_0(X)$ admits a $\mathbb{C}^\times$ action, so we can compute pushforward via localization. We view the projection

$$\mathbb{P}^1 \times X \to \mathbb{P}^1$$

as a fibre bundle. Every stable map in $G_0(X, \beta)$ is a union of a section and vertical curves.

4.3.2. *Fixed component.* For any $x \in X$, we denote $[x]$ the graph of constant map

$$[x] = \left[\mathbb{P}^1 \to \mathbb{P}^1 \times \{x\} \subset \mathbb{P}^1 \times X\right].$$

Assume $\beta > 0$. Let $\beta_1, \beta_2 > 0$. We have a morphism

$$i_{\beta_1,\beta_2} : \overline{\mathcal{M}}_1(X, \beta_1) \times_X \overline{\mathcal{M}}_1(X, \beta_2) \longrightarrow G_0(X, \beta_1 + \beta_2)$$

by putting two stable maps with same marked point on $X$ horizontally at $0$ and $\infty$ respectively, and gluing them by $[x]$. We also have

$$i_{\beta,0} : \overline{\mathcal{M}}_1(X, \beta) \longrightarrow G_0(X, \beta)$$

by putting a stable map at $0$. We similarly define $i_{0,\beta}$. Then

$$G_0(X,\beta)^{\mathbb{C}^\times} = \bigcup_{\beta_1+\beta_2=\beta} (\text{image of } i_{\beta_1,\beta_2}).$$

4.3.3. *Dimension estimation.* Let us estimate the dimension. We have

$$\begin{aligned}\dim G_0(X,\beta) &= \dim X + 1 + \langle c_1(\mathcal{T}_X),\beta\rangle + \langle c_1(\mathcal{T}_{\mathbb{P}^1}),1\rangle + 0 - 3\\ &= \dim X + \langle c_1(\mathcal{T}_X),\beta\rangle.\end{aligned}$$

For $\beta_1,\beta_2 > 0$ with $\beta_1+\beta_2=\beta$,

$$\begin{aligned}\dim \overline{\mathcal{M}}_1(X,\beta_1)\times_X \overline{\mathcal{M}}_1(X,\beta_2) &= \dim X + \langle c_1(\mathcal{T}_X),\beta\rangle + 1 - 3 + 1 - 3\\ &= \dim X + \langle c_1(\mathcal{T}_X),\beta\rangle - 4.\end{aligned}$$

On the other hand,

$$\begin{aligned}\dim \overline{\mathcal{M}}_1(X,\beta) &= \dim X + \langle c_1(\mathcal{T}_X),\beta\rangle + 1 - 3\\ &= \dim X + \langle c_1(\mathcal{T}_X),\beta\rangle - 2.\end{aligned}$$

4.3.4. *Normal bundle.* Denote $\xi$ the natural representation of $\mathbb{C}^\times$. For $\beta_1,\beta_2>0$, the normal bundle along $i_{\beta_1,\beta_2}$.

$$\begin{aligned}(\text{smoothing the gluing point at } 0) &= (\mathbb{L}^{-1}\otimes\xi)\boxtimes\mathcal{O}.\\ (\text{moving the vertical curve at } 0) &= \xi\boxtimes\mathcal{O} = \xi.\end{aligned}$$

Similarly for the gluing point at $\infty$

$$\begin{aligned}(\text{smoothing the gluing point at } \infty) &= \mathcal{O}\boxtimes(\mathbb{L}^{-1}\otimes\xi^{-1}).\\ (\text{moving the vertical curve at } \infty) &= \mathcal{O}\boxtimes\xi^{-1} = \xi^{-1}.\end{aligned}$$

Thus the Euler class

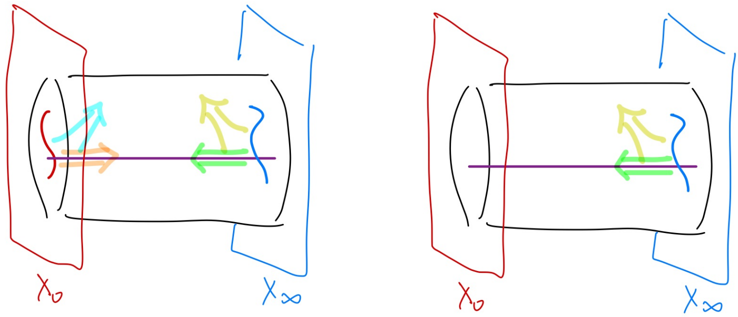
$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2})) = \text{restriction of } z(z-\psi)\otimes(-z(-z-\psi)).$$

When $\beta_2=0$, we do not need to smooth and move $\infty$, so

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta,0})) = \text{restriction of } z(z-\psi)\otimes 1.$$

Similarly, when $\beta_1=0$,

$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta})) = \text{restriction of } 1\otimes(-z(-z-\psi)).$$

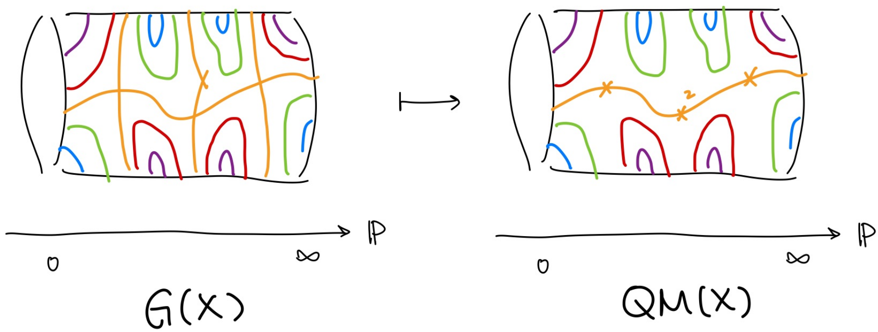



## 4.4. **Comparison.**

4.4.1. *Comparison.* Note that both $G(X, \beta)$ and $QM(X, \beta)$ are compatification of $\mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X)$. We actually have a birational morphism

$$\mu : G(X, \beta) \longrightarrow QM(X, \beta)$$

by changing the vertical curves by marked points.




4.4.2. *Localization.* Let

$$\phi = f(D_1, \ldots, D_m).$$

As $\mu$ is birational,

$$\begin{aligned}
&\int_{QM(X,\beta)} f(\epsilon v^* D_1, \ldots, \epsilon v^* D_m) \\
= &\int_{G(X,\beta)} \mu^* f(\epsilon v^* D_1, \ldots, \epsilon v^* D_m) \\
= &\sum_{\beta_1+\beta_2=\beta} \int \frac{i^*_{\beta_1,\beta_2} \mu^* f(\epsilon v^* D_1, \ldots, \epsilon v^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}.
\end{aligned}$$

When $\beta_1, \beta_2 > 0$, we have the following diagram

$$\begin{tikzcd}
G(X,\beta) \arrow[r, "\mu"] & QM(X,\beta) \\
\text{image of } i_{\beta_1,\beta_2} \arrow[u] \arrow[r] & \chi^{\beta_1}X \arrow[u] \\
\overline{\mathcal{M}}_1(X,\beta_!) \times_X \overline{\mathcal{M}}_1(X,\beta_2) \arrow[u, equal] \arrow[r, "\mathrm{ev}\boxtimes 1 = 1\boxtimes \mathrm{ev}"] & X \arrow[u, equal]
\end{tikzcd}$$

Thus

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta_1)\times_X \overline{\mathcal{M}}_1(X,\beta_2)} \frac{i^*_{\beta_1,\beta_2}\mu^* f(\epsilon\upsilon^* D_1, \ldots, \epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))} \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times_X \overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*\big(f(D_1,\ldots,D_m)\big)}{z(z-\psi)\otimes(-z)(-z-\psi)} \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times \overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*(\phi)}{z(z-\psi)\otimes(-z)(-z-\psi)} (\mathrm{ev}\boxtimes\mathrm{ev})^*(\Delta_X) \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times \overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*(\phi)}{z(z-\psi)\otimes(-z)(-z-\psi)} \sum_w (\mathrm{ev}\boxtimes\mathrm{ev})^*(\sigma_w\boxtimes\sigma^w) \\
&= \sum_w \int_{\overline{\mathcal{M}}_1(X,\beta_1)} \frac{\mathrm{ev}^*(\phi\cdot\sigma_w)}{z(z-\psi)} \int_{\overline{\mathcal{M}}_1(X,\beta_2)} \frac{\mathrm{ev}^*(\sigma^w)}{z(z-\psi)} \\
&= \sum_w \left\langle \mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right), \phi\cdot\sigma_w \right\rangle \left\langle \mathrm{ev}_*\left(\frac{1}{-z(-z-\psi)}\right), \sigma^w \right\rangle
\end{aligned}$$

Similarly, when $\beta' = \beta$,

$$\begin{tikzcd}
\text{image of } i_{\beta,0} \arrow[r] & \chi^{\beta}X \\
\overline{\mathcal{M}}_1(X,\beta) \arrow[u, equal] \arrow[r, "\mathrm{ev}_2"] & X. \arrow[u, equal]
\end{tikzcd}$$

We have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i^*_{\beta,0}\mu^* f(\epsilon\upsilon^* D_1, \ldots, \epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))} \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{\mathrm{ev}^*(\phi)}{z(z-\psi)} = \left\langle \mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right), \phi \right\rangle \\
&= \sum_w \left\langle \mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right), \phi\sigma_w \right\rangle \langle 1, \sigma^w \rangle
\end{aligned}$$

Similarly,

$$\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i_{0,\beta}^*\mu^* f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}$$

$$= \int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{\mathrm{ev}^*(\phi)}{-z(-z-\psi)} = \left\langle \mathrm{ev}_* \left( \frac{1}{-z(-z-\psi)} \right), \phi \right\rangle$$

$$= \sum_w \langle 1, \phi \cdot \sigma_w \rangle \left\langle \mathrm{ev}_* \left( \frac{1}{-z(-z-\psi)} \right), \sigma^w \right\rangle .$$

4.4.3. *J-function again.* Let us denote

$$\tilde{J}(z) = 1 + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \, \mathrm{ev}_* \left( \frac{1}{z(z-\psi)} \right) .$$

Recall that

$$J = e^{p \ln q / z} \tilde{J}.$$

Then above computation shows

$$\sum_{q \in \mathrm{Eff}(X)} q^\beta \int_{QM(X,\beta)} f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m)$$

$$= \sum_w \langle \tilde{J}(z), \phi \cdot \sigma_w \rangle \langle \tilde{J}(-z), \sigma^w \rangle = \langle \tilde{J}(z) \tilde{J}(-z), \phi \rangle.$$

# 5. Properties

## 5.1. **Quantum connection.**

5.1.1. *Remark.* Recall that a connection of a vector bundle $\mathcal{V}$ over a real manifold $M$ is an $\mathbb{R}$-bilinear morphism

$$\nabla : \mathcal{V} \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{V}.$$

with the Leibniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla s.$$

For a local vector field $X \in \mathcal{T}_M$, we deonte $\nabla_X s = \langle X, \nabla s\rangle$, with the pairing induced by the natural pairing $\langle\, , \,\rangle : \mathcal{T}_M \otimes \Omega_M \otimes \mathcal{V} \longrightarrow \mathcal{V}$. Then $\nabla_X s$ satisfies

- $\nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s;$ (linearity)
- $\nabla_X(fs + t) = (Xf)s + f\nabla_X s + \nabla_X t;$ (Leibinize rule)

To define a connection locally, it suffices to define $\nabla_X$ for those $X$ forming a basis of $\mathcal{T}_M$ over $\mathcal{O}_M$ (called a frame) and check the second condition.

5.1.2. *Quantum connection.* Let us consider

the trivial vector bundle $\mathcal{V}$ over $M = H^2(X)$ with fibre $H^*(X)$.

Note that we can view $q^\beta$ as a function over $H^2(X)$ for $\beta \in \mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$. Thus

$$H^0(M, \mathcal{V}) = H^*(X) \otimes_{\mathbb{C}} \mathcal{O}(M) = QH^*(X) \otimes_{\mathbb{C}(q)} \mathcal{O}(M).$$

The *quantum connection* is defined to be ($z$ is a formal variable)

$$\nabla_\lambda = \partial_\lambda - \frac{1}{z}\lambda *,$$

where

- $\partial_\lambda$ is the differential operator over $M$ such that $\partial_\lambda q^\beta = \langle\lambda, \beta\rangle q^\beta$;
- $\lambda*$ is the $\mathcal{O}$-linear map of quantum product with divisor $\lambda \in H^2(X)$ fibrewise.

This is a connection:

$$\begin{aligned}
\nabla_\lambda(fs + t) &= \partial_\lambda(fs + t) - \frac{1}{z}\lambda * (fs + t)\\
&= (\partial_\lambda f) + f(\partial_\lambda s) + \partial_\lambda t - \frac{1}{z} f\lambda * s - \frac{1}{z}\lambda * t\\
&= (\partial_\lambda f) + f\nabla_\lambda s + \nabla_\lambda t.
\end{aligned}$$

Here we use the fact that the quantum product is $\mathbb{C}(q)$-linear.

5.1.3. *Remark.* For a connection $\nabla$ of a vector bundle $\mathcal{V}$ over $M$, we can extend

$$0 \longrightarrow \mathcal{V} \xrightarrow{\nabla} \Omega_M \otimes_{\mathcal{O}_M} \mathcal{V} \xrightarrow{\nabla} \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{V} \xrightarrow{\nabla} \cdots$$

by

$$\nabla(\omega \wedge s) = d\omega \otimes s + (-1)^{\deg \alpha} \omega \wedge \nabla s.$$

The map $\nabla^2 : \mathcal{V} \to \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{V}$ is $\mathcal{O}_M$-linear, called the curvature. A connection is flat if $\nabla^2 = 0$, equivalently, the above chain is a complex. In terms of $\nabla_X s$, it is equivalent to say

$$\langle X \wedge Y, \nabla^2 s \rangle = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = 0.$$

If we define $\nabla_X$ for a frame forming a basis of $\mathcal{T}_M$, then it suffice to check for all pairing of vector fields from the frame. For a flat connection $\nabla$, the following differential equation has a local solution

$$\nabla(f) = 0, \qquad f \in H^0(M, \mathcal{V})$$

for any given initial value of $f$ at a point $x \in M$.

5.1.4. *Flatness.* The quantum connection is flat.

$$\begin{aligned}
&\nabla_\lambda \nabla_\mu s - \nabla_\mu \nabla_\lambda s - \nabla_{[\lambda,\mu]} s \\
&= \nabla_\lambda \nabla_\mu s - \nabla_\mu \nabla_\lambda s \\
&= (\partial_\lambda - \tfrac{1}{z}\lambda *)(\partial_\mu - \tfrac{1}{z}\mu *)s - (\partial_\mu - \tfrac{1}{z}\mu *)(\partial_\lambda - \tfrac{1}{z}\lambda *)s \\
&= \big(\partial_\lambda \partial_\mu s - \tfrac{1}{z}\mu * \partial_\lambda s - \tfrac{1}{z}\lambda * \partial_\mu s + \tfrac{1}{z^2}\lambda * \mu * s\big) \\
&- \big(\partial_\mu \partial_\lambda s - \tfrac{1}{z}\lambda * \partial_\mu s - \tfrac{1}{z}\mu * \partial_\lambda s + \tfrac{1}{z^2}\mu * \lambda * s\big) \\
&= \frac{1}{z^2}(\lambda * \mu * s - \mu * \lambda * s) = 0.
\end{aligned}$$

Here we use the associativity and commutativity of the quantum product.

5.1.5. *Remark.* As we mentioned, $S(\gamma')$ solves the quantum differential equation,

$$\nabla_\lambda(f) = 0, \quad \text{i.e.} \quad \partial_\lambda f = \frac{1}{z}\lambda * \gamma.$$

It is actually the fundamental solution.

5.1.6. *Remark.* Note that if we replace quantum product by usual product, then the fundamental solution is easy seen to be

$$S(\gamma') = e^{p \ln q / z}\gamma'.$$

5.2. **Applications.**

5.2.1. *Remark.* Let $F$ be a component of $X^T$. Then the push forward

$$i_* : H^*_T(F) \longrightarrow H^*_T(X)$$

is an isomorphism after localization. The inverse is given by

$$H^*_T(X) \longrightarrow H^*_T(F), \qquad \gamma \longmapsto \frac{\gamma|_F}{\mathrm{Eu}(\mathrm{Nm}_F X)}.$$

5.2.2. *Embedding.* We have an embedding

$$i_{\beta,0} : \overline{\mathcal{M}}_1(X,\beta) \longrightarrow G_0(X,\beta).$$

For two varieties $X$ and $Y$, we have

$$\begin{tikzcd}
G_0(X \times Y, (\beta_X, \beta_Y)) \arrow[r, "\text{birational}"] & G_0(X, \beta_X) \times G_0(Y, \beta_Y) \\
\overline{\mathcal{M}}_1(X \times Y, (\beta_X, \beta_Y)) \arrow[r, "\Pi"] \arrow[u, "i_{X\times Y}"'] \arrow[d, "\mathrm{ev}"'] & \overline{\mathcal{M}}_1(X, \beta_X) \times \overline{\mathcal{M}}_1(Y, \beta_Y) \arrow[u, "i_X \times i_Y"'] \arrow[d, "\mathrm{ev}"] \\
X \times Y \arrow[r] & X \times Y.
\end{tikzcd}$$

This implies

$$\Pi_*\left(\frac{1}{z(z-\psi)} \boxtimes \frac{1}{z(z-\psi)}\right) = \frac{1}{z(z-\psi)}.$$

This shows the J-function of the product is the product of J-functions.

5.2.3. *J-function of projective space.* Recall we have

$$\begin{tikzcd}
G(\mathbb{P}^N, d) \arrow[r, "\text{birational}"] & QM(\mathbb{P}^N, d) \\
\overline{\mathcal{M}}_1(X, d) \arrow[r, "\mathrm{ev}"] \arrow[u] & \mathbb{P}^N \arrow[u, "i"']
\end{tikzcd}$$

As a result,

$$\mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right) = \frac{1}{\mathrm{Eu}(i)}.$$

Recall

$$QM(\mathbb{P}^N, d) = \mathbb{P}(H^0(\mathbb{C}[x]_{\deg \le d})^{N+1}).$$

Note that $\mathbb{P}^N \subset QM(\mathbb{P}^N, d)$ is induced by

$$\mathbb{C}^{N+1} \simeq (\mathbb{C}x^d)^{N+1} \subset (\mathbb{C}[x]_{\deg \le d})^{N+1}.$$

So it is defined by

$$\text{coefficients of } x^0, \ldots, x^{d-1} \text{ of every } N+1 \text{ component} = 0.$$

So

$$\mathrm{Eu}(i) = \prod_{k=1}^{d} (H + kz).$$

As a result, we have

$$\tilde{J} = 1 + \sum_{d>1} \frac{q^d}{\prod_{k=1}^{d} (H + kz)}.$$

That is,

$$J = q^{H/z} \left( 1 + \sum_{d>1} \frac{q^d}{\prod_{k=1}^{d} (H + kz)^{N+1}} \right).$$

5.2.4. *Remark.* Let us compute

$$\partial_H J = \frac{H}{z} q^{H/z} + \sum_{d>1} \frac{(d + \frac{H}{z}) q^{d+H/z}}{\prod_{k=1}^{d} (H + kz)^{N+1}}.$$

Similarly,

$$\begin{aligned}(z\partial_H)^{N+1} J &= H^{N+1} q^{H/z} + \sum_{d>1} \frac{(H + dz)^{N+1} q^{d+H/z}}{\prod_{k=1}^{d} (H + kz)^{N+1}} \\ &= \sum_{d>1} \frac{q^{d+H/z}}{\prod_{k=1}^{d-1} (H + kz)^{N+1}} = qJ.\end{aligned}$$

So we have

$$H^{N+1} = q \qquad \text{(quantum product)}.$$

## 5.3. **Unitary property.**

5.3.1. *A twisted fundamental solution.* Let us denote

$$\mathcal{M}(\gamma, \gamma') = \langle \gamma, \gamma' \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes \gamma') \frac{1}{z - \psi_2}.$$

Let us denote the operator $M$ by

$$\langle M(\gamma), \gamma' \rangle = \mathcal{M}(\gamma, \gamma').$$

5.3.2. *Equation for* $M$. Then

$$\partial_\lambda \langle M(\gamma), \gamma' \rangle = \frac{1}{z} \langle M(\lambda * \gamma), M(\gamma') \rangle - \frac{1}{z} \langle \lambda M(\gamma), \gamma' \rangle.$$

Thus

$$\partial_\lambda M(\gamma) = \frac{1}{z} M(\lambda * \gamma) - \frac{1}{z} \lambda M(\gamma).$$

For general $f$, i.e. possibly involving quantum parameters,

$$\partial_\lambda M(f) = \frac{1}{z} M(\lambda * f) - \frac{1}{z} \lambda M(f) + M(\partial_\lambda f).$$

5.3.3. *Summary.* We have the following commutative diagram

$$\begin{tikzcd} H_{\mathbb{T}}(X) \arrow[r, "M(-{,}z)"] \arrow[d, "\partial_\lambda + \frac{1}{z}\lambda *"'] & H_{\mathbb{T}}(X)(q) \arrow[d, "\partial_\lambda + \frac{1}{z}\lambda"] \\ H_{\mathbb{T}}(X) \arrow[r, "M(-{,}z)"] & H_{\mathbb{T}}(X)(q) \end{tikzcd}$$

5.3.4. *Equation for inverse.* By substituting $f$ by $M^{-1}(f)$, we get

$$\partial_\lambda f = \frac{1}{z}M(\lambda * M^{-1}(f)) - \frac{1}{z}\lambda f + M(\partial_\lambda M^{-1}(f)).$$

Applying $M^{-1}$, we get

$$M^{-1}(\partial_\lambda f) = \frac{1}{z}\lambda * M^{-1}(f) - \frac{1}{z}M^{-1}(\lambda f) + \partial_\lambda M^{-1}(f).$$

That is,

$$\partial_\lambda M^{-1}(f) = -\frac{1}{z}\lambda * M^{-1}(f) + \frac{1}{z}M^{-1}(\lambda f) + M^{-1}(\partial_\lambda f).$$

5.3.5. *Equation for adjoint.* On the other hand, denote the operator $M'$ by

$$\langle \gamma, M'(\gamma') \rangle = M(\gamma, \gamma').$$

Then

$$\partial_\lambda \langle \gamma, M'(\gamma') \rangle = \frac{1}{z}\langle \gamma, \lambda * M'(\gamma') \rangle - \frac{1}{z}\langle \gamma, M'(\lambda\gamma') \rangle.$$

Thus

$$\partial_\lambda M'(\gamma') = \frac{1}{z}\lambda * M'(\gamma') - \frac{1}{z}M'(\lambda\gamma').$$

For general $f$, i.e. possibly involving quantum parameters,

$$\partial_\lambda M'(f) = \frac{1}{z}\lambda * M'(f) - \frac{1}{z}M'(\lambda f) + M'(\partial_\lambda f).$$

5.3.6. *Conclusion.* Let us denote $M(\gamma) = M(\gamma, z)$ to empathise the dependence of $z$. By comparing the differential equation, we have

$$M'(\gamma, z) = M^{-1}(\gamma, -z).$$

As a result, we have

$$\langle M(\gamma, z), M(\gamma', -z) \rangle = \langle \gamma, \gamma' \rangle.$$

In the rest of this section, we are going to give a geometric proof of this identity.

5.4. **Gromov–Witten invariant over graph space.**

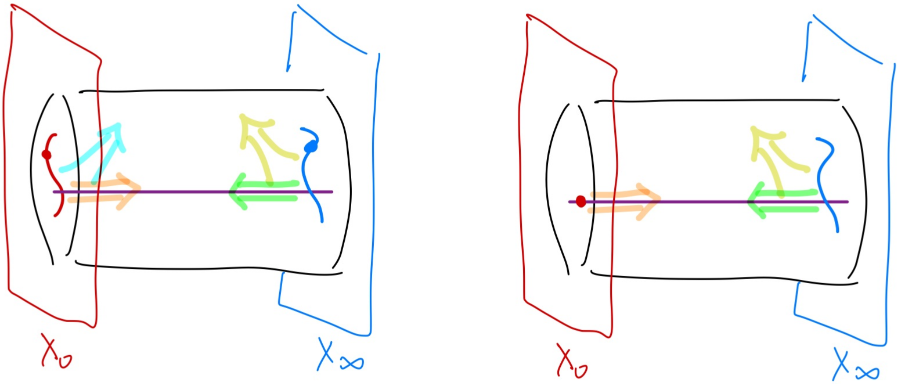
5.4.1. *A pairing.* Let us denote similarly
$$G_2(X,\beta)=\overline{\mathcal{M}}_2(\mathbb{P}^1\times X,(1,\beta)).$$
We define for $\gamma_1,\gamma_2\in H^*(X)$
$$G(\gamma_1,\gamma_2)=\langle\gamma_1,\gamma_2\rangle+\sum_{\beta>0}q^\beta\int_{G_2(X,\beta)}\mathrm{ev}^*(i_{0*}\gamma_1\boxtimes i_{\infty*}\gamma_2)$$
where $i_0:X\to\mathbb{P}^1\times X$ and $i_\infty:X\to\mathbb{P}^1\times X$ the inclusion of the fibre at $0$ and $\infty$ respectively.

5.4.2. *Remark.* Note that by 2.3.2, we have $G(\gamma_1,\gamma_2)=\langle\gamma_1,\gamma_2\rangle$.

5.4.3. *Components.* Let us use localization to compute this pairing. Let us denote for $\beta_1,\beta_2>0$
$$i_{\beta_1,\beta_2}:\overline{\mathcal{M}}_2(X,\beta_1)\times_X\overline{\mathcal{M}}_2(X,\beta_2)\longrightarrow G_0(X,\beta_1+\beta_2)$$
by gluing the second marked points. Similarly we define $i_{\beta,0}$ and $i_{0,\beta}$. Then
$$G_2(X,\beta)^{\mathbb{C}^\times}=(\cdots)\cup\bigcup_{\beta_1+\beta_2=\beta}\big(\text{image of } i_{\beta_1,\beta_2}\big).$$
Here $(\cdots)$ is the component does not contribute the pushforward.

5.4.4. *Dimension estimation.* Let us estimate the dimension. We have
$$\begin{aligned}\dim G_2(X,\beta)&=\dim X+1+\langle c_1(\mathcal{T}_X),\beta\rangle+\langle c_1(\mathcal{T}_{\mathbb{P}^1}),1\rangle+2-3\\&=\dim X+\langle c_1(\mathcal{T}_X),\beta\rangle+2.\end{aligned}$$
For $\beta_1,\beta_2>0$ with $\beta_1+\beta_2=\beta$,
$$\begin{aligned}\dim\overline{\mathcal{M}}_2(X,\beta_1)\times_X\overline{\mathcal{M}}_2(X,\beta_2)&=\dim X+\langle c_1(\mathcal{T}_X),\beta\rangle+2-3+2-3\\&=\dim X+\langle c_1(\mathcal{T}_X),\beta\rangle-2.\end{aligned}$$
On the other hand,
$$\begin{aligned}\dim\overline{\mathcal{M}}_2(X,\beta)&=\dim X+\langle c_1(\mathcal{T}_X),\beta\rangle+2-3\\&=\dim X+\langle c_1(\mathcal{T}_X),\beta\rangle-1.\end{aligned}$$

5.4.5. *Normal bundle.* Similarly, when $\beta_1,\beta_2>0$,
$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))=\text{restriction of } z(z-\psi)\otimes(-z(-z-\psi)).$$
When $\beta_2=0$, we do not need to smooth the marked point on $\infty$, so
$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta,0}))=\text{restriction of } z(z-\psi)\otimes(-z).$$
Similarly, when $\beta_1=0$,
$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta}))=\text{restriction of } z\otimes(-z(-z-\psi)).$$




5.4.6. *Localization.* When $\beta > 0$, using localization, we have

$$\int_{G_2(X,\beta)} \mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2) = \sum_{\beta_1+\beta_2=\beta} \int \frac{i^*_{\beta_1,\beta_2}(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}.$$

When $\beta_1, \beta_2 > 0$, we have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_2(X,\beta_1)\times_X\overline{\mathcal{M}}_2(X,\beta_2)} \frac{i^*_{\beta_1,\beta_2}(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}\\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times_X\overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(i_0^* i_{0*}\gamma_1 \boxtimes i_\infty^* i_{\infty*}\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))}\\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times\overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(i_0^* i_{0*}\gamma_1 \boxtimes i_\infty^* i_{\infty*}\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))}(\mathrm{ev}_2\boxtimes\mathrm{ev}_2)^*(\Delta_X)\\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times\overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(z\gamma_1 \boxtimes (-z)\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))}\sum_w(\mathrm{ev}_2\boxtimes\mathrm{ev}_2)^*(\sigma_w\boxtimes\sigma^w)\\
&= \sum_w \int_{\overline{\mathcal{M}}_2(X,\beta_1)} \frac{\mathrm{ev}^*(\gamma_1\boxtimes\sigma_w)}{z-\psi_2}\int_{\overline{\mathcal{M}}_2(X,\beta_2)}\frac{\mathrm{ev}^*(\gamma_2\boxtimes\sigma^w)}{-z-\psi_2}.
\end{aligned}$$

Here $\{\sigma_w\} \subset H^*(X)$ is a basis and $\{\sigma^w\} \subset H^*(X)$ is its dual basis. Similarly, when $\beta_2$, we have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i^*_{\beta,0}\left(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)\right)}{\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta}))} \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \frac{\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)}{-z-\psi_2} = \sum_w \int_{\overline{\mathcal{M}}_2(X,\beta_1)} \frac{\mathrm{ev}^*(\gamma_1 \boxtimes \sigma_w)}{z-\psi_2} \langle \gamma_2, \sigma^w \rangle, \\
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i^*_{0,\beta}\left(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)\right)}{\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta}))} = \sum_w \langle \gamma_1, \sigma_w \rangle \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \frac{\mathrm{ev}^*(\gamma_2 \boxtimes \sigma^w)}{-z-\psi_2}.
\end{aligned}$$

5.4.7. *Conclusion.* As a result,

$$\begin{aligned}
\langle \gamma_1, \gamma_2 \rangle = G(\gamma_1, \gamma_2) &= \sum_w \mathcal{M}(\gamma_1, \sigma_w)\mathcal{M}(\gamma_2, \sigma^w)|_{z \mapsto -z} \\
&= \sum_w \langle M(\gamma_1, z), \sigma_w \rangle \langle M(\gamma_2, -z), \sigma^w \rangle \\
&= \langle M(\gamma_1, z), M(\gamma_2, -z) \rangle.
\end{aligned}$$

# 6. Shift operators

## 6.1. Shift operator.

6.1.1. *Setup.* Assume $T$ acts on $X$. We are going to define a family of operators for any $k \in 1\mathsf{PS}(T)$. Let $\mathbb{T} = T \times \mathbb{C}^\times$. We denote $z$ the canonical generator in $H^2_{\mathbb{C}^\times}(\mathsf{pt})$.

6.1.2. *Twisted action.* For any $k \in 1\mathsf{PS}(T)$, we have a twisted $\mathbb{T}$-action by

$$\rho_k(t, u) \cdot x = t \cdot k(u) \cdot x.$$

We have

$$\begin{tikzcd} H^*_{\mathbb{T}}(X, \rho_0) \arrow[r, "\sim"] & H^*_{\mathbb{T}}(X, \rho_k) \\ H^*_{\mathbb{T}}(\mathsf{pt}) \arrow[u] \arrow[r, "\lambda \mapsto \lambda + \langle k, \lambda \rangle z", "\sim"'] & H^*_{\mathbb{T}}(\mathsf{pt}) \arrow[u] \end{tikzcd}$$

Let us denote the isomorphism by $\gamma \mapsto \gamma[k]$.

6.1.3. *Bundle.* Let us denote

$$E_k = \big(\mathbb{C}^2 \setminus \{0\}\big) \underset{\mathbb{C}^\times}{\times} X,$$

with the action induced by $k$. Then $\mathbb{T}$ acts on $E_k$. We have a projection

$$\pi : E_k \to \big(\mathbb{C}^2 \setminus \{0\}\big)/\mathbb{C}^\times = \mathbb{P}^1$$

with

$$\pi^{-1}(0) \simeq (X, \rho_0) =: X_0, \qquad \pi^{-1}(\infty) \simeq (X, \rho_k) =: X_\infty.$$

6.1.4. *Section class.* Let us denote

$$\mathrm{Eff}(E_k)_{\mathrm{sec}} = \text{preimage of } [\mathbb{P}^1 \xrightarrow{\mathrm{id}} \mathbb{P}^1] \in \mathrm{Eff}(\mathbb{P}^1) \text{ under } \pi_* : \mathrm{Eff}(E_k) \to \mathrm{Eff}(\mathbb{P}^1).$$

6.1.5. *Shift operator.* Let us define

$$\iota_0 : X_0 \to E_k, \qquad \iota_\infty : X_\infty \to E_k.$$

Let us define the *shifted operator*

$$\tilde{\mathbb{S}}_k : H^*_{\mathbb{T}}(X, \rho_0) \longrightarrow H^*_{\mathbb{T}}(X, \rho_k)$$

by

$$\langle \tilde{\mathbb{S}}_k(\gamma), \gamma'[k] \rangle = \sum_{\tilde{\beta} \in \mathrm{Eff}(E_k)_{\mathrm{sec}}} q^{\tilde{\beta}} \int_{\overline{\mathcal{M}}_2(E_k, \tilde{\beta})} \mathrm{ev}^*(\iota_{0*}\gamma, \boxtimes \iota_{\infty*}\gamma'[k]).$$

Let us use localization to compute $\tilde{\mathbb{S}}_k$.

6.1.6. *Example.* When $k = 0$, then

$$E_k = \mathbb{P}^1 \times X.$$

Applying the same trick to $\mathbb{C}^\times$ fixed locus as in the previous section, we get

$$\langle \tilde{\mathbb{S}}_k(\gamma), \gamma' \rangle = \langle M(\gamma, z), M(\gamma', -z) \rangle = \langle \gamma, \gamma' \rangle.$$

Thus $\tilde{\mathbb{S}}_0 = \mathrm{id}$. In general, we have to consider the T-fixed locus.

6.1.7. *Fixed locus.* Let $F \in \pi_0(X^T)$ be a connected component of $X^T$. We denote $\sigma_F \in \mathrm{Eff}(E_k)$ to be the class of $\sigma_x$ for any $x \in F$. For $\beta_1, \beta_2 > 0$, let us denote

$$\overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2) = (\mathrm{ev}_2 \boxtimes \mathrm{ev}_2)^{-1}(\Delta_F)$$

the space of stable maps with the second marked points the same in $F$. For $(C_1, C_2)$ in this space with $\mathrm{ev}_2(C_1) = \mathrm{ev}_2(C_2) = x \in F$, by gluing $\sigma_x \subset E_k$, we have a $\mathbb{T}$-invariant stable maps over $E_k$. This defines

$$i_{\beta_1,\beta_2} : \overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2) \longrightarrow \overline{\mathcal{M}}_2(E_k, i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F).$$

It induces

$$\overline{\mathcal{M}}_2(X_0, \beta_1)^T \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2)^T \longrightarrow \overline{\mathcal{M}}_2(E_k, i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F)^{\mathbb{T}}.$$

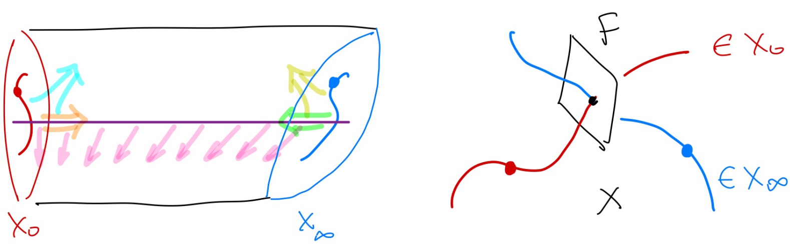
We similarly denote

$$i_{\beta_1,0}, i_{0,\beta_2} : \overline{\mathcal{M}}_2(X_0, \beta) \cap \mathrm{ev}_2^{-1}(F) \longrightarrow \overline{\mathcal{M}}_2(E_k, \beta_1 + \sigma_F).$$

We have the following decomposition

$$\overline{\mathcal{M}}_2(E_k, \tilde{\beta})^{\mathbb{T}} = (\cdots) \cup \bigcup_{i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F = \tilde{\beta}} \text{image of } i_{\beta_1,\beta_2}.$$

Here $(\cdots)$ are those components not in $\mathrm{ev}^{-1}(X_0 \times X_\infty)$, which does not contribute the integral.



6.1.8. *Computation.* Let us compute the normal bundle of

$$\overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2).$$

It contains the fixed component. Denote $\xi$ the natural representation of $\mathbb{C}^\times$.

$$(\text{smoothing the gluing point at } 0) = (\mathbb{L}_2^{-1} \otimes \xi) \boxtimes \mathcal{O}.$$
$$(\text{moving the gluing point at } 0) = \xi \boxtimes \mathcal{O} = \xi.$$

Similarly for the gluing point at $\infty$

$$(\text{smoothing the gluing point at } \infty) = \mathcal{O} \boxtimes (\mathbb{L}_2^{-1} \otimes \xi^{-1}).$$
$$(\text{moving the gluing point at } \infty) = \mathcal{O} \boxtimes \xi^{-1} = \xi^{-1}.$$

Thus the Euler class

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2})) = z(z - \psi_2) \otimes (-z(-z - \psi_2)).$$

When $\beta_1 = 0$, the computation will be different. Now $0$ is a marked point, so we do not need to smooth it. The Euler class

$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta_2})) = z \otimes (-z(-z - \psi_2)).$$

Similarly for $\beta_2 = 0$,

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,0})) = z(z - \psi) \otimes (-z).$$

6.1.9. *Lemma.* The normal bundle of $F \times \mathbb{P}^1$ is

$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda \boxtimes \mathcal{O}_{\mathbb{P}^1}(-\langle\lambda, k\rangle),$$

where $(\mathrm{Nm}_F X)_\lambda = \mathrm{Hom}_T(\mathbb{C}_\lambda, \mathrm{Nm}_F X)$. Actually, it is characterized by (as $\mathbb{C}^\times$-equivariant bundles)

$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k|_{F\times 0} = \mathrm{Nm}_F X_0 = \mathrm{Nm}_F X = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda$$
$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k|_{F\times\infty} = \mathrm{Nm}_F X_\infty = (\mathrm{Nm}_F X)[k] = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda(\langle\lambda, k\rangle z).$$

6.1.10. *Moving the horizontal cruve.* Now let us compute the part of moving the horizontal curve. We have

$$\begin{aligned}&(\text{moving the horizontal curve})\\ &= (\text{moving to be non-constant inside } F) \oplus (\text{moving out of } F)\end{aligned}$$

Note that

$$(\text{moving to be non-constant inside } F) = \mathrm{Mor}(\mathbb{P}^1, H^0(F, \mathcal{T}_F))/\text{constant} = 0.$$

Note that

$$(\text{moving out of } F) = \bigoplus_{\lambda\in\mathsf{char}(T)} \mathrm{ev}^*(\mathrm{Nm}_F X)_\lambda \cdot \chi\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-\langle\lambda,k\rangle)\big)$$

where $\mathrm{ev} = \mathrm{ev}_2 \boxtimes 1 = 1 \boxtimes \mathrm{ev}_2$. Here $(\mathrm{Nm}_F X)_\lambda$ has trivial $\mathbb{C}^\times$-action, so $\mathrm{ev}^*$ induced by two maps do not differ. By localization theorem, we have

$$\chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) = \frac{1-\xi^{-i-1}}{1-\xi^{-1}} = \sum_{c\le 0}\xi^c - \sum_{c<-i}\xi^c.$$

So

$$(\text{moving the horizontal curve}) = \bigoplus_{\lambda\in\mathsf{char}(T)} \mathrm{ev}^*(\mathrm{Nm}_F X)_\lambda \cdot \left(\sum_{c\le 0}\xi^c - \sum_{c<\langle\lambda,k\rangle}\xi^c\right).$$

Note that its Euler class is

$$\prod_{\lambda\in\mathsf{char}(T)} \prod_{x\in\sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c\le 0}(\mathrm{ev}^* x+\lambda+cz)}{\prod_{c<\langle\lambda,k\rangle}(\mathrm{ev}^* x+\lambda+cz)} = (\mathrm{ev}_2\boxtimes 1)^*(\cdots),$$

$$= (\mathrm{ev}_2\boxtimes 1)^*\left(\prod_{\lambda\in\mathsf{char}(T)} \prod_{x\in\sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c\le 0}(x+\lambda+cz)}{\prod_{c<\langle\lambda,k\rangle}(x+\lambda+cz)}\right) =: (\mathrm{ev}_2\boxtimes 1)^*(\cdots)$$

where $\sqrt{(\mathrm{Nm}_F X)_\lambda}$ means the Chern roots of the bundle.

6.1.11. *Computation.* Now, let us evaluate

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_2(E_k,\tilde\beta)} \mathrm{ev}^*(\iota_{0*}\gamma, \boxtimes\iota_{\infty*}\gamma'[k])\\
&= \sum_{\beta_1,\beta_2,F}\int_{\overline{\mathcal{M}}_2(X_0,\beta_1)\times_F\overline{\mathcal{M}}_2(X_\infty,\beta_2)} \frac{(\mathrm{ev}_1\boxtimes\mathrm{ev}_1)^*(\iota_0^*\iota_{0*}\gamma\boxtimes\iota_\infty^*\iota_{\infty*}\gamma'[k])}{\mathrm{Nm}(\cdots)}\\
&= \sum_{\beta_1,\beta_2,F}\int_{\overline{\mathcal{M}}_2(X_0,\beta_1)\times\overline{\mathcal{M}}_2(X_\infty,\beta_2)} \frac{(\iota_0^*\iota_{0*}\gamma\boxtimes\iota_\infty^*\iota_{\infty*}\gamma'[k])}{\mathrm{Nm}(\cdots)}(\mathrm{ev}_2\boxtimes\mathrm{ev}_2)^*(\Delta_F)\\
&= \sum_{\beta_1,\beta_2,F}\sum_{w}\int_{\overline{\mathcal{M}}_2(X_0,\beta_1)} \frac{z\,\mathrm{ev}^*(\gamma\boxtimes i_{F*}\sigma_w^F)}{z(z-\psi_1)}\cdot\mathrm{ev}_2^*\frac{1}{(\cdots)}\int_{\overline{\mathcal{M}}_2(X_\infty,\beta_2)}\frac{-z\,\mathrm{ev}^*(\gamma'[k]\boxtimes i_{F*}\sigma_F^w)}{-z(-z-\psi_1)}\\
&= \sum_{\beta_1,\beta_2,F}\sum_{u,w}\int_{\overline{\mathcal{M}}_2(X_0,\beta_1)} \frac{z\,\mathrm{ev}^*(\gamma\boxtimes i_{F*}\sigma_w^F)}{z(z-\psi_2)}\int_{\overline{\mathcal{M}}_2(X_\infty,\beta_2)}\frac{-z\,\mathrm{ev}^*(\gamma'[k]\boxtimes i_{F*}\sigma_F^u)}{-z(-z-\psi_2)}\int_F\frac{\sigma_F^w\sigma_u^F}{(\cdots)}
\end{aligned}$$

Here we omit the summand of $\beta_1,\beta_2=0$. Here we assume

$$[\Delta_F] = \sum_w \sigma_w^F\boxtimes\sigma_F^w \in H^*(F)\subset H_T^*(X).$$

We find

$$\begin{aligned}
\langle \tilde{\mathbb{S}}_k(\gamma), \gamma'[k]\rangle &= \sum_F q^{\sigma_F} \sum_{w,u} \langle M(\gamma,z), i_{F*}\sigma^F_w\rangle \langle M(\gamma',-z), i_{F*}\sigma^u_F\rangle[k] \int_F \frac{\sigma^w_F \sigma^F_u}{(\cdots)} \\
&= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \sum_w i_{F*}\sigma^F_w \int_F \frac{\sigma^w_F\sigma^F_u}{(\cdots)} \right\rangle \langle M(\gamma',-z), i_{F*}\sigma^u_F\rangle[k] \\
&= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \frac{i_{F*}\sigma^F_u}{(\cdots)} \right\rangle \langle M(\gamma',-z), \sigma^u_F\rangle[k].
\end{aligned}$$

We have

$$\begin{aligned}
\langle (\tilde{\mathbb{S}}_k(\gamma))[-k], \gamma'\rangle &= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \frac{i_{F*}\sigma^F_u}{(\cdots)} \right\rangle [-k]\langle M(\gamma',-z), \sigma^u_F\rangle \\
&= \sum_u \left\langle M(\gamma,z)[-k], \sum_F q^{\sigma_F} \frac{i_{F*}\sigma^F_u}{(\cdots)}[-k] \right\rangle \langle M(\gamma',-z), \sigma^u_F\rangle
\end{aligned}$$

Let us compute

$$\frac{i_{F*}\sigma^F_u}{(\cdots)}[-k] = \frac{i_{F*}\sigma^F_u}{\mathrm{Eu}(\mathrm{Nm}_F X)} \prod_{\lambda\in\mathrm{char}(T)} \prod_{x\in\sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c\le 0}(x+\lambda+cz)}{\prod_{c\le -\langle\lambda,k\rangle}(x+\lambda+cz)}.$$

Let us denote

$$\Delta_F = \prod_{\lambda\in\mathrm{char}(T)} \prod_{x\in\sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c\le 0}(x+\lambda+cz)}{\prod_{c\le -\langle\lambda,k\rangle}(x+\lambda+cz)}.$$

Note that $\{i_{F*}\sigma^u_F\}$ is dual to $\left\{\frac{i_{F*}\sigma^F_u}{\mathrm{Eu}(\mathrm{Nm}_F X)}\right\}$, so

$$\langle (\tilde{\mathbb{S}}_k(\gamma))[-k], \gamma'\rangle = \left\langle \sum_F q^{\sigma_F}\Delta_F M(\gamma,z)[-k], M(\gamma',-z)\right\rangle.$$

By 5.3.6,

$$(\tilde{\mathbb{S}}_k(\gamma))[-k] = M^{-1}\left(\sum_F q^{\sigma_F}\Delta_F\cdot M(\gamma,z)[-k], z\right).$$

6.1.12. *Summary.* Let us denote $\mathbb{S}_k$ by

$$\mathbb{S}_k(\gamma) = (\tilde{\mathbb{S}}_k\gamma)[-k].$$

We have the following commutative diagram

$$\begin{tikzcd} H_{\mathbb{T}}(X) \arrow[r, "M(-,z)"] \arrow[d, "\mathbb{S}_k"'] & H_{\mathbb{T}}(X)(q) \arrow[d, "\gamma\mapsto\bigoplus_F q^{\sigma_F}\Delta_F(\gamma[-k])"] \\ H_{\mathbb{T}}(X)(q) \arrow[r, "M(-,z)"] & H_{\mathbb{T}}(X)(q) \end{tikzcd}$$

6.1.13. *Corollary.* We have

$$\mathbb{S}_k \circ \mathbb{S}_\ell = q^{(\cdots)}\mathbb{S}_{k+\ell}.$$

Since M is non-degenerate, this reduces to the following easy identity

$$\Delta_F^\ell \cdot \Delta_F^k[-\ell] = \Delta_F^{k+\ell}.$$

6.1.14. *Seidel element.* Define

$$S_k = \lim_{z\to 0} \mathbb{S}_k(1) \in QH_T^*(X).$$

Note that

$$\Big[z\partial_\lambda + \lambda, \sum_F q^{\sigma_F}\Delta_F\Big] = z\sum_F (\partial_\lambda q^{\sigma_F})\Delta_F = o(z).$$

So

$$\big[\mathbb{S}_k, z\nabla_\lambda + \lambda *\big] = o(z).$$

Then by taking $z \to 0$, we see $\lim_{z\to 0}\mathbb{S}_k$ commutes with the quantum product with a divisor. When $H_T^*(X)$ is generated by divisor (after localization), it is given by the quantum product with $S_k$.

6.1.15. *Remark.* When $z = 1$, we can write $\Delta_F$ in terms of Gauss Gamma function

$$\Gamma(s) = \int_0^\infty t^s e^{-t}\frac{dt}{t}.$$

Recall that

$$\Gamma(s+1) = s\Gamma(s).$$

So when $a, b \in \mathbb{Z}$

$$\begin{aligned} \frac{\Gamma(s+a+1)}{\Gamma(s+b+1)} &= \frac{(s+a)\Gamma(s+a)}{(s+b)\Gamma(s+b)} = \cdots \\ &= \frac{(s+a)\cdots(s+c)\Gamma(s+c)}{(s+b)\cdots(s+c)\Gamma(s+c)} = \frac{\prod_{c\le a}(s+c)}{\prod_{c\le b}(s+c)}. \end{aligned}$$

As a result,

$$
\begin{aligned}
\Delta_F|_{z=1} &= \prod_{\lambda \in \mathrm{char}(T)} \prod_{x \in \sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\Gamma(x+\lambda+1)}{\Gamma(x+\lambda-\langle\lambda,k\rangle+1)} \\
&= \frac{\prod_{x \in \sqrt{\mathrm{Nm}_F(X,\rho_k)}} \Gamma(x+1)}{\prod_{x \in \sqrt{\mathrm{Nm}_F(X,\rho_0)}} \Gamma(x+1)} [-k] \\
&=: \frac{\Gamma\big(1+\mathrm{Nm}_F(X,\rho_k)\big)}{\Gamma\big(1+\mathrm{Nm}_F(X,\rho_0)\big)} [-k].
\end{aligned}
$$