Bumpless Pipe Dreams meet Puzzles

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1 Introduction

Knutson and Zinn-Justin recently found a puzzle rule for the expansion of the product $\mathfrak{G}_u(x,t)\cdot\mathfrak{G}_v(x,t)$ of two double Grothendieck polynomials indexed by permutations with separated descents. We establish its **triple Schubert calculus version** in the sense of Knutson and Tao, namely, a formula for expanding $\mathfrak{G}_u(x,y)\cdot\mathfrak{G}_v(x,t)$ in different secondary variables. Our rule is formulated in terms of pipe puzzles, incorporating both the structures of bumpless pipe dreams and classical puzzles. As direct applications, we recover the **separated-descent puzzle** formula by Knutson and Zinn-Justin (by setting $y=t$) and the **bumpless pipe dream** model of double Grothendieck polynomials by Weigandt (by setting $v=\mathrm{id}$ and $x=t$). Moreover, we utilize the formula to partially confirm a positivity conjecture of Kirillov about applying a skew operator to a Schubert polynomial.

2 Bumpless Pipe Dreams

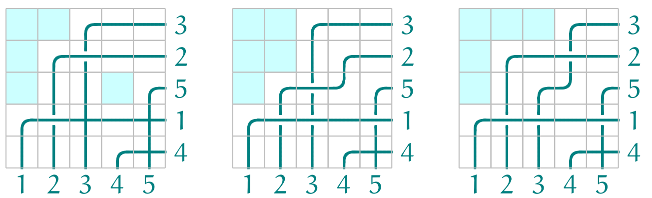
Double schubert polynomials $\mathfrak{S}_w$ for $w\in S_\infty$ are characterized by

$$\mathfrak{S}_{n\cdots 21}=\prod_{i+j\le n}(x_i-t_j);$$

$$\mathfrak{S}_{ws_i}=\frac{\mathfrak{S}_w-\mathfrak{S}_w|_{x_i\leftrightarrow x_{i+1}}}{x_i-x_{i+1}},\qquad \text{if } w(i)>w(i+1).$$

They are the stable representatives of Schubert varieties in torus equivariant cohomology of flag varieties.

In [4], Lam, Lee and Shimozono found a (new) combinatorial model of computing it, known as **bumpless pipe dream**. Here is an example



So

$$\begin{aligned}\mathfrak{S}_{32514}(x,t)=&(x_1-t_1)(x_1-t_2)(x_2-t_1)(x_3-t_1)(x_3-t_4)\\&+(x_1-t_1)(x_1-t_2)(x_2-t_1)(x_2-t_2)(x_3-t_1)\\&+(x_1-t_1)(x_1-t_2)(x_1-t_3)(x_2-t_1)(x_3-t_1).\end{aligned}$$

3 Puzzles

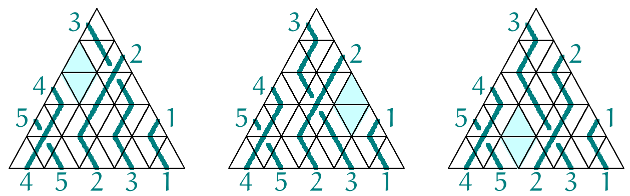
We would like to consider the **Littlewood–Ricchardson coefficients** c_{uv}^w in the expansion

$$\mathfrak{S}_u\cdot\mathfrak{S}_v=\sum_w c_{uv}^w(t)\cdot\mathfrak{S}_w.$$

Recently, Knutson and Zinn-Justin [3] found a model for permutations of separated descents:

$$\max(\mathrm{des}(u))\le k\le\max(\mathrm{des}(v)).$$

For example,



$$c_{42135,14532}^{53412}(t)=(t_4-t_1)+(t_5-t_3)+(t_3-t_2)$$

4 Pipe Puzzles

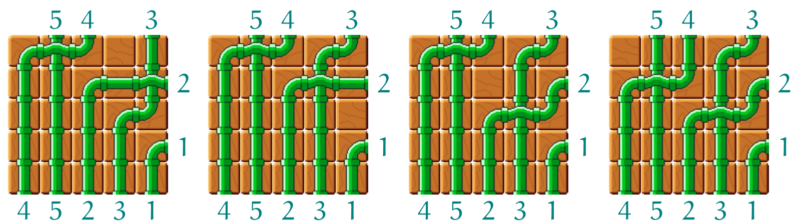
Our main result generalizes the aforementioned results (and also their K-theory analogy). We are considering the **triple version** of Littlewood–Richardson coefficients

$$\mathfrak{S}_u(x,y)\cdot\mathfrak{S}_v(x,t)=\sum_w c_{uv}^w(y,t)\cdot\mathfrak{S}_w(x,t).$$

Assume u,v with separated descents

$$\max(\mathrm{des}(u))\le k\le\max(\mathrm{des}(v)).$$

Here is an example,

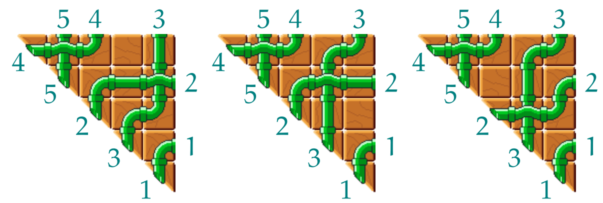


$$c_{42135,14532}^{53412}=(t_4-y_1)+(t_5-y_3)+(t_3-y_2)+(t_1-y_1).$$

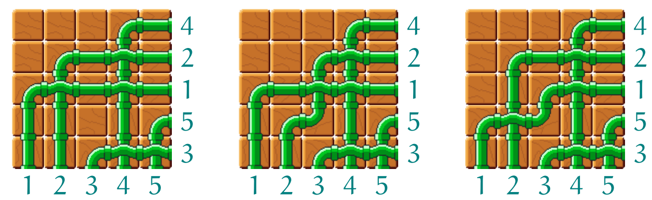
Our strategy is to check the model enjoys the same recurrence relations and initial condition. The proof follows from a classical trick of colored 6 vertex model. We remark that the above mentioned recurrence relations are no longer available in the case $y=t$. This means **in some sense that while the problem of computing triple Schubert structure constants is broader, its proof could be simpler**.

5 Applications

The case $y=t$ If we specialize $y=t$, we will recover the separated descents puzzle mentioned above.



The case $k=n$ In this case, $u\in S_n$ is arbitrary and v has to be id. We will recover the model of bumpless pipe dream.



The case $t=0$ In this case, it partially confirms the **conjecture** by Kirillov [1] which is equivalent to

$$\mathfrak{S}_u(x)\mathfrak{S}_v(x,t)\in\sum_w\mathbb{Z}_{\ge0}[t]\cdot\mathfrak{S}_w(x,t).$$

References

[1] A. Kirillov, Skew divided difference operators and Schubert polynomials, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007), Paper 072, 14.

[2] A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. 119 (2003), 221–260.

[3] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability III: separated descents, 2023, arXiv:2306.13855.

[4] T. Lam, S.J. Lee and M. Shimozono, Back stable Schubert calculus, Compos. Math. 157 (2021), 883–962.