The cohomology of a smooth projective toric variety is a permutation representation
(joint with Tao Gui, Chushi Qin, Kaizhe Shen)

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The Question by Stanley

Question (Stanley [Ste, Question 11.1])

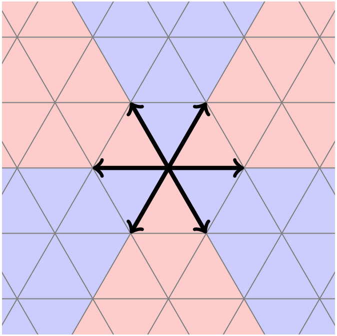
Let G act on a complete simplicial fan Σ through an action on the lattice. Is the G-representation $H^*(X_\Sigma)$ a permutation representation?

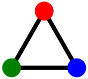
- Stembridge [Ste] showed this for Weyl group actions on Weyl fans, using case-by-case analysis.
- Browder [Bro] proved this for cyclic groups, under the properness assumption.

J. R. Stembridge, *Some permutation representations of Weyl groups*
Adv. Math. **106** (1994), no. 2, 244–301.

J. Browder, *Representations of cyclic groups acting on complete simp*
J. Combin. Theory Ser. A **117** (2010), no. 6, 766–775.

Example





This toric variety is a

- permutohedral variety
- Hessenberg variety
- del Pezzo surface...

It is a blow-up of 3 points over $\mathbb{P}^2$.
The symmetric group S_3 acts on it.

On cohomology,

- S_3 permutes the three exceptional fibers.
- S_3 fixes the cycles from $\mathbb{P}^2$, as $GL_3 \supset S_3$ is connected.

Thus it gives a permutation representation.

Permutation representations

A G-representation V is called a **permutation representation** of G if there is a finite G-set S with

$$V \simeq \mathsf{Map}(S, \mathbb{C}).$$

Example

For the cyclic group $\mathbb{Z}/3\mathbb{Z}$,

$$\mathbf{1} \text{ is a perm-rep}, \qquad \mathbf{1} \oplus \zeta \oplus \zeta^2 \text{ is a perm-rep},$$

$$\mathbf{1} \oplus \zeta \text{ is NOT a perm-rep}.$$

Permutation representations are very restricted, for example, it has to be self-dual, and its character

$$\chi_V(g) = |S^g| := \#\{s \in S : gs = s\} \geq 0.$$

Characterization for Cyclic Groups

The characterization of permutation representations for cyclic groups is included as a part of the theory of cyclic sieving phenomenon.

Theorem (Alexandersson–Amini [AA])

For cyclic groups $G = \langle g \rangle$, a representation V is permutation representation if and only if

$$\sum_{i|k} \mu(k/i)\chi_V(g^i) \geq 0, \qquad \forall k \geq 1.$$

Per Alexandersson and Nima Amini. The cone of cyclic sieving
Discrete Mathematics, 342(6):1581–1601, 2019.

Character Formulas

Assume a reductive G acts on a smooth projective variety X.

Character Formula

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(g \mid H^i(X)) = \chi(X^g).$$

This is a version of Atiyah–Bott.

Example

When G is a torus, we get $\chi(X) = \chi(X^g)$, this follows from equivariant formality and localization theorem.

Example

When G is finite, this also follows from the existence of equivariant triangulation of X.

Sketch of the proof

It follows from two ways of computing

$$\int_{X\times X}[\Delta_X]\cdot[\Gamma_g]$$

where Δ_X (resp. Γ_g) is the diagonal (resp. graph of g) in $X\times X$.

- On one hand, picking a basis and the dual basis of $H^*(X;\mathbb{Q})$, explicit computation shows this number is exactly the LHS.
- On the other hand, we can view X^g as the scheme-theoretic intersection $\Delta_X\cap\Gamma_g$, and excess intersection formula implies this number is exactly the RHS.

Our Theorem

Theorem (Gui–Qin–Shen–X.)

Let G act on a fan Σ through a linear action on the lattice. When X_Σ is smooth projective, then $H^*(X_\Sigma)$ is a permutation representation of G.

- The original conjecture requires a very strong constrain that the G-action is **proper**: the g-action on any g-stable cone is trivial, i.e. $g|_\sigma = \mathrm{id}$ if $g\sigma = \sigma$ for $\sigma \in \Sigma$. At the same time, the conjecture was made for X_Σ complete, rational smooth.
- By our discussion, we need to find a G-set S such that

$$|S^g| = \chi(X_\Sigma^g) \text{ and in particular } |S| = \chi(X_\Sigma) = |\Sigma_{\max}|.$$

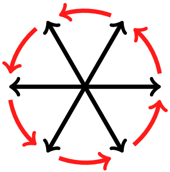
In some sense, we should take S to be "$X_\Sigma(\mathbb{F}_1)$".

Remarks

- Being a permutation representation does not depend on the field — by a standard Noether–Deuring argument:

$$M \simeq N \text{ as } R\text{-modules} \iff M_{\mathbb{C}} \simeq N_{\mathbb{C}} \text{ as } R_{\mathbb{C}}\text{-modules}.$$

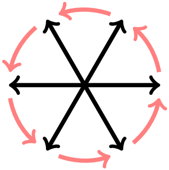
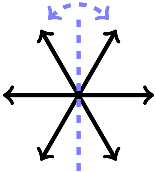
- It is necessary to take the ungraded representation. For example, the 60°-rotation of the A_2-fan:



$$\mathrm{Tr}\left(g \mid H^2(X_\Sigma)\right) = -1$$
$$\mathrm{Tr}\left(g \mid H^*(X_\Sigma)\right) = 1$$

Example

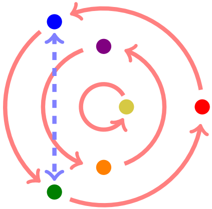
Let us take the A_2-fan as an example.

$r =$  $s =$ 

The following set does the job:

rotations	Tr
1	6
g, g^5	1
g^2, g^4	3
g^3	4

reflections	Tr
s, sg^2, sg^4	4
sg, sg^3, sg^5	2



Being Permutation Representations

Assume that G acts by algebra automorphisms on a finite-dimensional commutative $\mathbb{C}$-algebra R.

Principle

If R is semisimple, then R is a permutation representation of G.

Proof. The spectrum $\operatorname{Spec} R$ is a reduced G-scheme. □

This generalizes the normal basis theorem for Galois extensions.

Note that the cohomology ring $H^*(X_\Sigma)$ is **never semisimple**, since any element of positive degree is nilpotent.

To apply the Principle, we shall deform the cup product.

Some Candidates

There are at least two ways of deforming cohomology to a semisimple one.

Equivariant Cohomology

Equivariant cohomology $H_T^*(X)$ is a deformation parametrized by equivariant parameters from characters $X^*(T)$ of T.

If X has isolated T-fixed points, the generic fiber is semisimple.

Quantum Cohomology

Quantum cohomology $QH^*(X)$ is a deformation parametrized by quantum parameters from curve classes $\mathrm{Eff}(X)$ over X.

Quantum cohomology is typically semisimple (generic fiber).

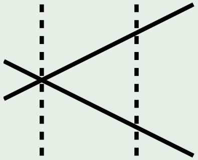
Example

Cohomology

$$H^*(\mathbb{P}^1) = \mathbb{C}[x]/\langle x^2\rangle.$$

Equivariant Cohomology

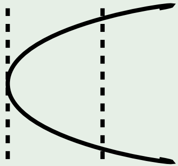
$$H_T^*(\mathbb{P}^1) = \mathbb{C}[x,t]/\langle x^2 - t^2\rangle$$



$t = 0$ $t \neq 0$

Quantum Cohomology

$$QH^*(\mathbb{P}^1) = \mathbb{C}[x,q]/\langle x^2 - q\rangle$$



$q = 0$ $q \neq 0$

Deforming Representations

Lemma

Let B be a connected complex scheme with trivial G-action. Let $\mathcal{E}$ be a G-equivariant vector bundle on B. Then all closed fibers of $\mathcal{E}$ are isomorphic as G-representations.

Proof. This is because the isotypic component $\mathcal{E}_\chi$ for any irreducible representation χ of G is a direct summand of $\mathcal{E}$ hence again a vector bundle on the connected base B. □

When proving

$$H^*(G/B) = \text{coinvariant ring}$$

is a regular representation of the Weyl group W, the equivariant cohomology provides a desired deformation.

Quantum Cohomology

In our case, we are not able to use the equivariant cohomology since G acts nontrivially on the equivariant parameters.

So we will use the quantum cohomology.

Note that G also acts nontrivially on $\mathrm{Eff}(X_\Sigma)$. But since the G-action is algebraic, we can choose a G-invariant specialization of quantum parameters. That is, we can always pick a G-equivariant ample divisor by averaging an ample divisor (here we use the projectivity).

Two Obstructions

However, there are two obstructions to making this argument work in general:

- semisimplicity is sensitive under specialization, a specific choice of G-invariant direction need not meet the semisimple locus;
- in general, even in the Fano case, the semisimplicity of the small quantum cohomology is not generally true;

see [OT] for explicit examples.

$QH^*(X_\Sigma)$ is not sufficiently semisimple.

Y. Ostrover and I. Tyomkin, *On the quantum homology algebra of*
Selecta Math. (N.S.) **15** (2009), no. 1, 121–149.

Mirror Symmetry

We need to deform further — That is how

Mirror Symmetry

comes into the picture.

A-side		B-side
quantum cohomology of T-toric varieties	$\longleftrightarrow$	Jacobian algebra of potential on $T^\vee$

Theorem (Givental [Giv])

This is true in the Fano case.

A. B. Givental, *A mirror theorem for toric complete intersections*, *Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996)*, Progr

Jacobian algebra

For a function $f \in \mathbb{C}[x_1, \ldots, x_n]$, the Jacobian algebra (a.k.a. Milnor algebra) is

$$M_f = \mathbb{C}[x_1, \ldots, x_n] / \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle .$$

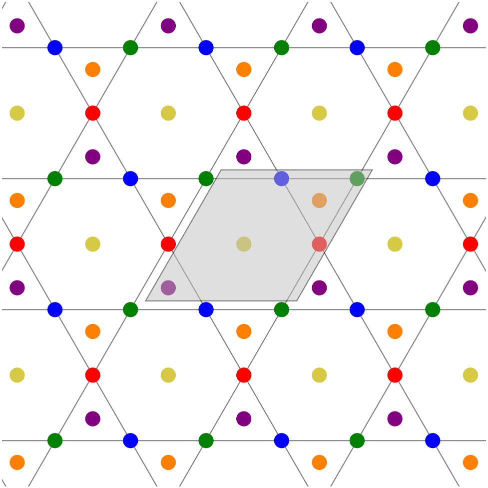
Clearly, M_f is the coordinate ring of the scheme of critical points.

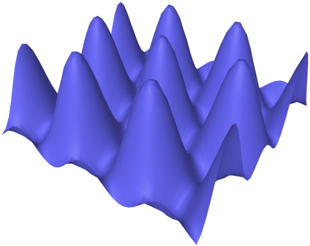
Theorem (Morse)

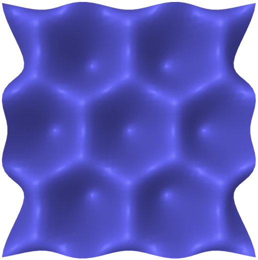
Every function with isolated critical points can be approximated by Morse functions (i.e. the critical points are non-degenerate).

For a Morse function, M_f is semisimple.

Example

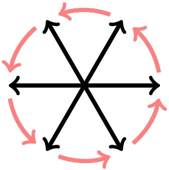
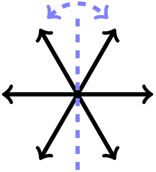
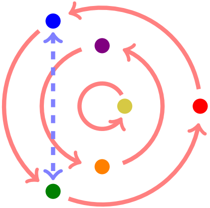




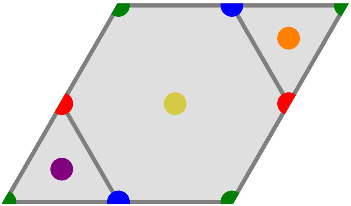


Example

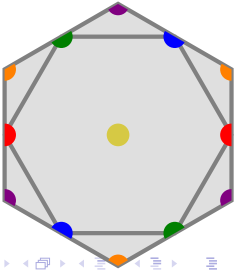
Recall

$r =$  $s =$  $S =$ 

Compare:



or equivalently



Equivariant Morse Approximation

The question is, can we do the Morse approximation equivariantly? The answer is yes if the G-action is **real**.

Theorem (Roberts [Rob])

Assume we have an $\mathbb{R}G$-module V, then any G-invariant $f \in \mathbb{C}[V_{\mathbb{C}}]$ admits a G-equivariant Morse approximation.

Corollary

Under this condition, M_f is a permutation representation of G.

M. Roberts, *Equivariant Milnor numbers and invariant Morse appr*
J. London Math. Soc. (2) **31** (1985), no. 3, 487–500.

Example

Let $G = \mathbb{Z}/3\mathbb{Z}$ and ζ a primitive cube root of unity.

Let G act on $\mathbb{C}$ by $z \mapsto \zeta z$ (not real). Then $f = z^3$ is G-invariant.

$$M_f = \mathbb{C}[z]/\langle z^2 \rangle \simeq \mathbb{C} \oplus \mathbb{C}z$$

is not a permutation representation.

Let G act on $\mathbb{C}^2$ by $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$. This is induced from the rotation representation on $\mathbb{R}^2$. Then $f = x^3 + y^3$ is G-invariant.

$$M_f = \mathbb{C}[x, y]/\langle x^2, y^2 \rangle \simeq \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$$

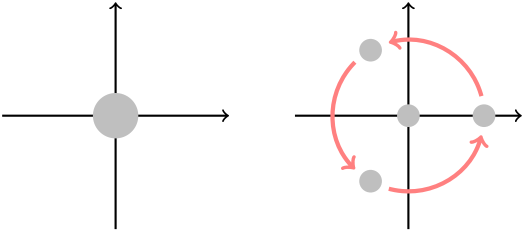
is a permutation representation $\mathrm{Map}(S, \mathbb{C})$ with $S = G \sqcup \{*\}$.

Example

An equivariant Morse approximation can be taken to be $f_t = x^3 + y^3 - 3txy$. Then for $t \neq 0$,

$$\mathbb{C}[x, y]/\langle x^2 - ty, y^2 - tx\rangle = \mathbb{C}[x]/\langle x^4 - t^3x\rangle$$

It spectrum is $t \cdot \mu_3 \sqcup \{0\} \simeq S$.



The Proof

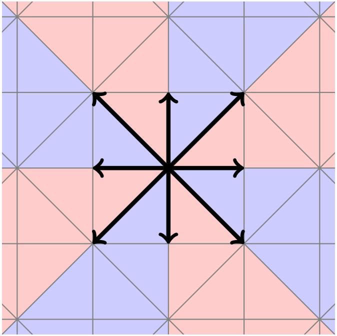
- The proof in the Fano case is clear.

$$H^*(X_\Sigma) \quad \rightsquigarrow \quad QH^*(X_\Sigma) \quad \simeq \quad M_f \quad \overset{\text{semisimple}}{\rightsquigarrow} \quad M_{f_s}$$

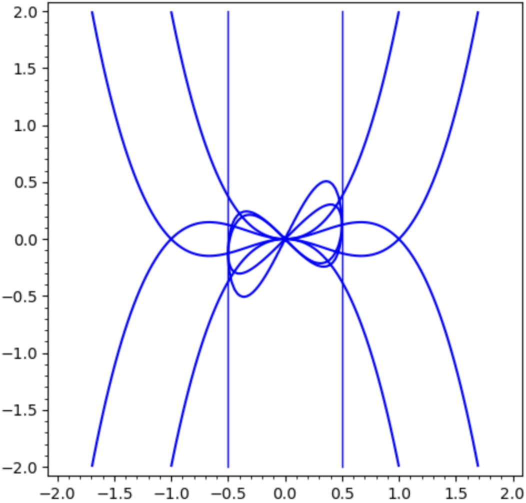
- For the general case, we can define a replacement of $QH^*(X_\Sigma)$, but it is not necessarily free (equivalently, finite and flat) over $\mathbb{C}[q]$.

  Let us see some examples.

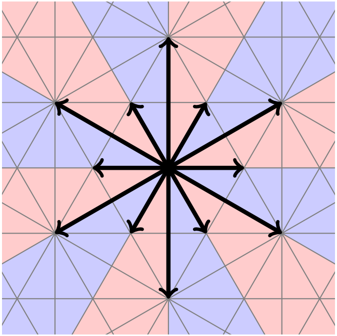
Example



$$\dim(\text{generic fiber}) = 8$$
$$\dim(\text{special fiber}) = \infty$$

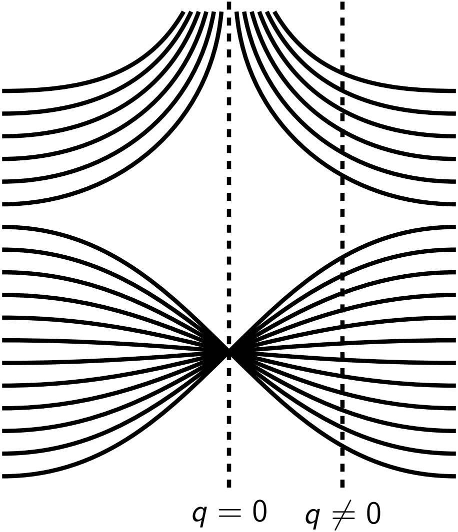


Example



$\dim(\text{central fiber}) = 12$

$\dim(\text{generic fiber}) = 18$



Some Commutative Algebra

In the above two examples, we see there are two defects for our replacement of QH to be free:

- there are some annoying fiber — but it is far from $q = 0$;
- there are some branch creating a pole at $q = 0$.

Actually, they are the only defects.

We can isolate the finite flat branch near $q = 0$ that specializes to ordinary cohomology using either

analytic topology	étale topology

This is how we proved the general case.

Thank You!