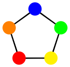
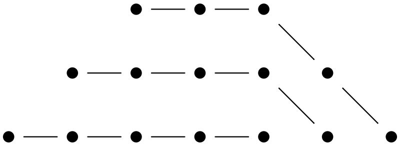
Motivic Lefschetz Theorem for twisted Milnor Hypersurfaces arXiv:2404.07314

with Kirill Zainoulline

Rui Xiong





Simple algebras

Let F be a field. The study of simple algebras can be traced as follows.

- A simple F-algebra A, by Artin–Wedderburn theorem, is a matrix algebra over a *division algebra*:

$$A = M_n(D).$$

- A division algebra D is a central division algebra over its center:

$$Z(D) = E \supset F.$$

- central division algebras are classified by the Brauer group:

$$[D] \in \mathrm{Br}(E) = H^2(E, \mathbb{G}_m).$$

Brauer Groups

The classical definition of Brauer group is

$$\mathrm{Br}(F) = \frac{\{\text{central simple algebras over } F\}}{A \sim B \iff M_m(A) \cong M_n(B)}.$$

Let A be a central simple algebra of degree n over F. Then $\mathrm{Aut}(A)$ is a twisted form of PGL_n, thus defines a class in

$$[A] \in H^1(F, PGL_n) \subset H^2(F, \mathbb{G}_m).$$

Here the inclusion is induced by the long exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1.$$

Severi–Brauer variety

Let us denote the *Severi–Brauer* variety

$$\mathrm{SB}(A) = \{I \subset A\}, \qquad I \lhd_r A, \ \dim I = n.$$

This is a twisted form of the projective space $\mathbb{P}^{n-1}$. But the geometry is quite different from $\mathbb{P}^{n-1}$. For example, in general,

- There is no rational points. Actually,

$$\mathrm{SB}(A)(E) \neq \varnothing \iff A_E \simeq M_n(E) \iff \mathrm{SB}(A)_E \cong \mathbb{P}^{n-1}_E.$$

- There is no bundle behavior like $\mathcal{O}(1)$ over $\mathrm{SB}(A)$ in general. Otherwise, the intersection of hyperplane sections will produce a rational point.

Twisted Milnor Hypersurfaces

We can identify

$$\mathrm{SB}(A^{op}) = \{I \subset A\}, \qquad I \lhd_r A,\ \dim I = n(n-1)$$

which is also a twisted form of $\mathbb{P}^{n-1}$. Let us define the *twisted Milnor hypersurface*

$$X = \{I_1 \subset I_{n-1} \subset A\} \subset \mathrm{SB}(A) \times \mathrm{SB}(A^{op})$$

cut by the section of the line bundle

$$[\mathcal{I}_1 \subset A \to A/\mathcal{I}_{n-1}] \in \mathcal{H}om_A(\mathcal{I}_1, A/\mathcal{I}_{n-1})$$

Note that this is a twisted form of the *incidence variety*

$$X_0 := Fl(1, n-1; n) = \{V_1 \subset V_{n-1} \subset F^n\}, \qquad \dim V_i = i.$$

Cyclic algebras

There is a huge source of central simple algebras known as *cyclic algebras*.

Assume F contains a n-th primitive roots of unity ζ. We pick $a, b \in F^\times$. Let A be a *cyclic algebra* of degree n

$$A = F\langle \mathbf{u}, v\rangle / \langle \mathbf{u}^n = a, v^n = b, \mathbf{u}v = \zeta v\mathbf{u}\rangle.$$

This algebra is known to be a central simple algebra.

Problem

Given a prime p, it is an open problem of a construction of a non-cyclic division algebra of degree p over some field F.

Asher Auel, Eric Brussel, Skip Garibaldi, Uzi Vishne. Open Problems on Central Simple Algebras. Transformation Groups. June 2010.

The Hyperplane section

We define a hyperplane section of the twisted Milnor hypersurfaces X to be

$$Y = \{(I_1 \subset I_{n-1}) \in X : \mathbf{u} I_1 \subset I_{n-1}\} \subset X$$

cut by the section of the line bundle

$$[\mathcal{I}_1 \subset A \xrightarrow{\mathbf{u}} A \to A/\mathcal{I}_{n-1}] \in \mathcal{H}om_A(\mathcal{I}_1, A/\mathcal{I}_{n-1}).$$

In other word, Y is a complete intersection of two sections from the same line bundle over $\mathrm{SB}(A) \times \mathrm{SB}(A^{op})$.

We will study the *motivic decomposition* of Y.

Motives

A *Chow motive* is a pair

$$(X, p) : \begin{array}{l} X \text{ is a smooth complete variety over } F, \\ p \in \mathrm{CH}(X \times X) \text{ is an idempotent.} \end{array}$$

A morphism $(X, p) \to (Y, q)$ is

$$q \circ \mathrm{CH}(X \times Y) \circ p.$$

The Chow motives form an additive category, so we want to study how to decompose

$$\mathcal{M}(X) = (X, \Delta_X) = (X, \mathrm{id}_X)$$

into smaller direct summands.

Motivic decomposition

It is known that $\mathcal{M}(X) =$

Theorem (Calmès, Petrov, Semenov, Zainoulline, 2006)

$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\mathrm{SB}(A))(n-2).$

Our result is $\mathcal{M}(Y) =$

Theorem (Xiong, Zainoulline, 2024)

$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\mathrm{Spec}\, L)(n-2).$

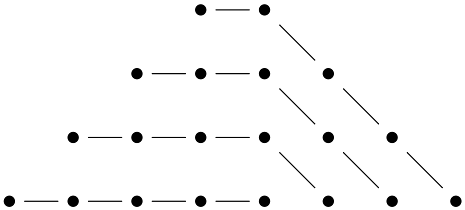
Here $L = F[\sqrt[n]{a}]$ is a field extension of F of degree n.

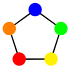
Since when A is a division algebra, $\mathcal{M}(\mathrm{SB}(A))$ is indecomposable, this is the best we can prove for general A.

Calmès, B.; Petrov, V.; Semenov, N.; Zainoulline, K. Chow motives of twisted flag varieties. *Compos. Math.* **142** (2006), no. 4, 1063–1080.

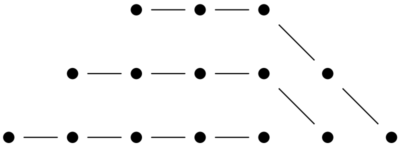
Example ($n = 5$)

$\mathcal{M}(X)$





$\mathcal{M}(Y)$



Hard Lefschetz

Recall the Hard Lefschetz theorem in complex algebraic geometry.

Theorem (Hard Lefschetz)

Let $\iota : Y \subset X$ be an ample smooth divisor. Then

the pullback

$$\iota^* : H^*(X) \longrightarrow H^*(Y)$$

is an isomorphism for $* < \dim Y$

the pushforward

$$\iota_* : H^*(Y) \longrightarrow H^{*+2}(X)$$

is an isomorphism for $* > \dim Y$.

The diagram is like this

$$\begin{tikzcd}
H^0(X) \arrow[dr, equal] & & H^2(X) \arrow[dr, equal] & & H^4(X) \arrow[dr, equal] & & H^6(X) \arrow[dr, dotted] & & H^8(X) \arrow[dr, dotted] & & H^{10}(X) \arrow[dr, dotted] & & H^{12}(X) \arrow[dr, dotted] & & H^{14}(X) \\
& H^0(Y) \arrow[ur, dotted] & & H^2(Y) \arrow[ur, dotted] & & H^4(Y) \arrow[ur, dotted] & & H^6(Y) \arrow[ur, dotted] & & H^8(Y) \arrow[ur, equal] & & H^{10}(Y) \arrow[ur, equal] & & H^{12}(Y) \arrow[ur, equal] &
\end{tikzcd}$$

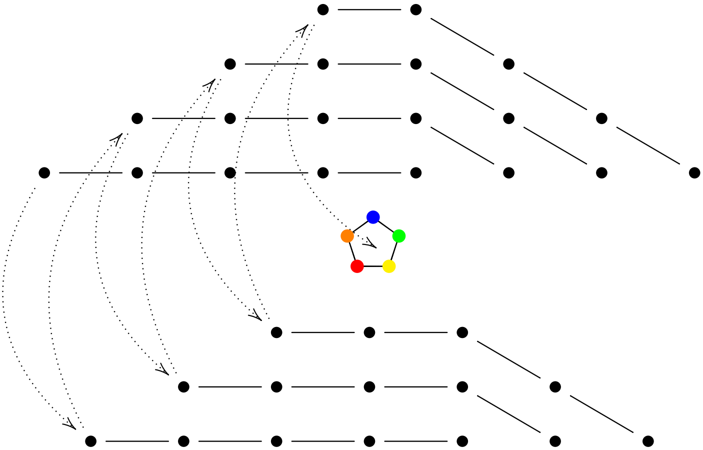
Severi–Brauer part

The Severi–Brauer part of the decompostion can be viewed as an analogy of this theorem.

- In the motivic decomposition of $\mathcal{M}(X)$, the idempotents is given by
$$g_i \circ f_i, \qquad 0 \leq i \leq n-2.$$
- In the motivic decomposition of $\mathcal{M}(Y)$, the idempotents is given by
$$\overline{g}_i \circ \overline{f}_{i+1}, \qquad 0 \leq i \leq n-3$$
where $\overline{*}$ is the restriction from X to Y.

Note that the shift of index is a feature of Lefschetz type theorem.

Example ($n = 5$)



Monodromy Actions (I)

Hodge theory also gives us hint about how the middle dimension cohomology supposed to be.

Theorem (Deligne invariant cycle theorem)

$$\mathrm{im}\left[H^*(X) \longrightarrow H^*(Y)\right] = \textit{monodromy invariant component}.$$

This is over $\mathbb{C}$. While in our case, the Galois group $\Gamma_L = \mathrm{Gal}(L/F)$ will be a part of monodromy.

$$\begin{tikzcd}
Y_L \arrow[loop above, "\sigma"] \arrow[r] \arrow[d] & \mathrm{Spec}\, L \arrow[loop above, "\sigma"] \arrow[d] \\
Y \arrow[r] & \mathrm{Spec}\, F
\end{tikzcd}$$

Universal family

We can consider the *universal family of hyperplane sections*

$$\mathcal{Y} = \left\{(I_1 \subset I_{n-1}, x) \in X \times A : xI_1 \subset I_{n-1}\right\} \xrightarrow{\mathrm{pr}_2} A.$$

Thus Y is the fibre at $u \in A$.

Let us assume at this moment $F = \mathbb{C}$.

Theorem (Ehresmann's fibration theorem)

The space $\mathcal{Y}$ is a topological fibre bundle over the smooth locus A^{sm}.

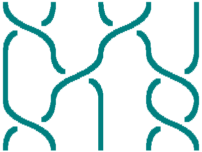
In particular , the fundamental group $\pi_1(A^{sm})$ acts on the cohomlogy of the any smooth fibre (called the *monodromy*).

Artin braid group

It is not hard to see, $a \in A^{sm}$ if and only if the reduced characteristic polynomial of u has no multiple roots (i.e. a is a regular semisimple element).

Over $\mathbb{C}$, it is well-known that

$$\pi_1(A^{sm}) = \mathfrak{B}_n = \left\{ \right.$$



$$\left. , \cdots \right\}$$

The monodromy action factors through the symmetric group S_n.

For general F, philosophically speaking, Galois group plays the role of (at least, a part of) monodromy.

1-cocycle

Recall $L = F[\sqrt[n]{a}]$. We know there is an L-algebra isomorphism

$$\rho : A_L \simeq M_n(L).$$

But this does not commute with the obvious Galois group actions. The obstruction is recorded in a 1-cocycle

$$\begin{aligned} \mathfrak{a} \colon \Gamma_L &\longrightarrow \mathrm{Aut}_L(M_n(L)) \simeq PGL_n(L), \\ \sigma &\longmapsto \mathfrak{a}_\sigma = \rho \circ \sigma \circ \rho^{-1} \circ \sigma^{-1}. \end{aligned}$$

In our case, we can compute this 1-cocycle explicitly:

- $\mathfrak{a}_\sigma \in N_{PGL_n}(T)(F) \subset PGL_n(F) \subset PGL_n(L)$
- $\mathfrak{a}$ is actually a group homomorphism.

Monodromy Actions (II)

As we mentioned, X is a twisted form of

$$X_0 = Fl(1, n-1; n) = \{V_1 \subset V_{n-1} \subset F^n\}, \qquad \dim V_i = i.$$

Thus ρ induces an isomorphism $X_L \simeq X_{0L}$. On the geometric side, we have

$$\begin{tikzcd} Y_L \arrow[r, "\subset"] \arrow[d, "\sigma"'] & X_L \arrow[r, "\simeq", "\rho"'] \arrow[d, "\sigma"'] & X_{0L} \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] \\ Y_L \arrow[r, "\subset"] & X_L \arrow[r, "\simeq", "\rho"'] & X_{0L} \end{tikzcd}$$

This will allow us computing the monodromy action.

Torus fixed points

Note that X_0 admits an action of the standard maximal torus $T \subset PGL_n$. The torus fixed point are given by

$$[ij] = \big(\operatorname{span}(e_i) \subseteq \operatorname{span}(e_1, \ldots, \not{e}_j, \ldots, e_n)\big), \qquad 1 \leq i \neq j \leq n.$$

Then by the isomorphism $X_L \simeq X_{0L}$, the torus T_L acts on X_L. Since L contains all eigenvalues of $u \in A$, we can assume the image of u is diagonal. So T_L also acts on Y_L and one can check directly that

$$Y_L^{T_L} = X_L^{T_L}$$

In particular, $\operatorname{rank} \mathrm{CH}(Y_L) = \operatorname{rank} \mathrm{CH}(X_L)$.

Monodromy Actions (III)

Now by taking torus fixed points, we get the following diagram

$$\begin{tikzcd}
Y_L^{T_L} \arrow[r, equal] \arrow[d, "\sigma"'] & X_L^{T_L} \arrow[r, "\rho"] \arrow[d, "\sigma"'] & X_0^T \times \operatorname{Spec} L \arrow[r, equal] \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] & \coprod_{1\leq i\neq j\leq n}[ij]_L \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] \\
Y_L^{T_L} \arrow[r, equal] & X_L^{T_L} \arrow[r, "\rho"] & X_0^T \times \operatorname{Spec} L \arrow[r, equal] & \coprod_{1\leq i\neq j\leq n}[ij]_L .
\end{tikzcd}$$

It is obvious that $\mathfrak{a}_\sigma$ permutes $[ij]$. Explicit computation shows that it is induced by the n-cycle

$$1 \xmapsto{\eta} 2 \xmapsto{\eta} \cdots n \xmapsto{\eta} 1$$

where $\eta \in \Gamma_L$ such that $\eta(\sqrt[n]{a}) = \zeta\sqrt[n]{a}$.

Equivariant Chow ring

The T-invariance of the varieties allows us to consider T-equivariant Chow rings. Assume T splits. We have

Theorem (Brion, [Br97])

- $\mathrm{CH}_T(\mathrm{pt}) = \mathrm{Sym}_{\mathbb{Z}}\, T^*$;
- *the usual Chow ring* $\mathrm{CH}(X)$ *is a quotient of* $\mathrm{CH}_T(X)$.

The main benefit of considering equivariantly is the localization theorem.

Theorem (Brion, [Br97])

Let X be a projective, nonsingular variety with an action of T. Then the restriction $\mathrm{CH}_T(X) \longrightarrow \mathrm{CH}_T(X^T)$ *is injective.*

Brion, M. Equivariant Chow groups for torus actions. *Transform. Groups* **2** (1997), no. 3, 225–267.

Monodromy Actions (IV)

As a result, we can lift the Galois group action to equivariant Chow ring where we can play the trick of localization theorem.

$$\begin{tikzcd}
\mathrm{CH}_{T_L}(Y_L) \arrow[d, "\sigma"'] & \mathrm{CH}_{T_L}(X_L) \arrow[l, "\iota^*"'] \arrow[d, "\sigma"'] & \mathrm{CH}_{T_L}(X_{0L}) \arrow[l, "\rho^*"'] \arrow[r, hook] \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] & \bigoplus_{1\le i\neq j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^* \arrow[d, "\widehat{\sigma}"] \\
\mathrm{CH}_{T_L}(Y_L) & \mathrm{CH}_{T_L}(X_L) \arrow[l, "\iota^*"'] & \mathrm{CH}_{T_L}(X_{0L}) \arrow[l, "\rho^*"'] \arrow[r, hook] & \bigoplus_{1\le i\neq j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^*.
\end{tikzcd}$$

where

$$\widehat{\sigma}(\varphi_{ij})_{ij} = (\sigma\,\varphi_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij}, \qquad \varphi_{ij} \in \mathrm{Sym}_{\mathbb{Z}}\, T^*.$$

This will allow us computing the monodromy action combinatorially.

T-stable curves

Let us consider the following T-stable curves over X_0

(i) a root-conic curve connecting $[ij]$ and $[ji]$:

$$\mathbb{P}^1 \ni [x\colon y] \mapsto \big(\operatorname{span}(xe_i+ye_j) \subset \operatorname{span}(e_1, \dots, \cancel{e_i}, \dots, \cancel{e_j}, \dots, e_n, xe_i+ye_j)\big),$$

(ii) a plane curve connecting $[ij]$ and $[ik]$ (for distinct i, j, k):

$$\mathbb{P}^1 \ni [x\colon y] \mapsto \big(\operatorname{span}(e_i) \subset \operatorname{span}(e_1, \dots, \cancel{e_j}, \dots, \cancel{e_k}, \dots, e_n, ye_j+xe_k)\big),$$

(iii) a plane curve connecting $[ij]$ and $[kj]$ (for distinct i, j, k):

$$\mathbb{P}^1 \ni [x\colon y] \mapsto \big(\operatorname{span}(xe_i + ye_k) \subset \operatorname{span}(e_1, \dots, \cancel{e_j}, \dots, e_n)\big).$$

Theorem (Benedetti, Perrin, [BP22])

All T_L-stable curves over Y_L are plane curves.

Benedetti, V.; Perrin, N. Cohomology of hyperplane sections of (co)adjoint varieties. *arXiv:2207.02089*.

Equivariant cohomology

Let us define a graph with $n(n-1)$ vertices denoted $[ij]$, $1 \leq i \neq j \leq n$, which has two types of labelled edges

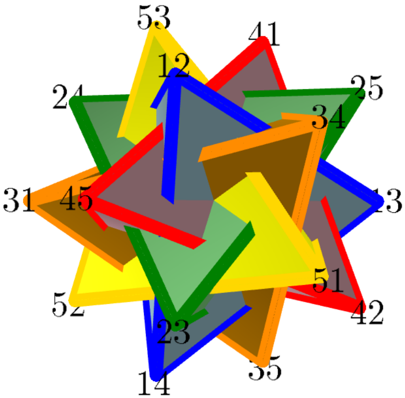
$$[ij] \xrightarrow{\alpha_{jk}} [ik] \quad \text{and} \quad [ij] \xrightarrow{\alpha_{ik}} [kj], \quad \text{where all } i, j, k \text{ are distinct,}$$

and $\alpha_{ij} = t_i - t_j \in T^*$.

Theorem

$$\mathrm{CH}_{T_L}(Y_L) \simeq \left\{ (\varphi_{ij})_{ij} : \alpha \mid \varphi_{ij} - \varphi_{kh} \textit{ for any edge } [ij] \xrightarrow{\alpha} [kh] \right\}.$$

This is a particular case [Br97, §3].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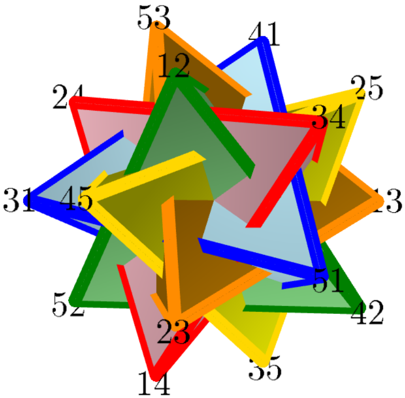
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Poincaré pairing

We are now able to construct classes in $\mathrm{CH}_T(Y_L)$, but we need make sure they are non-zero under the forgetful map

$$\mathrm{CH}_T(Y_L) \twoheadrightarrow \mathrm{CH}(Y_L).$$

The answer is the Poincaré pairing, which can be computed equivariantly.

Theorem

The Poincaré pairing is given by

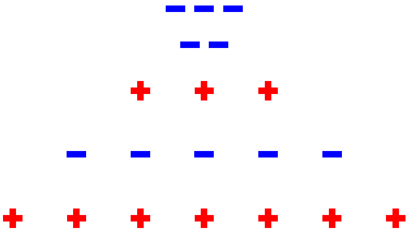
$$\langle \varphi, \psi \rangle_{Y_L} = \sum_{1 \le i \ne j \le n} \frac{\varphi_{ij}\psi_{ij}}{\prod_{s \ne i,j} \alpha_{is}\alpha_{sj}}.$$

Note that $\pm\alpha_{is}$ and $\pm\alpha_{sj}$ are exactly the labels of edges joint $[ij]$.

Hodge–Riemann relation

The last piece of $\mathcal{M}(Y)$, the Artin motive $\mathcal{M}(\operatorname{Spec} L)$, is supported on the primitive space (in terms of Hodge theory).

The Hodge–Riemann relations predicts the index of the intersection form over the primitive space

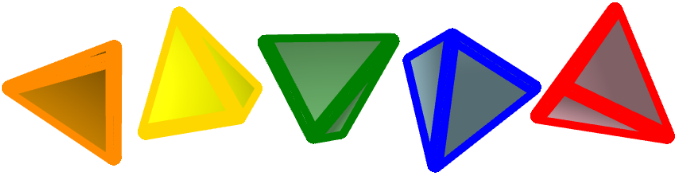


So the intersection form should be of $(-1)^{n-2}$-definite.

Monodromy actions (V)

We then constructed cycles

$$\gamma_\ell \in \bigoplus_{1\le i\ne j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^* \quad \text{with properties} \begin{cases} \gamma_\ell \in \mathrm{CH}_T^{n-2}(Y_L), \\ \widehat{\sigma}\gamma_\ell = \gamma_{\sigma(\ell)},\ \sigma \in \Gamma_L, \\ \langle \gamma_k, \gamma_\ell \rangle_{Y_L} = (-1)^{n-2}\delta_{k,\ell}. \end{cases}$$



One can show further that they are orthogonal to the Severi–Brauer parts, and give the last motive $\mathcal{M}(\mathrm{Spec}\, L)$. Q.E.D.

Thanks