NOTES ON MACDONALD POLYNOMIALS

Contents

1. Double affine Hecke Algebras

1.1. Affine Weyl groups. Let us fix a finite root system. The *affine Weyl group* is

$$\widehat{W} = Q^\vee \rtimes W = W \ltimes Q^\vee \qquad \begin{aligned} Q^\vee &= \text{coroot lattice} \\ W &= \text{Weyl group} \end{aligned}$$

The group $\widehat{W}$ acts

on $Q^\vee = \widehat{W}/W$ *affinely*:	on $\widehat{Q} = Q \oplus \mathbb{Z}\boldsymbol{\delta}$ *linearly*:
$(wt_\lambda) \cdot \mu = w(\lambda + \mu)$.	$(wt_\lambda) \cdot (\alpha + k\boldsymbol{\delta}) = w\alpha + (k - \langle \lambda, \alpha \rangle)\boldsymbol{\delta}$.

The set of (positive) real roots is

$$\widehat{\Delta} = \left\{ \alpha + k\boldsymbol{\delta} : \begin{array}{l} \alpha \in \Delta \\ k \in \mathbb{Z} \end{array} \right\}, \qquad \widehat{\Delta}^+ = \left\{ \alpha + k\boldsymbol{\delta} : \begin{array}{l} k > 0 \text{ or} \\ k = 1, \alpha > 0 \end{array} \right\}.$$

For any $\widehat{\alpha} = \alpha + k\boldsymbol{\delta}$, we can define the *reflection*

$$r_{\widehat{\alpha}} = r_\alpha t_{k\alpha^\vee} = t_{-k\alpha^\vee} r_\alpha.$$

Assume the Dynkin diagram of R is connected, then there is a highest root θ. We denote

$$s_0 = r_{-\theta+\boldsymbol{\delta}} = r_\theta t_{-\theta^\vee} = t_{\theta^\vee} r_\theta.$$

Then $\widehat{W}$ is Coxeter system with $\widehat{I} = I \cup \{0\}$. For $x = wt_\lambda \in \widehat{W}$, the *length* is

$$\ell(x) = \#\,\mathrm{Inv}(x), \qquad \text{where} \quad \begin{aligned} \mathrm{Inv}(x) &= \widehat{\Delta}^+ \cap x^{-1}\widehat{\Delta}^- \\ &= \{\widehat{\alpha} \in \widehat{\Delta}^+ : x\,\widehat{\alpha} \in \widehat{\Delta}^-\} \end{aligned}$$

We have a very famous formula

$$\ell(wt_\lambda) = \sum_{\alpha \in \Delta^+} \Big| \langle \alpha, \lambda \rangle + [w\alpha < 0] \Big|.$$

Actually, if we denote for $\alpha > 0$, the set

$$\mathrm{Inv}_\alpha(x) = \{\pm\alpha + k\boldsymbol{\delta} \in \mathrm{Inv}(x)\},$$

we have

$$\mathrm{Inv}_\alpha(wt_\lambda) = \begin{cases} \{\alpha + k\boldsymbol{\delta}\}_{0 \le k < \langle \alpha, \lambda \rangle + [w\alpha < 0]}, & \langle \alpha, \lambda \rangle \ge 0, \\ \{-\alpha + k\boldsymbol{\delta}\}_{0 < k \le -\langle \alpha, \lambda \rangle - [w\alpha < 0]}, & \langle \alpha, \lambda \rangle < 0, \end{cases}$$

1.2. **Affine Hecke Algebras.** Let $\widehat{H}_t(W)$ be the Hecke algebra for the Coxeter system $\widehat{W}$. We have

$$\widehat{H}_t(W) = \bigoplus_{x \in \widehat{W}} \mathbb{Q}_t \cdot T_x, \qquad \begin{array}{c} T_x T_y = T_{xy} \text{ if} \\ \ell(x) + \ell(y) = \ell(xy). \end{array}$$

Let us denote Y^λ for $\lambda \in Q^\vee$ as follows. For dominant λ, we define $Y^\lambda = t^{-\langle \lambda, \rho \rangle} T_{t_\lambda}$; for general λ, we define $Y^\lambda = Y^{\lambda_1}(Y_{\lambda_2})^{-1}$ if we can write $\lambda = \lambda_1 - \lambda_2$ with λ_1, λ_2 dominant. This is well-defined since for dominant $\lambda \in Q^\vee$

$$\ell(t_\lambda) = \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle| = 2\langle \rho, \lambda \rangle.$$

Denote

$$H_t(W) = \bigoplus_{w \in W} \mathbb{Q}_t \cdot T_w = \text{Hecke algebra for } W$$

$$\mathbb{Q}_t[Y] = \bigoplus_{\lambda \in Q^\vee} \mathbb{Q}_t \cdot Y^\lambda = \text{group ring of } Q^\vee.$$

Then $\widehat{H}_t(W)$ contains them as subalgebras and

$$\widehat{H}_t(W) = \mathbb{Q}_t[Y] \otimes H_t(W) \qquad \text{(as a vector space)},$$

with intertwine

$$T_i Y^\lambda - Y^{s_i \lambda} T_i = (t-1) \frac{Y^{s_i \lambda} - Y^\lambda}{Y^{-\alpha_i^\vee} - 1}.$$

Here our convention of quadratic relation for Hecke algebras is

$$(T - t)(T + 1) = 0.$$

We will check this relation soon after introducing extended affine Hecke algebras.

Consider the Bernstein representation of $\widehat{H}_t(W)$ on $\mathbb{Q}_t[Y]$:

$$T_i \mapsto \text{Demazure–Lusztig operator} = t\, s_i + (t-1) \frac{s_i - 1}{Y^{-\alpha_i^\vee} - 1}$$

$$Y^\lambda \mapsto \text{multiplication by } Y^\lambda.$$

It defines a faithful representation of $\widehat{H}_t(W)$. Actually, it is isomorphic to $\widehat{H}_t(W) \otimes_{H_t(W)} \mathbb{Q}_t$ with $T_i \mapsto t$ on $\mathbb{Q}_t$ $(i \in I)$.

1.3. **Extended affine Hecke algebras.** Define *extended affine Weyl group*

$$\widehat{W}_e = W \ltimes P^\vee = P^\vee \rtimes W.$$

It acts on $P^\vee$ and $\widehat{Q} = Q \oplus \mathbb{Z}\boldsymbol{\delta}$. We can extend the length function to $\widehat{W}_e$ using the same expression. We have

$$\widehat{W}_e = \widehat{W} \rtimes \Omega = \Omega \ltimes \widehat{W}.$$

$$\Omega = \{x \in \widehat{W}_e : \ell(x) = 0\} = \mathrm{Aut}(\substack{\text{affine Dynkin} \\ \text{diagram}}).$$

Define the *extended affine Hecke algebra*

$$H_{\boldsymbol{t}}(\widehat{W}_e) = \Omega \ltimes H_{\boldsymbol{t}}(\widehat{W}) \xlongequal{\text{as v.s.}} \mathbb{Q}_{\boldsymbol{t}}[\Omega] \otimes H_{\boldsymbol{t}}(\widehat{W}).$$

Then we have

$$H_{\boldsymbol{t}}(\widehat{W}_e) = \bigoplus_{x \in \widehat{W}_e} \mathbb{Q}_{\boldsymbol{t}} \cdot T_x, \qquad \begin{array}{c} T_x T_y = T_{xy} \text{ if} \\ \ell(x) + \ell(y) = \ell(xy). \end{array}$$

We can define Y^λ in the same manner. To check the intertwine, it suffices to check for fundamental coweights $\in P^\vee$ since it is true for $\lambda - \mu$ if it is true for λ and μ. It reduces to check the following (λ is dominant)

$\langle \lambda, \alpha_i \rangle = 0$	$\langle \lambda, \alpha_i \rangle = 1$
$H_i Y^\lambda = Y^\lambda H_i$	$H_i Y^{s_i \lambda} H_i = Y^\lambda$

$$H_w = \boldsymbol{t}^{-\ell(w)/2} T_w$$
$$(H_i - \boldsymbol{t}^{1/2})(H_i + \boldsymbol{t}^{1/2}) = 0.$$

Sketch for the second case. Denote $\lambda' = s_i \lambda$. Then $\lambda' + \lambda$ is dominant. Using the length formula, we can check

$$t_{\lambda' + \lambda} = (t_{\lambda'} s_i)(s_i t_\lambda)$$

is a reduced decomposition i.e. $\ell(t_{\lambda'+\lambda}) = \ell(t_{\lambda'} s_i) + \ell(s_i t_\lambda)$. Then

$$Y^{\lambda' + \lambda} = H_{t_{\lambda'+\lambda}} = (H_{t_{\lambda'}} H_i^{-1})(H_i^{-1} H_{t_\lambda}) = (H_{t_{\lambda'}} H_i^{-1})(H_i^{-1} Y^\lambda).$$

Thus $Y^{\lambda'} = T_{t_{\lambda'}} H_i^{-2}$, so

$$H_i Y^{\lambda'} H_i = H_i T_{t_{\lambda'}} H_i^{-1} = H_{t_\lambda} = Y^\lambda.$$

1.4. **Double affine Hecke algebras.** Let us denote $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}} = \mathbb{Q}(\boldsymbol{q},\boldsymbol{t})$ for short. Let us denote $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[X] = \bigoplus_{\alpha\in Q} \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}} \cdot X^\alpha$. For $\alpha + k\boldsymbol{\delta} \in \widehat{Q} = Q \oplus \mathbb{Z}\boldsymbol{\delta}$, we denote

$$X^{\alpha+k\boldsymbol{\delta}} = \boldsymbol{q}^k X^\alpha.$$

i.e. $\boldsymbol{q} = X^{\boldsymbol{\delta}}$. So we can identify

$$\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[X] = \text{a localization of the group ring of } \widehat{Q} = Q \oplus \mathbb{Z}\boldsymbol{\delta}.$$

We define *double affine Hecke algebras*

$$\ddot{H}_{\boldsymbol{q},\boldsymbol{t}}(W) = \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[X] \otimes \widehat{H}_{\boldsymbol{t}}(W)$$

with intertwine for $i \in I \cup \{0\}$

$$T_i X^\lambda - X^{s_i\lambda} T_i = (\boldsymbol{t}-1)\frac{X^{s_i\lambda} - X^\lambda}{X^{\alpha_i} - 1}.$$

Here, $X^{\alpha_0} = \boldsymbol{q}\, X^{-\theta}$. Note that there is no minus. Note that $\boldsymbol{q}$ is central, thus we can just record it in the base field.

We define the following *Cherednik's representation* of $\ddot{H}_{\boldsymbol{q},\boldsymbol{t}}(W)$ on $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[X]$ by

$$T_i \mapsto \text{Demazure–Lusztig operator} = \boldsymbol{t} s_i + (\boldsymbol{t}-1)\frac{s_i - 1}{X^{\alpha_i} - 1}$$

$$X^\lambda \mapsto \text{multiplication by } X^\lambda$$

It is a faithful representation isomorphic to $\ddot{H}_{\boldsymbol{q},\boldsymbol{t}}(W) \otimes_{\widehat{H}_{\boldsymbol{q},\boldsymbol{t}}(W)} \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$ with $T_i \mapsto \boldsymbol{t}$ on $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$ $(i \in I \cup \{0\})$.

It is clear that T_i $(i \in i)$ and X^λ $(\lambda \in Q)$ generate the affine Hecke algebra $\widehat{H}_{\boldsymbol{t}}(W^\vee)$ of the dual root system. Let us denote $T_i^\vee = T_i$ for $i \in I$. Denote $T_0^\vee$ such that for *anti*-dominant weight λ, $X^\lambda = \boldsymbol{t}^{-\langle\rho^\vee,\lambda\rangle} T_{t_\lambda}^\vee$. Then $T_i^\vee$ $(i \in I \cup \{0\})$ generate $\widehat{H}_{\boldsymbol{q},\boldsymbol{t}}(W^\vee)$.

In summary, we have

$$\begin{tikzcd}[column sep=small]
 & & \ddot{H}_{\boldsymbol{q},\boldsymbol{t}}(W) & & \\
 & \widehat{H}_{\boldsymbol{q},\boldsymbol{t}}(W) \arrow[ur, hook] & & \widehat{H}_{\boldsymbol{q},\boldsymbol{t}}(W^\vee) \arrow[ul, hook'] & \\
\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y] \arrow[ur, hook] & & H_{\boldsymbol{q},\boldsymbol{t}}(W) \arrow[ul, hook'] \arrow[ur, hook] & & \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[X] \arrow[ul, hook']
\end{tikzcd}$$

Actually, the following is an isomorphism

$$\ddot{H}_{q,t}(W) \longrightarrow \ddot{H}_{q,t}(W^\vee), \qquad \begin{cases} T_i \mapsto T_i^{\vee\,-1}, \\ q \mapsto q^{-1}, t \mapsto t^{-1}, \\ X^\mu \mapsto Y^\mu, Y^\lambda \mapsto X^\lambda, \end{cases}$$

Proof is technical and can be found in [Mac2, §3.5-3.7] and [Hai, §4]. The duality switches two copies of affine Hecke algebra induced from the bar-involution.

Compare:

$$Y^\mu = t^{\langle \rho, \mu \rangle} T_{t_\mu} \quad (\mu \in Q^\vee \text{ dominant}),$$
$$X^\lambda = t^{-\langle \rho^\vee, \lambda \rangle} T^\vee_{t_\lambda} \quad (\lambda \in Q \text{ anti-dominant}).$$

where $\phi \in R$ is the root with the coroot $\phi^\vee$ highest. Let θ be the highest root, and ϕ the root such that $\phi^\vee$ is the highest coroot. Note that $\phi = \theta$ if and only if the Dynkin diagram is simply connected. Then

$$T_0 = t^{\langle \rho, \theta \rangle} Y^{\theta^\vee} T_{r_\theta}^{-1} \quad (t_{\theta^\vee} = s_0 r_\theta),$$
$$T_0^\vee = t^{\langle \rho^\vee, \phi \rangle} T_{r_\phi}^{-1} X^{-\phi} \quad (t_{-\phi} = r_\phi s_0).$$

2. Macdonald Polynomials

2.1. Cherednik's representation. Let P be the weight lattice. Let us denote

$$R = \bigoplus_{\lambda \in P} \mathbb{Q}_{q,t} \cdot e^\lambda, \qquad e^{\delta} = q.$$

We twist the *Cherednik's representation* $\widehat{H}_t(W)$ on R by

$$T_i \mapsto \text{Demazure–Lusztig operator} = t s_i + (t-1)\frac{s_i - 1}{e^{\alpha_i} - 1}$$

Recall that $e^{\alpha_0} = q\, e^{-\theta}$. Note that

$$\mathbb{Q}_t[Y] \subset \widehat{H}_t(W)$$

is a family of commutative operators over R, thus can be upper triangulated simultaneously. Actually, we are going to prove the eigenvalues are different and thus can be diagonalized simultaneously.

Let us define an order over P. We denote $\prec$ the dominant order.

$$\lambda < \mu \iff \begin{array}{l} \lambda^+ \prec \mu^+ \text{ or} \\ \lambda^+ = \mu^+, \mu \prec \lambda. \end{array}$$

Here λ^+ stands the dominant weight in the W-orbit of λ. We are going to show

$$Y^\mu e^\lambda = \boldsymbol{t}^{-\langle \rho_\lambda, \mu \rangle} \boldsymbol{q}^{-\langle \lambda, \mu \rangle} e^\lambda + (\text{lower terms}),$$

where $\rho_\lambda = \dfrac{1}{2} \sum_{\alpha>0} \begin{cases} \alpha, & \langle \lambda, \alpha \rangle > 0, \\ -\alpha, & \langle \lambda, \alpha \rangle \le 0. \end{cases}$. We remark that if $w \in W$ is the maximal element such that $\lambda = w\lambda^+$, then $w\rho = \rho_\lambda$.

The proof goes as follows. Denote for any root $\widehat{\alpha}$ an operator

$$G(\widehat{\alpha}) = \boldsymbol{t} + (\boldsymbol{t} - 1)\frac{1 - r_\alpha}{e^{-\alpha} - 1}.$$

Note that $T_i \mapsto s_i G(\alpha_i)$ and $wG(\alpha)w^{-1} = G(w\alpha)$. For any $x \in \widehat{W}$, if we pick a reduced word $x = s_{i_1} s_{i_2} \cdots s_{i_\ell}$,

$$\begin{aligned} T_x &= T_{i_1} T_{i_2} \cdots T_{i_\ell} \\ &\mapsto s_{i_1} G(\alpha_{i_1}) s_{i_2} G(\alpha_{i_2}) \cdots s_{i_\ell} G(\alpha_{i_\ell}) \\ &= s_{i_1} s_{i_2} G(s_{i_2}\alpha_{i_1}) G(\alpha_{i_2}) \cdots s_{i_\ell} G(\alpha_{i_\ell}) \\ &= xG(s_{i_\ell} \cdots s_{i_2}\alpha_{i_1}) G(s_{i_\ell} \cdots s_{i_3}\alpha_{i_2}) \cdots G(\alpha_{i_\ell}). \end{aligned}$$

Note that

$$\{s_{i_\ell} \cdots s_{i_2}\alpha_{i_1}, s_{i_\ell} \cdots s_{i_3}\alpha_{i_2}, \cdots, \alpha_{i_\ell}\} = \mathrm{Inv}(x).$$

Thus for $x = t_\mu$ with μ dominant, we have

$$Y^\mu = \boldsymbol{t}^{-\langle \mu, \rho \rangle} T_{t_\mu} \mapsto \boldsymbol{t}^{-\langle \mu, \rho \rangle} t_\mu G(\beta_1) G(\beta_2) \cdots G(\beta_\ell)$$

such that

$$\{\beta_1, \beta_2, \ldots, \beta_\ell\} = \mathrm{Inv}(t_\mu).$$

Note that any positive root of $\mathrm{Inv}(t_\mu)$ is of the form $\alpha + k\boldsymbol{\delta}$ for $\alpha > 0$. Let us study $G(\alpha)$ for $\alpha \bmod \boldsymbol{\delta} > 0$. We can compute obtain directly

$$G(\alpha)e^\lambda = \begin{cases} e^\lambda + (\text{lower terms}), & \langle \lambda, \alpha \rangle > 0, \\ \boldsymbol{t}\, e^\lambda + (\text{lower terms}), & \langle \lambda, \alpha \rangle \le 0. \end{cases}$$

Here is an example in A_1, we have

$\langle\alpha,\lambda\rangle$	-2	-1	0	1	2
e^λ	1				
$G(\alpha)e^\lambda$	$\boldsymbol{t}$		$\boldsymbol{t}-1$		$\boldsymbol{t}-1$

$\langle\alpha,\lambda\rangle$	-2	-1	0	1	2
e^λ					1
$G(\alpha)e^\lambda$	0		$1-\boldsymbol{t}$		1

Thus

$$\begin{aligned} Y^\mu e^\lambda &= \boldsymbol{t}^{-\langle\rho_\lambda,\mu\rangle} e^{-\lambda-\langle\mu,\lambda\rangle\boldsymbol{\delta}} + (\text{lower terms}) \\ &= \boldsymbol{t}^{-\langle\rho_\lambda,\mu\rangle}\boldsymbol{q}^{-\langle\lambda,\mu\rangle} e^\lambda + (\text{lower terms}). \end{aligned}$$

By definition of Y^μ, this extends to all $\mu \in Q^\vee$. This shows $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y]$ has different eigenvalues.

2.2. **Nonsymmetric Macdonald polynomials.** By above, there exists a unique $E_\lambda \in R$ called *non-symmmetric Macdonald polynomials* such that

(1) $E_\lambda = e^\lambda + (\text{lower terms})$;
(2) $Y^\mu E_\lambda = \boldsymbol{t}^{-\langle\rho_\lambda,\mu\rangle}\boldsymbol{q}^{-\langle\lambda,\mu\rangle} E_\lambda$.

Actually, E_λ can be constructed by the standard diagonalization trick. Since (2) determines E_λ up to a scalar, the condition of (1) by requiring the coefficient of e^λ is 1:

(1') $[e^\lambda]E_\lambda = 1$.

For example:

- $E_0 = 1$
- for a minuscule weight λ, we have $E_\lambda = e^\lambda$.
- for any weight λ, we have $E_\lambda = e^\lambda \bmod (t-1)$.

Next, let us describe an induction formula for E_λ. If $s_i\lambda > \lambda$ for some $i \in I$, i.e. $\langle\lambda,\alpha_i^\vee\rangle > 0$, then

$$E_{s_i\lambda} = \left(T_i + \frac{\boldsymbol{t}-1}{\boldsymbol{t}^{\langle\rho_\lambda,\alpha_i^\vee\rangle}\boldsymbol{q}^{\langle\lambda,\alpha_i^\vee\rangle}-1}\right)E_\lambda.$$

Let us check the two conditions. (1') is obvious by direct computation. Let us check (2). Let us denote

$$\tau_i = T_i + \frac{\boldsymbol{t}-1}{Y^{-\alpha_i^\vee}-1} \in \text{a localization of } \widehat{H}_{\boldsymbol{t}}(W).$$

Note that the right-hand side is nothing but $\tau_i E_\lambda$. The key observation is

$$\tau_i Y^\mu = Y^{s_i\mu}\tau_i \in \text{a localization of } \widehat{H}_{\boldsymbol{t}}(W).$$

Actually, it suffices to check this under the Bernstein representation:

$$\tau_i \mapsto \boldsymbol{t}\, s_i + (\boldsymbol{t}-1)\frac{s_i-1}{Y^{-\alpha_i^\vee}-1} + \frac{\boldsymbol{t}-1}{Y^{-\alpha_i^\vee}-1} = \frac{\boldsymbol{t}Y^{-\alpha_i^\vee}-1}{Y^{-\alpha_i^\vee}-1}s_i.$$

The standard trick tells

$$\begin{aligned} Y^\mu(\tau_i E_\lambda) &= \tau_i Y^{s_i\mu}E_\lambda = \tau_i \boldsymbol{t}^{-\langle\rho_\lambda, s_i\mu\rangle}\boldsymbol{q}^{-\langle\lambda, s_i\mu\rangle}E_\lambda \\ &= \boldsymbol{t}^{-\langle\rho_{s_i\lambda},\mu\rangle}\boldsymbol{q}^{-\langle s_i\lambda,\mu\rangle}(\tau_i E_\lambda). \end{aligned}$$

The proof is complete.

We can extend the induction formula to $i = 0$ by introducing a similar operator τ_0. But for type A, it is simpler to make use of the symmetry of the affine root system, see [Hai §2].

2.3. **Center characters.** The argument above tells for any λ, we still have

$$E_{s_i\lambda} \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}\cdot\left(T_i + \frac{\boldsymbol{t}-1}{\boldsymbol{t}^{\langle\rho_\lambda,\alpha_i^\vee\rangle}\boldsymbol{q}^{\langle\lambda,\alpha_i^\vee\rangle}-1}\right)E_\lambda.$$

Note that $\boldsymbol{t}\neq 1$, so that the denominator never vanishes. This implies

$$\mathcal{A}(\lambda) = \bigoplus_{\lambda'\in W\lambda}\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}\cdot E_{\lambda'}$$

is closed under actions of T_i. Since E_λ's are eigenvalues of $\mathbb{Q}_{\boldsymbol{t}}[Y]$, $\mathcal{A}(\lambda)$ is a representation of $\widehat{H}_{\boldsymbol{t}}(W)$. So this gives the composition

$$R = \bigoplus_{\text{dom }\lambda}\mathcal{A}(\lambda) \quad (\text{as a } \widehat{H}_{\boldsymbol{q},\boldsymbol{t}}(W)\text{-module}).$$

Actually $\mathcal{A}(\lambda)$ can be characterized by center characters. Note that

$$Z(\widehat{H}_{\boldsymbol{t}}(W)) = \mathbb{Q}_{\boldsymbol{t}}[Y]^W.$$

We have

$$\mathcal{A}(\lambda) = \left\{ g\in R : \begin{array}{l} \forall f(Y)\in\mathbb{Q}_{\boldsymbol{t}}[Y]^W \\ f(Y)g = f(\boldsymbol{t}^{-\rho}\boldsymbol{q}^{-\lambda})g \end{array}\right\}.$$

Recall that

$$Y^\mu E_{\lambda'} = \boldsymbol{t}^{-\langle \rho_{\lambda'}, \mu\rangle} \boldsymbol{q}^{-\langle \lambda', \mu\rangle} E_{\lambda'}$$

$$f(Y) E_{\lambda'} = f(\boldsymbol{t}^{-\rho_{\lambda'}} \boldsymbol{q}^{-\lambda'}) E_{\lambda'}$$

for $f(Y) \in \mathbb{Q}_{\boldsymbol{t}}[Y]^W$, Note that we can always find $w \in W$ such that $u\rho_\lambda = \rho_{u\lambda}$, thus $f(\boldsymbol{t}^{-\rho_{\lambda'}} \boldsymbol{q}^{-\lambda'})$ does not depend on the choice of $\lambda' \in W\lambda$. Then the easiest choice is $\lambda' = w_0\lambda$ and $f(\boldsymbol{t}^{-\rho_{w_0\lambda}} \boldsymbol{q}^{-w_0\lambda}) = f(\boldsymbol{t}^{\rho} \boldsymbol{q}^{-w_0\lambda}) = f(\boldsymbol{t}^{-\rho} \boldsymbol{q}^{-\lambda})$, as desired.

Let us consider R^W, the ring of symmetric polynomials over $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$. We have

$$\begin{tikzcd}
\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y] \arrow[r, bend left] & R \\
\cup & \cup \\
\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y]^W \arrow[r, bend left] & R^W
\end{tikzcd}$$

Firstly, we can rewrite the condition of being symmetric in terms of DL operators:

$$s_i f = f \iff T_i f = \boldsymbol{t} s_i f + (\boldsymbol{t} - 1)\frac{s_i f - f}{e^{\alpha_i} - 1} = \boldsymbol{t} f.$$

$$R^W = \{f \in R : \forall i \in I, T_i f = \boldsymbol{t} f\} = \bigcap_{i \in I} \ker(T_i - \boldsymbol{t}).$$

Secondly, since $\mathbb{Q}_{\boldsymbol{t}}[Y]^W$ is the center of $\widehat{H}_{\boldsymbol{t}}(W)$, so R^W is $\mathbb{Q}_{\boldsymbol{t}}[Y]^W$-equivariant.

Let us denote for a dominant weight λ the *monomial symmetric polynomials*

$$m_\lambda = \sum_{\lambda' \in W\lambda} e^{\lambda'} = \sum_{w \in W^\lambda} e^{w\lambda} \in R^W.$$

Note that by assumption

$$m_\lambda = e^{w_0\lambda} + (\text{lower terms})$$

$$Y^\mu m_\lambda = \boldsymbol{t}^{\langle \rho, \mu\rangle} \boldsymbol{q}^{\langle -w_0\lambda, \mu\rangle} e^{w_0\lambda} + (\text{lower terms})$$

$$f(Y) m_\lambda = f(\boldsymbol{t}^{-\rho} \boldsymbol{q}^{-\lambda}) m_\lambda + (\text{lower terms})$$

for $f \in \mathbb{Q}_{\boldsymbol{t}}[Y]^W$.

2.4. **Symmetric Macdonald polynomials.** By above, there exists a unique $P_\lambda \in R^W$ called *symmmetric Macdonald polynomials* such that

(1) $P_\lambda = m_\lambda + (\text{lower terms})$;
(2) $f(Y)P_\lambda = f(\boldsymbol{t}^{-\rho}\boldsymbol{q}^{-\lambda})P_\lambda$ for any symmetric $f(Y) \in \mathbb{Q}_{\boldsymbol{t}}[Y]^W$.

Let us state the relation between E_λ and P_λ. We have

$$P_\lambda = \frac{1}{W_\lambda(\boldsymbol{t})}\sum_{w\in W} T_w E_\lambda = \frac{1}{W_\lambda(\boldsymbol{t})}\sum_{w\in W} w\left(E_\lambda \prod_{\alpha>0}\frac{e^\alpha - \boldsymbol{t}}{e^\alpha - 1}\right)$$

where $W_\lambda(\boldsymbol{t}) = \sum_{w\in W_\lambda} \boldsymbol{t}^{\ell(w)}$.

Let us denote the symmetrizer $\Pi = \sum_{w\in W} T_w$. Since $\Pi = (T_i + 1)\sum_{s_i w > w} T_w$, we have $T_i\Pi = t\Pi$. It defines an operator $R \to R^W$. It acts as the following operator

$$\Pi f = \sum_w w\left(f\prod_{\alpha>0}\frac{e^\alpha - \boldsymbol{t}}{e^\alpha - 1}\right).$$

For example, for A_1,

$$\begin{aligned}
\Pi f &= Tf + f = \boldsymbol{t}sf + f + (\boldsymbol{t}-1)\frac{sf - f}{e^\alpha - 1}\\
&= \left(1 - \frac{\boldsymbol{t}-1}{e^\alpha - 1}\right)f + s\left(\boldsymbol{t} + \frac{(\boldsymbol{t}-1)}{e^{-\alpha}-1}\right)sf\\
&= \frac{e^\alpha - \boldsymbol{t}}{e^\alpha - 1} + s\frac{\boldsymbol{t}e^{-\alpha} - 1}{e^{-\alpha}-1} = (1+s)\left(\frac{e^\alpha - \boldsymbol{t}}{e^\alpha - 1}f\right).
\end{aligned}$$

By direct computation, we see SE_λ satisfies (2). Thus it suffices to prove the property (1). It suffices to

$$[e^{w_0\lambda}]\big(\Pi e^\lambda\big) = W_\lambda(\boldsymbol{t}).$$

The trick is polarization.

$$\begin{aligned}\sum_{w\in W} w\left(e^{\mu}\prod \frac{X^{\alpha}-t}{X^{\alpha}-1}\right) &= \sum_{u\in W^{\lambda}} u\left(\sum_{w\in W_{\lambda}} w\left(e^{\mu}\prod_{\alpha>0}\frac{e^{\alpha}-\boldsymbol{t}}{e^{\alpha}-1}\right)\right)\\ &= \sum_{u\in W^{\mu}} u\left(e^{\mu}\sum_{w\in W_{\mu}} w\left(\prod_{\alpha>0}\frac{e^{\alpha}-\boldsymbol{t}}{e^{\alpha}-1}\right)\right)\\ &\overset{(*)}{=} W_{\mu}(\boldsymbol{t})\sum_{u\in W^{\mu}} u\left(e^{\mu}\prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\mu}}\frac{e^{\alpha}-t}{e^{\alpha}-1}\right).\end{aligned}$$

Here $(*)$ is a very famous identity on the Poincaré polynomial of a Weyl group, we will prove it in the appendix. Let us denote R^{+} be the polynomial ring generated by e^{α} for $\alpha>0$. Then for $\alpha>0$,

$$\frac{e^{\alpha}-\boldsymbol{t}}{e^{\alpha}-1}=\frac{\boldsymbol{t}-e^{\alpha}}{1-e^{\alpha}}=(\boldsymbol{t}-e^{\alpha})(1+e^{\alpha}+e^{2\alpha}+\cdots)\in \text{completion of } R^{+};$$

for $\alpha<0$,

$$\frac{e^{\alpha}-\boldsymbol{t}}{e^{\alpha}-1}=\frac{1-\boldsymbol{t}e^{-\alpha}}{1-e^{-\alpha}}=(1-\boldsymbol{t}e^{-\alpha})(1+e^{-\alpha}+e^{-2\alpha}+\cdots)\in \text{completion of } R^{+}.$$

Then

$$\sum_{u\in W^{\mu}} u\left(e^{\mu}\prod_{\alpha\in\Delta\setminus\Delta_{\mu}}\frac{e^{\alpha}-\boldsymbol{t}}{e^{\alpha}-1}\right)=e^{w_0\mu}(1+R^{+}_{>0}).$$

An identity on Poincaré polynomials. Let us prove

$$\sum_{w\in W} w\prod_{\alpha>0}\frac{1-\boldsymbol{t}e^{-\alpha}}{1-e^{-\alpha}}=\sum_{w\in W}\boldsymbol{t}^{\ell(w)}.$$

Actually,

$$
\begin{aligned}
LHS &= \frac{1}{\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})}\sum_{w\in W}(-1)^{\ell(w)}\prod_{\alpha>0}(e^{\alpha/2}-te^{-\alpha/2})\\
&= \frac{1}{\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})}\sum_{w\in W}(-1)^{\ell(w)}\sum_{u\in W}(-t)^{\ell(u)}e^{u\rho}\\
&= \sum_{u\in W} t^{\ell(u)} = RHS.
\end{aligned}
$$

In the second equality, $\prod_{\alpha>0}(e^{\alpha/2}-te^{\alpha/2})$ is supported over weights in the convex hull of $\{u\rho\}_{u\in W}$. For any such weight λ, if e^{λ} is not killed by $\sum(-1)^{\ell(w)}$, then it has to be $u\rho$ for some $u\in W$. The third equality follows from Weyl character formula.

3. Cheridnik Pairing

3.1. Analogy of Discriminant. Recall the pairing over $\mathrm{Rep}(G)$ is

$$\langle U,V\rangle = \dim \mathrm{Hom}_G(U,V) = \begin{array}{c}\text{the multiplicity of the}\\ \text{trivial component of } V\otimes U^{\vee}.\end{array}$$

If G is reductive, then by Weyl character formula

$$\chi(\mathbb{V}(\lambda)) = \frac{1}{\Delta}\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}, \qquad \Delta=\prod_{\alpha>0}(1-e^{-\alpha}).$$

In particular, for any representation V, the multiplicity of the trivial component is

$$[e^0]\big(\operatorname{char}(V)\Delta\big) = \text{constant term of } \operatorname{char}(V)\Delta.$$

Thus the Hom-pairing induces the following pairing over $\mathrm{Rep}(G)$

$$\langle f,g\rangle = [e^0](\Delta f\overline{g}), \qquad \langle U,V\rangle = \langle \operatorname{char}(U),\operatorname{char}(V)\rangle.$$

Here is the example for SL_2:

dim	1	2	3	$\cdots$
χ	1	$e+e^{-1}$	e^2+1+e^{-2}	$\cdots$
$(1-e^{-2})\chi$	$1-e^{-2}$	$e-e^{-3}$	e^2-e^{-4}	$\cdots$

We are going to construct the $\boldsymbol{t}$-analogy and affine analogy of Δ. Define

$$\Delta^\circ_{\boldsymbol{q},\boldsymbol{t}} = \prod_{\alpha\in\widehat{\Delta}_+} \frac{1-e^\alpha}{1-\boldsymbol{t}\,e^\alpha} = \prod_{\alpha>0}\prod_{k=1}^{\infty} \frac{(1-\boldsymbol{q}^{k-1}e^\alpha)(1-\boldsymbol{q}^k e^{-\alpha})}{(1-\boldsymbol{t}\boldsymbol{q}^{k-1}e^\alpha)(1-\boldsymbol{t}\boldsymbol{q}^k e^{-\alpha})}.$$

We shall understand it as an element in

$$\sum_{\lambda\in P} \mathbb{Q}[\![\boldsymbol{q},\boldsymbol{t}]\!]\cdot e^\lambda \qquad \text{(possibly infinite sum).}$$

We normalize the constant term to be

$$\Delta_{\boldsymbol{q},\boldsymbol{t}} = \Delta^\circ_{\boldsymbol{q},\boldsymbol{t}}/([e^0]\Delta^\circ_{\boldsymbol{q},\boldsymbol{t}}).$$

We will show that $[e^\lambda]\Delta_{\boldsymbol{q},\boldsymbol{t}} \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$.

For any $i\in I\cup\{0\}$

$$\begin{aligned} s_i\Delta_{\boldsymbol{q},\boldsymbol{t}} &= \tfrac{1}{[e^0]\Delta^\circ_{\boldsymbol{q},\boldsymbol{t}}} \textstyle\prod_{\alpha\in\widehat{\Delta}_+} \tfrac{1-e^{s_i\alpha}}{1-\boldsymbol{t}\,e^{s_i\alpha}} \\ &= \tfrac{1}{[e^0]\Delta^\circ_{\boldsymbol{q},\boldsymbol{t}}} \tfrac{1-e^{-\alpha_i}}{1-\boldsymbol{t}\,e^{-\alpha_i}} \tfrac{1-\boldsymbol{t}\,e^{\alpha_i}}{1-e^{\alpha_i}} \textstyle\prod_{\alpha\in\widehat{\Delta}_+} \tfrac{1-e^\alpha}{1-\boldsymbol{t}\,e^\alpha} \\ &= \tfrac{1-\boldsymbol{t}\,e^{\alpha_i}}{\boldsymbol{t}-e^{\alpha_i}}\Delta_{\boldsymbol{q},\boldsymbol{t}}. \end{aligned}$$

This relation can be expressed as a system of linear equations over $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$ in $[e^\lambda]\Delta_{\boldsymbol{q},\boldsymbol{t}}$. Since we already have a solution in $\mathbb{Q}[\![q,t]\!]$, it has a solution in $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$. Let Δ' be the solution over $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$. We can assume $[e^0]\Delta'=1$ by normalization. Then $\Delta'/\Delta_{\boldsymbol{q},\boldsymbol{t}}$ is $\widehat{W}$-invariant. But

$$t_\mu e^\lambda = e^{\lambda-\langle\lambda,\mu\rangle\boldsymbol{\delta}} = \boldsymbol{q}^{-\langle\lambda,\mu\rangle}e^\lambda.$$

This shows $\Delta'/\Delta_{\boldsymbol{q},\boldsymbol{t}}=1$.

3.2. **Non-symmetric case.** Define the *Cherednik's inner product* on R by

$$\langle f,g\rangle_{\boldsymbol{q},\boldsymbol{t}} = [e^0](f\overline{g}\Delta_{\boldsymbol{q},\boldsymbol{t}})$$

where $\bar{\cdot}$ is the involution $e^\lambda\mapsto e^{-\lambda}$, $\boldsymbol{q}\mapsto\boldsymbol{q}^{-1}$, $\boldsymbol{t}\mapsto\boldsymbol{t}^{-1}$. Note that $\overline{\Delta_{\boldsymbol{q},\boldsymbol{t}}} = \Delta_{\boldsymbol{q},\boldsymbol{t}}$ So we have $\langle g,f\rangle_{\boldsymbol{q},\boldsymbol{t}} = \overline{\langle f,g\rangle_{\boldsymbol{q},\boldsymbol{t}}}$. Actually, we showed $\Delta_{\boldsymbol{q},\boldsymbol{t}}$ is the unique element in

$$\sum_{\lambda\in P}\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}\cdot e^\lambda \qquad \text{(possibly infinite sum).}$$

such that

$$s_i\Delta_{q,t} = \frac{1-t\,e^{\alpha_i}}{t-e^{\alpha_i}}\Delta_{q,t},\ \forall i\in I\cup\{0\}, \qquad [e^0]\Delta_{q,t}=1.$$

The conditions are bar-invariant.

Let us compute the adjoint of several operators

$$\begin{aligned}
\langle s_i f, g\rangle_{q,t} &= [e^0](s_i f\,\overline{g}\Delta_{q,t}) = [e^0](f\,\overline{s_i g}s_i\Delta_{q,t})\\
&= [e^0]\left(f\,\overline{s_i g}\frac{1-t\,e^{\alpha_i}}{t-e^{\alpha_i}}\Delta_{q,t}\right) = [e^0]\left(f\,\overline{\frac{1-t\,e^{\alpha_i}}{t-e^{\alpha_i}}s_i g}\Delta_{q,t}\right)\\
&= \left\langle f, \frac{1-t\,e^{\alpha_i}}{t-e^{\alpha_i}}s_i g\right\rangle_{q,t}
\end{aligned}$$

Thus the adjoint of

$$T_i = t s_i + (t-1)\frac{s_i-1}{e^{\alpha_i}-1} = \frac{t\,e^{\alpha_i}-1}{e^{\alpha_i}-1}s_i + \frac{1-t}{e^{\alpha_i}-1}$$

is

$$\begin{aligned}
&\frac{1-t\,e^{-\alpha_i}}{t-e^{-\alpha_i}}s_i\frac{t^{-1}\,e^{-\alpha}-1}{e^{-\alpha_i}-1} + \frac{1-t^{-1}}{e^{-\alpha_i}-1}\\
&= \frac{1-t^{-1}\,e^{-\alpha_i}}{t^{-1}-e^{-\alpha_i}}\frac{t^{-1}\,e^{\alpha_i}-1}{e^{\alpha_i}-1}s_i + \frac{1-t^{-1}}{e^{-\alpha_i}-1}\\
&= \frac{t^{-1}\,e^{-\alpha_i}-1}{e^{-\alpha_i}-1}s_i + \frac{1-t^{-1}}{e^{-\alpha_i}-1} = T_i^{-1}.
\end{aligned}$$

As a result, $\langle T_i f, g\rangle_{q,t} = \langle f, T_i^{-1}g\rangle_{q,t}$, i.e. $\langle T_i f, T_i g\rangle_{q,t} = \langle f, g\rangle_{q,t}$. As a result,

$$\langle Y^\mu f, Y^\mu g\rangle_{q,t} = \langle f, g\rangle_{q,t}.$$

As a result, nonsymmetric Macdonald polynomials can be characterized by

(1) $E_\lambda = e^\lambda + (\text{lower terms})$
(2') $\langle E_\lambda, E_\mu\rangle_{q,t} = 0$ if $\lambda\neq\mu$.

3.3. **Symmetric case.** Similarly, symmetric Macdonald polynomials can be characterized by

(1) $P_\lambda = m_\lambda + (\text{lower terms})$
(2') $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ if $\lambda\neq\mu$.

But we can simplify $\langle\cdot,\cdot\rangle_{\boldsymbol{q},\boldsymbol{t}}$.

Here is a useful trick when doing computation. Note that for $f \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}} = \mathbb{Q}(\boldsymbol{q},\boldsymbol{t})$,

$$f(\boldsymbol{q},\boldsymbol{t}) = 0 \iff f(\boldsymbol{q},\boldsymbol{q}^{\boldsymbol{\kappa}}) = 0 \text{ for } \boldsymbol{\kappa} = 1,2,3,4,\ldots.$$

Thus assuming $\boldsymbol{t} = \boldsymbol{q}^{\boldsymbol{\kappa}}$ is harmless. For example,

$$\begin{aligned}\Delta^{\circ}_{\boldsymbol{q},\boldsymbol{t}} &= \prod_{\alpha\in\widehat{\Delta}_+} \frac{1-e^{\alpha}}{1-\boldsymbol{t}\,e^{\alpha}} = \prod_{\alpha>0}\prod_{k=1}^{\infty} \frac{(1-\boldsymbol{q}^{k-1}e^{\alpha})(1-\boldsymbol{q}^{k}e^{-\alpha})}{(1-\boldsymbol{t}\boldsymbol{q}^{k-1}e^{\alpha})(1-\boldsymbol{t}\boldsymbol{q}^{k}e^{-\alpha})}\\ &= \prod_{\substack{\alpha>0\\ 1\le k<\boldsymbol{\kappa}}} (1-\boldsymbol{q}^{k-1}e^{\alpha})(1-\boldsymbol{q}^{k}e^{-\alpha})\\ &= \prod_{\alpha>0}\frac{1-\boldsymbol{q}^{\boldsymbol{\kappa}}e^{-\alpha}}{1-e^{-\alpha}} \prod_{\substack{\alpha>0\\ 0\le k<\boldsymbol{\kappa}}} (1-\boldsymbol{q}^{k}e^{\alpha})(1-\boldsymbol{q}^{k}e^{-\alpha}).\end{aligned}$$

Since

$$\sum_{w\in W} w\left(\prod_{\alpha>0}\frac{1-\boldsymbol{q}^{\boldsymbol{\kappa}}e^{-\alpha}}{1-e^{-\alpha}}\right) = W(\boldsymbol{q}^{\boldsymbol{\kappa}}) \text{ is a constant}$$

and

$$\Delta'^{\circ}_{\boldsymbol{q},\boldsymbol{t}} = \prod_{\alpha>0}\prod_{k\ge 0}\frac{1-\boldsymbol{t}\boldsymbol{q}^{k}e^{\alpha}}{1-\boldsymbol{q}^{k}e^{\alpha}}\frac{1-\boldsymbol{t}\boldsymbol{q}^{k}e^{-\alpha}}{1-\boldsymbol{q}^{k}e^{-\alpha}} = \prod_{\substack{\alpha>0\\ 0\le k<\boldsymbol{\kappa}}} (1-\boldsymbol{q}^{k}e^{\alpha})(1-\boldsymbol{q}^{k}e^{-\alpha})$$

is W-invariant. By denoting $\Delta'_{\boldsymbol{q},\boldsymbol{t}} := \Delta'^{\circ}_{\boldsymbol{q},\boldsymbol{t}}/[e^0]\Delta'^{\circ}_{\boldsymbol{q},\boldsymbol{t}}$, we have

$$\langle f,g\rangle_{\boldsymbol{q},\boldsymbol{t}} = [e^0]\big(f\,\overline{g}\,\Delta'_{\boldsymbol{q},\boldsymbol{t}}\big).$$

We remark that in type A, for $f,g\in\Lambda$, we have

$$\lim_{n\to\infty}\langle f[X_n], g[X_n]\rangle_{\boldsymbol{q},\boldsymbol{t}} = \langle f,g\rangle_{\boldsymbol{q},\boldsymbol{t}}.$$

(If we want to extend to $\Lambda_{\boldsymbol{q},\boldsymbol{t}}$, we need to replace g by $g|_{\boldsymbol{q}\mapsto\boldsymbol{q}^{-1},\boldsymbol{t}\mapsto\boldsymbol{t}^{-1}}$) This is proved by the computation of $\langle P_\lambda, P_\lambda\rangle_{\boldsymbol{q},\boldsymbol{t}}$ and $\langle P_\lambda[X_n], P_\lambda[X_n]\rangle_{\boldsymbol{q},\boldsymbol{t}}$. The computation shows the left-hand side is not a constant when $\boldsymbol{t}\neq\boldsymbol{q}$, even for $n\gg 0$, see [Mac3].

References.

[Mac1] Affine Hecke algebras and orthogonal polynomials (Bourbaki seminar), by I.G. Macdonald.

[Mac2] Affine Hecke algebras and orthogonal polynomials (Cambridge University Press), by I.G. Macdonald.

[Kir] Lectures on the affine Hecke algebras and Macdonald conjectures, by A.A. Kirillov. Jr

[Hai] Cherednik algebras, Macdonald polynomials and combinatorics, by M. Haiman

4. Specializations

Recall that

The Demazure–Lusztig operator
$$T_i = t s_i + (t-1)\frac{s_i - \mathrm{id}}{e^{\alpha_i} - 1}.$$
When $\langle \lambda, \alpha_i^\vee \rangle > 0$,
$$E_{s_i\lambda} = \left(T_i + \frac{t-1}{t^{\langle \rho_\lambda, \alpha_i^\vee \rangle} q^{\langle \lambda, \alpha_i^\vee \rangle} - 1} \right) E_\lambda. \qquad (*)$$
For dominant λ,
$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(E_\lambda \prod_{\alpha > 0} \frac{e^\alpha - t}{e^\alpha - 1} \right)$$

4.1. The limit $q \to 0$. The result is
$$q \to 0, \qquad \begin{cases} \text{when } \lambda \text{ is domiannt} & E_\lambda = e^\lambda \\ \text{if } \langle \alpha_i^\vee, \lambda \rangle > 0 & E_{s_i\lambda} = t T_i^{-1} E_\lambda. \end{cases}$$
Actually, when $q \to 0$, the Cherednik pairing
$$\Delta^\circ_{q,t} = \prod_\alpha \frac{1 - e^\alpha}{1 - t e^\alpha} \in \sum_{\text{positive } \beta} \mathbb{Q}[t] e^\beta.$$
So for dominant λ, and any $\mu < \lambda$ (that is, $\mu^+ <_{\mathrm{dom}} \lambda$)
$$[e^0]\left(e^\lambda e^{-\mu} \Delta^\circ_{q,t}\right) = [e^\mu]\left(e^\lambda \Delta^\circ_{q,t}\right) = 0.$$
From the fact that
$$E_\mu = e^\mu + (\text{lower term}),$$
we see $E_\lambda = e^\lambda$ (from the construction, E_λ was constructed when E_μ for all $\mu < \lambda$ are constructed).

By $(*)$, when specialize $q \to 0$, we get
$$\begin{aligned} E_{s_i\lambda} &= (T_i - (t-1))E_\lambda = t T_i^{-1} E_\lambda \\ &= \left(s_i + (1-t)\frac{s_i - \mathrm{id}}{e^{-\alpha_i} - 1} \right) E_\lambda. \end{aligned}$$
Thus $E_\lambda|_{q=0}$ essentially gives the *Iwahori–Whittaker functions.*

In particular, if we specialize $\boldsymbol{q} \to 0, \boldsymbol{t} \to 0$, we will get

$$E_{s_i\lambda} = \left(s_i + \frac{s_i - \mathrm{id}}{e^{-\alpha_i} - 1}\right) E_\lambda = \frac{s_i - e^{-\alpha_i}}{1 - e^{-\alpha_i}} E_\lambda.$$

Thus $E_\lambda|_{\boldsymbol{q}=0,\boldsymbol{t}=0}$ gives the Demazure character of finite Lie algebra $\mathfrak{g}^\vee$.

Now we have

$$P_\lambda = \frac{1}{W_\lambda(\boldsymbol{t})} \sum_{w\in W} w\left(e^\lambda \prod_{\alpha>0} \frac{e^\alpha - \boldsymbol{t}}{e^\alpha - 1}\right).$$

This is known as Hall–Littlewood polynomials. The representation theoretic explanation is

$$\boldsymbol{t}^{\langle\rho,\lambda\rangle} P_\lambda|_{\boldsymbol{t}\mapsto\boldsymbol{t}^{-1},\boldsymbol{q}=0}$$

gives the spherical funtion in the dual group.

To explain the relation, we need an algebraic version of Satake equivalence. Let G be a reductive group. As usual, let $\mathbf{K}$ be a non-Archmedean local field with ring of integers $\mathbf{O}$ and residue field $\mathbf{k}$. Since $G_{\mathbf{O}}$ is compact, we take the Haar measure μ with $\mu(G_{\mathbf{O}}) = 1$. We have

$$G_{\mathbf{K}} = \bigsqcup_{\lambda\in P^\vee_{\mathrm{dom}}} G_{\mathbf{O}} t^\lambda G_{\mathbf{O}} = \bigsqcup_{\lambda\in P^\vee_{\mathrm{dom}}} O(\lambda).$$

We can define a convolution product over

$$\mathsf{Fun}({}_{G_{\mathbf{O}}}\backslash G_{\mathbf{K}}/{}_{G_{\mathbf{O}}}) := \bigoplus_{\lambda\in P^\vee_{\mathrm{dom}}} \mathbf{1}_{O(\lambda)}$$

with convolution product

$$(f * g)(x) = \int_{G_{\mathbf{O}}} f(xy^{-1})g(y)dy.$$

Explicitly, $\mathbf{1}_\lambda * \mathbf{1}_\mu = \sum_\nu c^\nu_{\lambda\mu} \mathbf{1}_\nu$ with

$$\begin{aligned} c^\nu_{\lambda\mu} &= \int_G [t^\nu y^{-1} \in O(\lambda)] \cdot [y \in O(\mu)]\, dx \\ &= \#\{y \in O(\mu) : t^\nu y^{-1} \in O(\lambda)\}/G_{\mathbf{O}} \\ &= \#\{(x, y) \in O(\lambda) \times O(\mu) : xy = t^\nu\}/(x, gy) \sim (xg, y), g \in G_\varnothing \\ &= \#\,\text{fibre of } O(\lambda) \underset{G_{\mathbf{O}}}{\times} O(\mu) \to G_{\mathbf{K}} \supset O(\nu). \end{aligned}$$

It is well known that this algebra is isomorphic to the spherical Hecke algebra:

$$\mathcal{H}_G \cong e\widehat{H}_{\boldsymbol{t}}(W)e\big|_{\boldsymbol{t}=\#\mathbf{k}} \quad \text{with} \quad e = \frac{1}{W(\boldsymbol{t})}\sum_{w\in Wt_\lambda W} T_w.$$

Under the isomorphism,

$$\mathbf{1}_{O(\lambda)} \longmapsto \frac{1}{W(\boldsymbol{t})}e\left(\sum_{w\in Wt_\lambda W} T_w\right)e.$$

Note that

$$Wt_\lambda W = \bigsqcup_{\lambda'\in W\lambda} t_{\lambda'}W.$$

We write $t_{\lambda'} = u_{\lambda'}v_{\lambda'}$ with $u_{\lambda'}$ minimal representative of $t_{\lambda'}W$, and $v_{\lambda'} \in W$. It is known that $v_{\lambda'}$ is the minimal element such that $v_{\lambda'}\lambda = \lambda'$. Thus

$$\bigsqcup_{\lambda'\in W\lambda} t_{\lambda'}W = \bigsqcup_{\lambda'\in W\lambda} t_{\lambda'}v_{\lambda'}^{-1}W.$$

So any element $w \in Wt_\lambda W$ can be uniquely written as $w = vt_\lambda u$ with $v \in W^\lambda$, $u \in W$ and $\ell(w) = -\ell(v) + \ell(t_\lambda) + \ell(u)$. Recall that $Y^\lambda = \boldsymbol{t}^{-\langle\rho,\lambda\rangle}T_{t_\lambda}$ for λ dominant. As a result,

$$\begin{aligned}
\mathbf{1}_{O(\lambda)} &\mapsto \frac{1}{W(\boldsymbol{t})}e\left(\sum_{w\in Wt_\lambda W} T_w\right)e \\
&= e\left(\sum_{v\in W^\lambda} T_v^{-1}\right)T_{t_\lambda}\left(\sum_{u\in W} T_u\right)e = \boldsymbol{t}^{\langle\rho,\lambda\rangle}\frac{W(\boldsymbol{t}^{-1})}{W_\lambda(\boldsymbol{t}^{-1})}eY^\lambda e \\
&= \boldsymbol{t}^{\langle\rho,\lambda\rangle}\frac{W(\boldsymbol{t}^{-1})}{W_\lambda(\boldsymbol{t}^{-1})}e\left(\frac{1}{W(\boldsymbol{t})}\sum_{w\in W} w\left(Y^\lambda\prod_{\alpha>0}\frac{Y^{-\alpha}-\boldsymbol{t}}{Y^{-\alpha}-1}\right)\right)e \\
&= e\left(\frac{\boldsymbol{t}^{\langle\rho,\lambda\rangle}}{W_\lambda(\boldsymbol{t}^{-1})}\sum_{w\in W} w\left(Y^\lambda\prod_{\alpha>0}\frac{1-\boldsymbol{t}^{-1}Y^{-\alpha}}{1-Y^{-\alpha}}\right)\right)e.
\end{aligned}$$

That is, under the identification $e\widehat{H}_{\boldsymbol{t}}e(W) = \mathbb{Q}_{\boldsymbol{t}}[Y]^W$, $\mathbf{1}_{O(\lambda)}$ is the function

$$K_\lambda := \frac{\boldsymbol{t}^{\langle\rho,\lambda\rangle}}{W_\lambda(\boldsymbol{t}^{-1})} \sum_{w\in W} w\left(Y^\lambda \prod_{\alpha>0} \frac{1-\boldsymbol{t}^{-1}Y^{-\alpha}}{1-Y^{-\alpha}}\right).$$

Denote

$$\chi_\lambda = \sum_{w\in W} w\left(Y^\lambda \prod_{\alpha>0} \frac{Y^\alpha}{Y^\alpha - 1}\right)$$

the Weyl character. By direct computation,

$$K_\lambda \in \boldsymbol{t}^{\langle\rho,\lambda\rangle}\chi_\lambda + \sum_{\mu<\lambda} \mathbb{Q}[\boldsymbol{t}]\chi_\mu.$$

Let us describe the bar-involution over $e\widehat{H}_{\boldsymbol{t}}(W)e$. Recall that $\overline{T_{t_\lambda}} = T_{t_{-\lambda}}^{-1}$. So

$$\overline{\frac{1}{W(\boldsymbol{t})} e\left(\sum_{w\in Wt_\lambda W} T_w\right) e} = \frac{1}{W(\boldsymbol{t})} e\left(\sum_{w\in Wt_\lambda W} T_{w^{-1}}^{-1}\right) e = \mathrm{const}\cdot eY^{w_0\lambda}e.$$

Do the same computation as above, we will see $\overline{K_\lambda} = K_\lambda|_{\boldsymbol{t}\mapsto\boldsymbol{t}^{-1}}$. In particular, for a symmetric $f \in \mathbb{Q}[Y]^W$, $\overline{f} = f$. As a result, the Kazhdan–Lusztig basis of $e\widehat{H}_{\boldsymbol{t}}(W)e$ is Weyl characters.

Perhaps let us state them in term of sheaves (geometric Satake). Denote

$$\Sigma_\lambda^\circ := G_{\mathbf{O}}t^\lambda G_{\mathbf{O}}/G_{\mathbf{O}} \subset \mathrm{Gr}_G := G_{\mathbf{K}}/G_{\mathbf{O}}.$$

Note that $\dim\Sigma_\lambda^\circ = 2\langle\rho,\lambda\rangle$. We have

$$K_\lambda \longleftrightarrow \mathbf{1}_{\Sigma_\lambda^\circ} \in D_{G_{\mathbf{O}}}(\mathrm{Gr}_G),$$
$$\chi_\lambda \longleftrightarrow \mathbf{IC}_{\Sigma_\lambda^\circ} \in \mathbf{SSPerv}(\mathrm{Gr}_G),$$

where the intersection complex is normalized such that $\mathbf{IC}(\lambda)|_{\Sigma_\lambda^\circ} = \mathbb{Q}[\dim\Sigma_\lambda^\circ]$.

References.

[Mac] Symmetric functions and Hall polynomials (Oxford Science Publication), by I.G. Macdonald.

[Ach] Perverse sheaves and applications to representation theory, by P. N. Achar.

4.2. **The limit $t \to 0$.** In this case, $E_\lambda \in R$ is the character of level 1 affine Demazure module. Let $\mathfrak{g}$ be a semisimple Lie algebra. Recall that the untwisted *affine Kac–Moody algebra* is

$$\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\partial \oplus \mathbb{C}c, \qquad L\mathfrak{g} = \mathfrak{g}[t^{\pm 1}] = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$$

with $\partial = t\frac{\partial}{\partial t}$ and c central. We are working in

$$\begin{array}{ccc} \widehat{\mathfrak{h}} & = & \overbrace{\mathbb{C}c \oplus \mathfrak{h}}^{\text{coroots}} \overbrace{\oplus\ \mathbb{C}\partial}^{\text{coweights}} \\ & & \updownarrow \qquad \updownarrow \qquad \updownarrow \\ \widehat{\mathfrak{h}}^* & = & \underbrace{\mathbb{C}\Lambda_0 \oplus \mathfrak{h}^*}_{\text{weights}} \underbrace{\oplus\ \mathbb{C}\boldsymbol{\delta}}_{\text{roots}} \end{array} \qquad \begin{array}{c} \langle c, \Lambda_0 \rangle = \langle \partial, \boldsymbol{\delta} \rangle = 1 \\ \langle c, \boldsymbol{\delta} \rangle = \langle \partial, \Lambda \rangle = 0 \end{array}$$

$$\begin{array}{cc} \text{simple roots} & \text{simple coroots} \\ \{\alpha_i\} \cup \{\alpha_0 = \boldsymbol{\delta} - \theta\} & \{\alpha_i^\vee\} \cup \{\alpha_0^\vee = c - \theta^\vee\} \end{array}$$

Note that the central element c can be written as a positive sum of simple coroots

$$c \in \alpha_0 + \sum \langle \theta^\vee, \omega_i \rangle \alpha_i.$$

Note that $\langle \theta^\vee, \omega_i \rangle$ is always positive. The fundamental weight Λ_i for $i \in I \cup \{0\}$ is normalized such that the coefficient of $\boldsymbol{\delta}$ is zero. In other word,

$$\Lambda_i = \begin{cases} \Lambda_0, & i = 0, \\ \omega_i + \langle \theta^\vee, \omega_i \rangle \Lambda_0, & i \neq 0. \end{cases}$$

For an affine weight $\lambda \in \mathbb{Z}\Lambda_0 \oplus P \oplus \mathbb{Z}\boldsymbol{\delta}$, we call the coefficient of Λ_0, i.e. $\langle c, \lambda \rangle$, the *level of* λ.

- If λ is a dominant affine weight of level 0, then

  $$\lambda \in \mathbb{Z}\boldsymbol{\delta}.$$

  Then the dimension of $\mathbb{V}(\widehat{\lambda})$ is one.

- If λ is a dominant affine weight of level 1, then
$$\lambda \in \Lambda_i + \mathbb{Z}\boldsymbol{\delta}$$
with $\langle \theta^\vee, \omega_i \rangle = 1$. This happens exactly when i is miniscule, equivalently, conjugate to the affine node under graph automorphism of the Dynkin diagram.

If λ has level ℓ, then the action of c on the irreducible integrable module $\mathbb{V}(\lambda)$ is always ℓ, thus to compute the character, specialization of $e^{\Lambda_0} = 1$ does not loss any generality if level is known.

Denote
$$\widehat{\mathfrak{b}} = \mathfrak{b} \oplus \oplus \mathbb{C}\partial \oplus \mathbb{C}c \oplus tL\mathfrak{g}.$$
For an affine dominant weight λ and affine Weyl group element $w \in \widehat{W}$, the Demazure module
$$\mathbb{V}_w(\lambda) = \text{the } \widehat{\mathfrak{b}}\text{-submodule generated by } w \cdot v_{\text{highest}}.$$
We have the following formula for its character
$$\begin{aligned} \mathbf{char}\mathbb{V}_{\text{id}}(\lambda) &= e^\lambda, \\ \mathbf{char}\mathbb{V}_w(\lambda) &= \pi_w\big(\mathbf{char}\mathbb{V}_{\text{id}}(\lambda)\big). \end{aligned}$$
Here π_w is the Demazure operators
$$\pi_i = \frac{\mathrm{id} - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}.$$
Since π_i satisfies the Braid relation, it is well-defined to denote π_w.

Now let us describe the level 1 action of $\widehat{W}$ on $\Lambda_0 + P$ mod $\boldsymbol{\delta}$. For a finite weight λ,
$$\begin{aligned} s_i(\Lambda_0{+}\lambda) &= \Lambda_0{+}\lambda + \langle \alpha_i^\vee, \Lambda_0{+}\lambda \rangle \alpha_i \\ &= \begin{cases} \Lambda_0{+}s_i\lambda, & i \in I, \\ \Lambda_0{+}(r_\theta\lambda - \theta) + \boldsymbol{\delta}, & i = 0. \end{cases} \end{aligned}$$
Let us denote for a level one weight $\lambda = \Lambda_0 + \lambda_0 + k\boldsymbol{\delta}$,
$$E_\lambda = \boldsymbol{q}^k E_{\lambda_0}.$$
The result is, for type ADE, we have
$$\mathbf{char}\mathbb{V}_w(\lambda) = e^{\Lambda_0} E_{w\lambda}|_{t=0},$$
for an affine domiant weight λ of level 1.

By $(*)$, when specializing $t \to 0$,

$$E_{s_i\lambda} = \left(-\frac{s_i - \mathrm{id}}{e^{\alpha_i} - 1} + 1\right) E_\lambda = \left(\frac{-s_i + e^{\alpha_i}\,\mathrm{id}}{e^{\alpha_i} - 1} + 1\right) E_\lambda$$
$$= \left(\frac{\mathrm{id} - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}\right) E_\lambda = \pi_i E_\lambda.$$

There are two methods of proving $i = 0$.

- By the Dynkin diagram automorphisms, we can transform affine node to a finite node, this proves for type A, D and E_6.
- In general, we need the $i = 0$ analogy of the induction formula (Cherednik intertwine theory).

We mention that the non-simply laced cases, we cannot apply the Dynkin diagram automorphisms since the affine Dynkin diagram of dual type are different. We remark that when λ is anti-dominant, $P_\lambda = E_\lambda$. This follows from the fact $\pi_i^2 = \pi_i$.

References.

[San] On the connection between Macdonald polynomials and Demazure characters by Yasmine B. Sanderson.

[Ion] Nonsymmetric Macdonald polynomials and Demazure characters by Ion.

5. Degeneration of DAHA

5.1. **Degeneration of Hecke Algebra.** Let u, v, β be three variables. Let us consider the Hecke algebra defined by

$$(T_i - u)(T_i + v) = 0.$$

Note that T_i/v satisfies the usual Hecke algebra relation with $t = u/v$:

$$(T_i/v - u/v)(T_i/v + 1) = 0.$$

Let us denote $R = \mathbb{Q}[e^{\beta\lambda}]_{\lambda \in P}$ the group algebra of P. We should understand the symbol

$$e^{\beta\lambda} = 1 + \beta\lambda + \frac{\beta^2}{2}\lambda^2 + \frac{\beta^3}{3!}\lambda^3 + \cdots \in \mathcal{O}(\mathfrak{t})[\![\beta]\!].$$

Denote cherednik representation on R

$$T_i f = us_i + (u-v)\frac{s_i f - f}{e^{\beta\alpha_i} - 1},$$
$$X_i f = e^{\beta x_i} f.$$

Then

$$T_i X^\lambda - X^{s_i\lambda} T_i = (u-v)\frac{X^{s_i\lambda} - X^\lambda}{X^{\alpha_i} - 1}.$$

Note that T_i is the unique operator such that T_i's satisfy the relations of Hecke algebra and $T_i 1 = u$.

Group algebra. Let us take $u = v = \beta = 1$. We see $T_i = s_i$, and $X^\lambda =$ mult by e^λ. The relations are

$$T_i^2 = 1, \quad X^\lambda X^{-\mu} = X^{\lambda-\mu},$$
$$T_i X^\lambda - X^{s_i\lambda} T_i = 0.$$

It gives the group algebra.

Degenerate Group algebra. Let us take $u = v = 1$ but set $\beta \to 0$. Then

$$T_i = s_i, \qquad x_\lambda = \lim_{\beta\to 0} \frac{X^\lambda - 1}{\beta} = \text{mult by } \lambda$$

. The relations are

$$T_i^2 = 1, \quad x_\lambda - x_\mu = x_{\lambda-\mu},$$
$$T_i x_\lambda - x_{s_i\lambda} T_i = 0.$$

It gives the group algebra.

Zero Hecke algebra. Let us take $u = -\beta$ and $v = 0$, i.e. $(T_i - \beta)T_i = 0$. Then

$$T_i = -\beta s_i - \beta\frac{s_i - 1}{e^{\beta\alpha_i} - 1} = \frac{1 - e^{\beta\alpha_i} s_i}{-(1 - e^{\beta\alpha_i})/\beta}.$$

If we specialize further $\beta = -1$, we get

$$T_i = \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}, \qquad X^\lambda = \text{mult-by } e^{-\lambda}.$$

The operator is originally found by Demazure. The relations are

$$T_i^2 = T_i, \qquad X^\lambda X^{-\mu} = X^{\lambda-\mu}$$

$$T_i X^\lambda - X^{s_i\lambda} T_i = \frac{X^{s_i\lambda} - X^\lambda}{X^{\alpha_i} - 1}$$

Nil-Hecke algebra. If we specialize $\beta = 0$ in stead, we get

$$T_i = \frac{1-s_i}{\alpha_i}, \qquad \lim_{\beta\to 0} \frac{X^\lambda - 1}{\beta} = x_\lambda = \text{mult by } \lambda.$$

The operator is the BGG Demazure operator. The relations are

$$T_i^2 = 0, \qquad x_\lambda - x_\mu = x_{\lambda-\mu}$$

$$T_i x_\lambda - x_{s_i\lambda} T_i = \langle \alpha_i^\vee, \lambda\rangle.$$

Note that by induction, it is not hard to prove

$$T_w x_\lambda = x_{w\lambda} T_w + \sum_{\substack{\alpha>0, w=ur_\alpha \\ \ell(w)=\ell(u)+1}} \langle \alpha^\vee, \lambda\rangle T_u.$$

Graded Hecke algebra. Let us take $u = e^{-\beta}$, and $v = 1$. We have

$$T_i = e^{-\beta} s_i + (e^{-\beta} - 1)\frac{s_i - 1}{e^{\beta\alpha_i} - 1}.$$

Let $\beta \to 0$, we get

$$T_i = s_i + \frac{1-s_i}{\alpha_i}, \qquad \lim_{\beta\to 0} \frac{X^\lambda - 1}{e^{-\beta} - 1} = x_\lambda = \text{mult by } \lambda.$$

This operator appears in the study of homology of Springer resolution. The relations are

$$T_i^2 = 1, \qquad x_\lambda - x_\mu = x_{\lambda-\mu}$$

$$T_i x_\lambda - x_{s_i\lambda} T_i = \langle \alpha_i^\vee, \lambda\rangle.$$

By induction, it is not hard to prove

$$T_w x_\lambda = x_{w\lambda} T_w + \sum_{\substack{\alpha>0, w=ur_\alpha \\ \ell(w)\geq\ell(u)+1}} \langle \alpha^\vee, \lambda\rangle T_u.$$

Note that the sum is over $\mathrm{Inv}(w) = \{\alpha > 0 : w\alpha < 0\}$. Moreover, we can rewrite $T_u = T_w T_{r_\alpha}$.

5.2. **Degeneration of DAHA.** Recall that a half of $\ddot{H}_{q,t}(W)$ is $\widehat{H}_{q,t}(W)$ and another half is $\widehat{H}_{q,t}(W^\vee)$. We can degenerate both of them.

Double affine Weyl group. We degenerate

$$\begin{aligned} \widehat{H}_{q,t}(W) &\longrightarrow \text{Group algebra}, \\ \widehat{H}_{q,t}(W^\vee) &\longrightarrow \text{Group algebra}. \end{aligned}$$

In this case, we have

$$Y^\mu X^\lambda = q^{-\langle \mu, \lambda \rangle} X^\lambda Y^\mu.$$

Actually, we can view Y^μ as the q-difference operator

$$e^\lambda \longmapsto q^{-\langle \lambda, \mu \rangle} e^\lambda.$$

Trigonometric degeneration. We degenerate

$$\begin{aligned} \widehat{H}_{q,t}(W) &\longrightarrow \text{Group algebra}, \\ \widehat{H}_{q,t}(W^\vee) &\longrightarrow \text{Degenerate Hecke algebra}. \end{aligned}$$

Let us denote $\hbar = x_\delta$. Then

$$\begin{aligned} Y^\mu x_\lambda &= (x_\lambda - \hbar\langle \lambda, \mu \rangle) Y^\mu + \sum_{\substack{\alpha > 0 \\ 0 \le k < \langle \alpha^\vee, \mu \rangle}} \langle \lambda, \alpha \rangle Y^\mu T_{r_{\alpha + k\delta}} \\ &= (x_\lambda - \hbar\langle \lambda, \mu \rangle) Y^\mu + \sum_{\alpha > 0} \langle \lambda, \alpha \rangle Y^\mu \frac{1 - Y^{-\langle \alpha^\vee, \mu \rangle \alpha_i^\vee}}{1 - Y^{-\alpha_i^\vee}} T_{r_\alpha} \\ &= (x_\lambda - \hbar\langle \lambda, \mu \rangle) Y^\mu + \sum_{\alpha > 0} \langle \lambda, \alpha \rangle \frac{Y^\mu - Y^{r_\alpha \mu}}{1 - Y^{-\alpha_i^\vee}} T_{r_\alpha}. \end{aligned}$$

Thus

$$x_\lambda Y^\mu = Y^\mu x_\lambda - \hbar\langle \lambda, \mu \rangle Y^\mu - \sum_{\alpha > 0} \langle \lambda, \alpha \rangle \frac{Y^\mu - Y^{r_\alpha \mu}}{1 - Y^{-\alpha_i^\vee}} T_{r_\alpha}.$$

Now let us consider

$$\begin{aligned} \widehat{H}_{q,t}(W) &\longrightarrow \text{Degenerate Hecke algebra}, \\ \widehat{H}_{q,t}(W^\vee) &\longrightarrow \text{Group algebra}. \end{aligned}$$

We have

$$y_\mu X^\lambda = X^\lambda y_\mu + \hbar\langle \lambda, \mu \rangle X^\lambda - \sum_{\alpha > 0} \langle \mu, \alpha \rangle \frac{X^\lambda - X^{r_\alpha \lambda}}{1 - X^{-\alpha_i}} T_{r_\alpha}.$$

Recall that X^λ acts by product with e^λ; T_{r_α} acts by r_α. In this case, y_λ is given by the *trigonometric Dunkl operator*

$$y_\lambda f = \hbar\,\partial_\mu f - \sum_{\alpha>0}\langle\mu,\alpha\rangle\frac{f - r_\alpha f}{1-e^{-\alpha}}.$$

Let us explain ∂_μ.

- Let us denote the differential operator,

$$(\partial_\mu f)(x) = \lim_{t\to 0}\frac{f(x+\mu t)-f(x)}{t}.$$

  Note that $f \in R$ are viewed as function over $\mathfrak{t}$, say e^λ are viewed as $x \mapsto e^{\langle\lambda,x\rangle}$. For example, $\partial_\mu e^\lambda = \langle\mu,\lambda\rangle e^\lambda$. We have Leibiniz rule

$$\partial_\mu(fg) = (\partial_\mu f)g + f(\partial_\mu g).$$

  In particular,

$$\hbar\partial_\mu X^\lambda = X^\lambda\hbar\partial_\mu + \hbar\langle\lambda,\mu\rangle e^\lambda.$$

- Let us denote the operator

$$G_\alpha f = \frac{f - r_\alpha f}{1-e^{-\alpha}}.$$

  Then we can check directly that

$$\begin{aligned} G_\alpha(fg) &= \frac{fg - (r_\alpha f)(r_\alpha g)}{1-e^{-\alpha}} \\ &= \frac{f(r_\alpha g) - (r_\alpha f)(r_\alpha g)}{1-e^{-\alpha}} + \frac{fg - f(r_\alpha g)}{1-e^{-\alpha}} \\ &= (G_\alpha f)(r_\alpha g) + f(G_\alpha g). \end{aligned}$$

  In particular,

$$G_\alpha X^\lambda - X^\lambda G_\alpha = \frac{X^\lambda - X^{r_\alpha\lambda}}{1-X^{-\alpha}}T_{r_\alpha}$$

From the above discussion, y_λ is given by the operator above.

Rational degeneration. Now let us consider

$$\widehat{H}_{q,t}(W) \longrightarrow \text{Degenerate Hecke algebra}$$
$$\widehat{H}_{q,t}(W^\vee) \longrightarrow \text{Degenerate group algebra.}$$

It can be computed by taking limit above,

$$\begin{aligned} y_\mu x_\lambda &= x_\lambda y_\mu + \hbar\langle\lambda,\mu\rangle - \sum_{\alpha>0}\langle\mu,\alpha\rangle\frac{x_\lambda - x_{r_\alpha\lambda}}{x_{\alpha_i}} r_\alpha \\ &= x_\lambda y_\mu + \hbar\langle\lambda,\mu\rangle - \sum_{\alpha>0}\langle\mu,\alpha\rangle\langle\lambda,\alpha^\vee\rangle r_\alpha. \end{aligned}$$

That is,

$$[y_\mu, x_\lambda] = \hbar\langle\lambda,\nu\rangle - \sum_{\alpha>0}\langle\mu,\alpha\rangle\langle\lambda,\alpha^\vee\rangle r_\alpha.$$

Moreover, y_μ is given by the *rational Dunkl operator*

$$y_\mu f = \hbar\,\partial_\mu f - \sum_{\alpha>0}\langle\lambda,\alpha\rangle\frac{f - r_\alpha f}{\alpha}.$$

References.

[Che] Double Affine Hecke Algebras by I Cherednik.

6. Macdonald functions

6.1. **Symmetric functions.** In type A, the Weyl group $W = \mathcal{S}_n$, we should deal with

$$R^W = \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathcal{S}_n}.$$

We have a smaller subring

$$\Lambda_n = \mathbb{Q}[z_1, \ldots, z_n]^{\mathcal{S}_n} = \text{symmetric functions}.$$

We define

$$\Lambda = \varprojlim[\cdots \xleftarrow{z_n=0} \Lambda_n \xleftarrow{z_{n+1}=0} \cdots].$$

Each element of Λ can be viewed as a function over the space

$$\{(z_i)_{i=0}^{\infty} : z_i = 0 \text{ for almost all } i\}$$

and thus is called a *symmetric function*. Recall the following functions:

$$m_\lambda(z) = \sum_{\lambda' \in \mathcal{S}_n \lambda} z^{\lambda'}, \qquad \text{monomial symmetric functions;}$$

$$p_r(z) = \sum_i z_i^r, \qquad \text{(Newton's) power sum;}$$

$$e_r(z) = \sum_{i_1 < \cdots < i_r} z_{i_1} \cdots z_{i_r}, \qquad \text{elementary symmetric functions;}$$

$$h_r(z) = \sum_{i_1 \leq \cdots \leq i_r} z_{i_1} \cdots z_{i_r}, \qquad \text{(complete) homogeneous symmetric functions.}$$

Note that

$$\Lambda = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots] = \mathbb{Q}[p_1, p_2, \ldots].$$

Let us include $\boldsymbol{q}, \boldsymbol{t}$: $\Lambda_{\boldsymbol{q},\boldsymbol{t}} = \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}} \otimes \Lambda$.

6.2. **Plethysm.** Let us define *plethysm*. Roughly speaking, plethysm is a notation for generalized substitution, i.e. for $f \in \Lambda_{\boldsymbol{q},\boldsymbol{t}}$, we can write

$$f(\square, \diamondsuit, \heartsuit, \triangle, \cdots) = f[\square + \diamondsuit + \heartsuit + \triangle + \cdots].$$

Since f is symmetric, the order does not matter. For example,

$$f[2x + y + 4] = f(x, x, y, 1, 1, 1, 1, 0, 0, \cdots).$$

To define it properly, we note that

$$\begin{aligned} p_r(\Box, \Diamond, \heartsuit, \triangle, \cdots) &= \Box^r + \Diamond^r + \heartsuit^r + \triangle^r + \cdots \\ &= (\Box + \Diamond + \heartsuit + \triangle + \cdots)|_{? \mapsto ?^r}. \end{aligned}$$

Now we give the strict definition.

For $f, A \in \Lambda_{\boldsymbol{q},\boldsymbol{t}}$, we define $f \mapsto f[A]$ to be the unique map $\Lambda_{\boldsymbol{q},\boldsymbol{t}} \to \Lambda_{\boldsymbol{q},\boldsymbol{t}}$ with the following preperties

(A) $(cf + gh)[A] = cf[A] + g[A]h[A]$ for $c \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$;
(P) $p_r[A] = A|_{z_i \mapsto z_i^r, \boldsymbol{q} \mapsto \boldsymbol{q}^r, \boldsymbol{t} \mapsto \boldsymbol{t}^r}$ for any $r \in \mathbb{Z}_{>0}$.

That is $f \mapsto f[A]$ is an $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$-algebra homomorphism.

In general, if $A = A(z, y, \boldsymbol{q}, x, \ldots)$ is any function and $f = f(z, y, \boldsymbol{q}, x, \cdots)$ is any function symmetric in z, we define $f[A]$ by

(A) $(cf + gh)[A] = cf[A] + g[A]h[A]$, where c does not contain z;
(P) $p_r[A] = A(z^r, y^2, \boldsymbol{q}^r, x^r, \ldots)$.

Note that under this notation, z is the special in the condition (A).

Let us denote

$$Z = p_1(z) = z_1 + z_2 + \cdots .$$

Then clearly, $Z[f] = f$ by (P). We actually have

$$f[Z] = f, \quad \text{since} \begin{cases} \text{(P)} \quad p_r[Z] = (z_1 + z_2 + \cdots)|_{z_i \mapsto z_i^r, \boldsymbol{q} \mapsto \boldsymbol{q}^r, \boldsymbol{t} \mapsto \boldsymbol{t}^r} \\ \qquad\qquad\quad = z_1^r + z_2^r + \cdots = p_r. \\ \text{(A)} \qquad f \mapsto f \text{ is an algebra homomorphism} \end{cases}$$

Let us give some examples to see the flavor of plethysm.

Example 1. For any f,

$$f[p_k] = f|_{z_i \mapsto z_i^k}.$$

Since

$$\begin{aligned} &\text{(A)} \qquad f \mapsto \text{RHS is an algebra homomorphism} \\ &\text{(P)} \quad p_r[p_k] = (z_1^k + z_2^k + \cdots)|_{z_i \mapsto z_i^r, \boldsymbol{q} \mapsto \boldsymbol{q}^r, \boldsymbol{t} \mapsto \boldsymbol{t}^r} \\ &\qquad\qquad = z_1^{kr} + z_2^{kr} + \cdots \\ &\qquad\qquad = (z_1^r + z_2^r + \cdots)|_{z_i \mapsto z_i^k} = \text{RHS when } f = p_r. \end{aligned}$$

Compare with:

$$p_k[f] = f|_{x_i \mapsto x_i^k, \boldsymbol{q} \mapsto \boldsymbol{q}^k, \boldsymbol{t} \mapsto \boldsymbol{t}^k}.$$

We remind this is the property $\mathsf{(P)}$. They coincides when $f \in \Lambda$ i.e. f only involving $x_1, x_2, \ldots$.

Example 2. For any f,

$$f[x_1 + \ldots + x_n] = f(x_1, \cdots, x_n, 0, \cdots).$$

Here $x_1, \ldots, x_n$ are viewed as variables, by default. Since

$$\begin{aligned} &\mathsf{(A)} \qquad f \mapsto \mathsf{RHS} \text{ is an algebra homomorphism} \\ &\mathsf{(P)} \quad \mathsf{LHS} = (x_1 + \cdots + x_n)|_{x_i \mapsto x_i^r} \\ &\qquad\qquad = z_1^r + \cdots + z_n^r = \mathsf{RHS} \text{ when } f = p_r. \end{aligned}$$

Example 3. Recall that the coproduct $\Lambda \to \Lambda \otimes \Lambda$ is defined as follows. We can always write

$$\begin{aligned} f(xy) &:= f(x_1, x_2, \ldots, y_1, y_2, \ldots) \\ &= \sum f_1(x_1, x_2, \ldots) f_2(y_1, y_2, \ldots) \end{aligned}$$

Note that the substitution makes sense by picking a bijection $\mathbb{Z}_{>0}$ between $\mathbb{Z}_{>0} \sqcup \mathbb{Z}_{>0}$. Then we define $\Delta f = \sum f_1 \otimes f_2$. Following the same principle as above examples, we have

$$\begin{aligned} f[X+Y] &= f(x_1, x_2, \ldots, y_1, y_2, \ldots) \\ &= \sum f_1[X] f_2[Y] = \sum f_1(x_1, x_2, \ldots) f_2(y_1, y_2, \ldots), \end{aligned}$$

where $X = x_1 + x_2 + \cdots$, $Y + y_1 + y_2 + \cdots$.

Example 4. Much generally, if we can expand $A = \sum_a c_a x^a$ with $c_a \in \mathbb{Z}_{>0}$, then

$$f[A] = f(\cdots, \overbrace{x^a, \ldots, x^a}^{c_a}, \cdots)$$

the substitution of f by the multiset $[A]$ such that multiplicity of x^a is c_a. Since

$$\begin{aligned} &\mathsf{(A)} \qquad f \mapsto \mathsf{RHS} \text{ is an algebra homomorphism} \\ &\mathsf{(P)} \quad \mathsf{LHS} = \left(\sum c_a x^a\right)|_{x_i \mapsto x_i^r} = \sum c_a x^{ra} \\ &\qquad\qquad = \cdots + \underbrace{x^{ar} + \cdots + x^{ar}}_{c_a} + \cdots = \mathsf{RHS} \text{ when } f = p_r. \end{aligned}$$

We basically achieve the goal in the motivation. For our purpose, we need to make sense of "substitution of negative many variables", e.g. $f[x-y]$. We first $f[-Y]$.

Example 5. Note that the power sum p_r is very special since

$$p_r[X \pm Y] = p_r[X] \pm p_r[Y].$$

$$p_r[XY] = p_r[X]p_r[Y].$$

Both of them can be checked directly by (P). In particular (or direct computation),

$$\begin{aligned} p_r[-Z] &= (-z_1 - z_2 - \cdots)|_{z_i \mapsto z_i^r, \boldsymbol{q} \mapsto \boldsymbol{q}^r, \boldsymbol{t} \mapsto \boldsymbol{t}^r} \\ &= -z_1^r - z_2^r - \cdots = -p_r. \end{aligned}$$

Example 6. Let

$$\begin{aligned} \Omega := \prod_i \frac{1}{1-z_i} &= 1 + h_1 + h_2 + \cdots \\ &= \exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right). \end{aligned}$$

We remark that even the sum is infinite, but we understand Ω as a formal sum of each degree component, which of them is in Λ. Note that by (A)

$$\begin{aligned} \Omega[A] &= 1 + h_1[A] + h_2[A] + h_3[A] + \cdots \\ &= \exp\left(p_1[A] + \tfrac{1}{2}p_2[A] + \tfrac{1}{3}p_3[A] + \cdots\right). \end{aligned}$$

Then

$$\begin{aligned} \Omega[-Z] &= \exp\left(-p_1 - \tfrac{1}{2}p_2 - \tfrac{1}{3}p_3 - \cdots\right) = \Omega^{-1} \\ &= \prod_i (1 - z_i) = 1 - e_1 + e_2 - \cdots. \end{aligned}$$

This shows $h_r[-Z] = (-1)^r e_r = e_r(-z_1, -z_2, \ldots)$.

Example 7. Denote $\omega : \Lambda \to \Lambda$ the ω-involution. It is an algebra homomorphism given by $p_r \leftrightarrow -(-1)^r p_r$. It is also characterized by $h_r \leftrightarrow e_r$. Then by the above computation, we have

$$f[-Z] = (\omega f)(-z_1, -z_2, \ldots).$$

Recall that $\omega s_\lambda = s_{\lambda'}$ for Schur functions.

Now we can compute $f[X - Y]$. This follows from a more general associativity.

Example 8. We have associativity

$$f\big[g[A]\big] = (f[g])[A].$$

Since:

$$\begin{aligned}
&\textsf{(A)} \quad f \mapsto \textsf{LHS} \text{ or } \textsf{RHS} \text{ are both algebra homomorphism}\\
&\textsf{(P)} \quad \textsf{LHS} = p_r[g[A]] = g[A]|_{?\mapsto ?^r}\\
&\qquad\ \ \textsf{RHS} = (p_r[g])[A] = (g|_{?\mapsto ?^r})[A] \qquad \text{when } f = p_r.
\end{aligned}$$

Note that $? \mapsto ?^r$ is a ring homomorphism but in general not linear:

$$\begin{aligned}
(cf + gh)|_{?\mapsto ?^r} &= (c|_{?\mapsto ?^r})(f|_{?\mapsto ?^r}) + (g|_{?\mapsto ?^r})(h|_{?\mapsto ?^r})\\
&\neq c(f|_{?\mapsto ?^r}) + (g|_{?\mapsto ?^r})(h|_{?\mapsto ?^r}).
\end{aligned}$$

As c would contain variables other than z. Therefore, we need to check two cases, $g = p_k$ and $g = c$ for c not relating to z. When $g = p_k$,

$$\begin{aligned}
\textsf{LHS} &= p_k[A]|_{?\mapsto ?^r} = (A|_{?\mapsto ?^r})|_{?\mapsto ?^k} = A|_{?\mapsto ?^{rk}}\\
\textsf{RHS} &= (p_k|_{?\mapsto ?^r})[A] = p_{kr}[A] = A|_{?\mapsto ?^{rk}}.
\end{aligned}$$

When $g = c$,

$$\begin{aligned}
\textsf{LHS} &= c[A]|_{?\mapsto ?^r} = c|_{?\mapsto ?^r}\\
\textsf{RHS} &= (c|_{?\mapsto ?^r})[A] = c|_{?\mapsto ?^r}.
\end{aligned}$$

So $\textsf{LHS} = \textsf{RHS}$.

Example 9. Assume $\Delta f = \sum f_1 \otimes f_2$, then

$$\begin{aligned} f[X-Y] &= \sum f_1[X] f_2[-Y] \\ &= \sum f_1(x_1, x_2, \ldots)(\omega f_2)(-y_1, -y_2, \ldots). \end{aligned}$$

Of course, this can be checked directly by setting $f = p_r$. But let us mention the following proof. We know

$$f[X+Y] = \sum f_1[X] f_2[Y],$$

equivalently,

$$f[X+Z] = \sum f_1[X] f_2.$$

We apply $[-Y]$ on both sides, we get

$$f[X-Y] = f\big[(X+Z)[-Y]\big] = \sum f_1[X] f_2[-Y].$$

This makes sense of "replacing Y by $-Y$".

Forget the next sentence if it looks confusing. Theoretically speaking, plethysm should be denoted by $f[Z \mapsto A]$, and "replacing Y by $-Y$" should be denoted by $f[Y] \mapsto f[Y \mapsto -Y]$.

Example 10. More generally, if we can expand $A = \sum_a c_a x^a$, then

$$f[A] = \sum f_1(\cdots, \overbrace{x^a, \ldots, x^a}^{c_a > 0}, \cdots)(\omega f_2)(\cdots, \overbrace{-x^a, \ldots, -x^a}^{-c_a > 0}, \cdots).$$

Let $A = A^+ - A^-$ in the obvious sense. Denote $(\omega f_2)(-z_1, -z_2, \ldots) = f_2'$.

$$\begin{array}{rrl} & f[X-Y] &= \sum f_1[X] f_2'[Y] \\ \Longleftrightarrow & f[Z-Y] &= \sum f_1[Z] f_2'[Y] \\ \Longrightarrow & f[A^+ - Y] &= \sum f_1[A^+] f_2'[Y] \\ \Longleftrightarrow & f[A^+ - Z] &= \sum f_1[A^+] f_2'[Z] \\ \Longleftrightarrow & f[A] &= \sum f_1[A^+] f_2'[A^-]. \end{array}$$

Example 11. It follows from the computation that

$$\begin{aligned}\Omega[X+Y] &= \exp\left(p_1[X+Y] + \tfrac{1}{2}p_2[X+Y] + \tfrac{1}{3}p_3[X+Y] + \cdots\right)\\ &= \exp\left((p_1[X]+p_1[Y]) + \tfrac{1}{2}(p_2[X]+p_2[Y]) + \cdots\right)\\ &= \Omega[X]\Omega[Y]\end{aligned}$$

and similarly

$$\begin{aligned}\Omega[X-Y] &= \exp\left(p_1[X-Y] + \tfrac{1}{2}p_2[X-Y] + \tfrac{1}{3}p_3[X-Y] + \cdots\right)\\ &= \exp\left((p_1[X]-p_1[Y]) + \tfrac{1}{2}(p_2[X]-p_2[Y]) + \cdots\right)\\ &= \Omega[X]/\Omega[Y].\end{aligned}$$

We thus have

$$\Omega[A] = \prod_a \frac{1}{(1-x^a)^{c_a}},$$

if we can expand $A = \sum_a c_a x^a$.

Example 12. Let us finally mention more useful computation.

$$\Omega[XY] = \prod_{i,j} \frac{1}{1-x_i y_j}.$$

$$\Omega\left[X\tfrac{1}{1-\boldsymbol{q}}\right] = \prod_i \prod_{k\geq 0} \frac{1}{1-\boldsymbol{q}^k x_i}.$$

Note that

$$\begin{aligned}f\left[X\tfrac{1}{1-\boldsymbol{q}}\right] &= f\left[X(1+\boldsymbol{q}+\boldsymbol{q}^2+\cdots)\right]\\ &= f(x_1, x_2, \ldots, \boldsymbol{q}x_1, \boldsymbol{q}x_2, \ldots, \boldsymbol{q}^2x_1, \boldsymbol{q}^2x_2, \ldots, \ldots)\\ &= f[XY]|_{y_i\mapsto \boldsymbol{q}^i}.\end{aligned}$$

But since we expand $\frac{1}{1-\boldsymbol{q}}$ with $|\boldsymbol{q}|<1$, it would be a few words to say. The result of $f\left[Z\frac{1}{1-\boldsymbol{q}}\right]$ must be with rational coefficients in $\boldsymbol{q}$. It gives the same answer as that over the ring of power series, this proves the validity of the expansion.

$$\Omega[X(1-\boldsymbol{t})] = \Omega[X(1-\boldsymbol{t})] - \Omega[\boldsymbol{t}X] = \prod_i \frac{1-\boldsymbol{t}x_i}{1-x_i}.$$

$$\Omega\left[X\tfrac{1-t}{1-q}\right] = \prod_i \prod_{k\geq 0} \frac{1-tq^k x_i}{1-q^k x_i}.$$

6.3. **Macdonald functions.** Recall the Hall inner product $\langle\cdot,\cdot\rangle$ is given by $\langle s_\lambda, s_\mu\rangle = \mathbf{1}_{\lambda=\mu}$. The kernel of the inner product is

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \prod_{i,j}\frac{1}{1-x_iy_j} = \Omega[XY].$$

Let us denote for partition λ

$$p_\lambda := p_1^{m_1}p_2^{m_2}\cdots, \qquad z_\lambda = 1^{m_1}m_1!2^{m_2}m_2!\cdots$$

for $m_j = \#\{j : \lambda_j = i\}$. Recall

$$\begin{aligned}\Omega &= \exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right)\\ &= \exp(p_1)\exp\left(\tfrac{p_2}{2}\right)\exp\left(\tfrac{p_3}{3}\right)\cdots = \textstyle\sum_\lambda \frac{1}{z_\lambda}p_\lambda.\end{aligned}$$

Since $p_\lambda[XY] = p_\lambda[X]p_\lambda[Y]$, $\Omega[XY] = \sum_\lambda \frac{1}{z_\lambda}p_\lambda(x)p_\lambda(y)$. So $\langle\cdot,\cdot\rangle$ is characterized by

$$\langle p_\lambda, p_\mu\rangle = \mathbf{1}_{\lambda=\mu}z_\lambda.$$

Let us equip $\Lambda_{q,t}$ a new inner product

$$\langle f,g\rangle_{q,t} := \left\langle f, g\left[Z\tfrac{1-q}{1-t}\right]\right\rangle.$$

Then the kernel is

$$\Omega\left[XY\tfrac{1-t}{1-q}\right] = \prod_{i,j}\prod_{k\geq 0}\frac{1-tq^kx_iy_j}{1-q^kx_iy_j}$$

and is characterized by

$$\langle p_\lambda, p_\mu\rangle_{q,t} = \mathbf{1}_{\lambda=\mu}z_\lambda(q,t)$$

where

$$z_\lambda(q,t) = p_\lambda\left[\tfrac{1-q}{1-t}\right] = \left(\tfrac{1-q}{1-t}\right)^{m_1}\left(\tfrac{1-q^2}{1-t^2}\right)^{m_2}\cdots.$$

We define *Macdonald functions* $\{P_\lambda\}_\lambda \subset \Lambda_{q,t}$ by

(1) $P_\lambda = m_\lambda + (\text{lower terms})$;
(2) $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ for $\lambda\neq\mu$.

This definition is actually compatible with the definition of Macdonald polynomials in type A.

7. Difference operators

Now let us restrict to type A. Now the Weyl group $W = \mathcal{S}_n$. The convention of the root system is weird:

$$\begin{array}{l} P = P^\vee = \{(x_1,\dots,x_n) \in \mathbb{Z}^n\}/\mathbb{Z}(1,\dots,1) \\ \twoheaduparrow \qquad \twoheaduparrow \\ \mathbb{Z}^n = \mathbb{Z}^n \\ \cup \qquad \cup \\ Q = Q^\vee = \{(x_1,\dots,x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\}. \end{array}$$

The identification is

$$\begin{array}{l} \widehat{W}_e = \tilde{\mathcal{S}}_n/\langle t_{(1,\dots,1)}\rangle, \quad \text{recall: } t_{(1,\dots,1)}(a) = a+n. \\ \twoheaduparrow \\ \tilde{\mathcal{S}}_n = \{\text{bijection } f : \mathbb{Z} \to \mathbb{Z} : f(a+n) = f(a)+n\} \\ \cup \\ \widehat{W} = \{f \in \tilde{S}_n : f(1) + \cdots + f(n) = 0\} \end{array}$$

The action is given by

$$wt_\lambda(i) = w(i) + n\lambda_i, \qquad i = 1,2,\dots,n.$$

We will use the Hecke algebra for $\tilde{S}_n$.

7.1. Diagramatics.

We denote

$$H_w = \boldsymbol{t}^{\ell(w)/2} T_w.$$

The Hecke algebra $\widehat{H}_n$ can be defined by

- $(H_i - \boldsymbol{t}^{1/2})(H_i + \boldsymbol{t}^{1/2}) = 0$ for all $i \in \mathbb{Z}/n$;
- $H_iH_j = H_jH_i$ for $j \neq i-1, i, i+1$ and $H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}$;
- $\omega H_i \omega^{-1} = H_{i-1}$.

Note that $\widehat{H}_{\boldsymbol{t}}(W) = \widehat{H}_n/\langle \omega^n = 1\rangle$. It has Bernstein's presentation

- $(H_i - \boldsymbol{t}^{1/2})(H_i + \boldsymbol{t}^{1/2}) = 0$ for $1 \leq i \leq n-1$;
- $H_iH_j = H_jH_i$ for $i \neq i-1, i, i+1$ and $H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}$;
- $Y_iY_j = Y_jY_i$;
- $H_iY_j = Y_jH_i$ for $j \neq i, i+1$ and $H_i^{-1}Y_iH_i^{-1} = Y_{i+1}$.

where

$$Y_i = H_i \cdots H_{n-1}\omega H_1^{-1} \cdots H_{i-1}^{-1}.$$

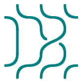
We can use a diagram on a cylinder to illustrate them

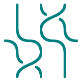
$H_i =$  ,

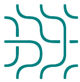
$\omega =$  ,

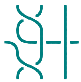
$Y_i =$  .

For example, when $n = 3$,

$\omega H_2 \omega^{-1} =$  $=$  $= H_1.$

$H_2 \omega H_1^{-1} =$  $=$  $= H_2.$

$\omega Y_2 \omega^{-1} =$  $=$  $= Y_1.$

$H_1^{-1} Y_1 H_1^{-1} =$  $=$  $= Y_2.$

$Y_1 Y_2 =$  $=$  $= Y_2 Y_1.$

7.2. **Computation.** Let us consider $\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[x_1, \dots, x_n]$. Note that our convention is $e^{\alpha_i} = x_i / x_{i+1}$. Then the Weyl group action is

$$\begin{aligned} wt_\lambda &: x_i \mapsto \boldsymbol{q}^{-\lambda_i} x_{w(i)}, \\ \omega &: x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_2 \mapsto x_1 \mapsto \boldsymbol{q} x_n, \\ s_0 &: x_1 \mapsto \boldsymbol{q} x_n,\ x_n \mapsto \boldsymbol{q}^{-1} x_1. \end{aligned}$$

Recall that

$$\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]^{\mathcal{S}_n} \curvearrowright \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathcal{S}_n}.$$

We will see that

$$\mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[Y_1, \dots, Y_n]^{\mathcal{S}_n} \curvearrowright \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[x_1, \dots, x_n]^{\mathcal{S}_n} := \Lambda_n.$$

Let us describe the action for the elementary symmetric polynomials $e_r(Y)$ in R^W.

Let $e = \frac{1}{\mathcal{S}_n(t)} \sum T_w$ be the symmetrizer. We have $H_i e = t^{1/2} e$.

Step 1. We have the following identity

$$eY_{n-r+1} \cdots Y_n e = \frac{\mathcal{S}_r(t)\mathcal{S}_{n-r}(t)}{\mathcal{S}_n(t)} \cdot e_r(Y) \cdot e \in \widehat{H}_n.$$

Since the Bernstein representation on $\widehat{H}_n \otimes_{H_n} 1$ is faithful, we check this by considering it as an operator over $\mathbb{Q}_{q,t}[Y_1, \ldots, Y_n]$. Since e is a symmetrizer, we have

$$e((Y_{n-r+1} \cdots Y_n)ef) = (eY_{n-r+1} \cdots Y_n)(ef).$$

Note that, as an operator,

$$ef = \frac{1}{\mathcal{S}_n(t)} \sum_{w \in \mathcal{S}_n} w \left(f \prod_{i<j} \frac{Y_j/Y_i - t}{Y_j/Y_i - 1} \right).$$

For example, when $n = 2$,

$$\begin{aligned}
ef &= \frac{1}{1+t}\left(1 + t s_i f + (t-1)\frac{s_i f - 1}{Y_2/Y_1 - 1}\right) \\
&= \frac{1}{1+t}\left(\frac{Y_2/Y_1 - t}{Y_2/Y_1 - 1} f + \frac{tY_2/Y_1 - 1}{Y_2/Y_1 - 1} s_i f\right) \\
&= \frac{1}{1+t}\left(\frac{Y_2/Y_1 - t}{Y_2/Y_1 - 1} f + s_i \left(f \frac{tY_1/Y_2 - 1}{Y_1/Y_2 - 1}\right)\right).
\end{aligned}$$

Since the r-th fundamental weight is minuscule, we get immediately that

$$eY_{n-r+1} \cdots Y_n = \frac{1}{\mathcal{S}_n(t)} \sum_{w \in W} T_w(Y_{n-r+1} \cdots Y_n) = \frac{\mathcal{S}_r(t)\mathcal{S}_{n-r}(t)}{\mathcal{S}_n(t)} e_r(Y).$$

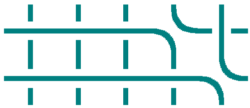
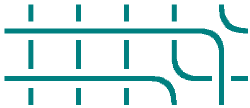
For example, when $n = 2$,

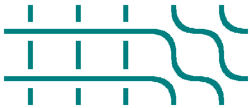
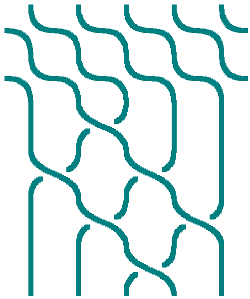
$$\begin{aligned}
eY_2 &= \frac{1}{1+t}\left(\frac{Y_2^2 - tY_1Y_2}{Y_2 - Y_1} + \frac{Y_1^2 - tY_1Y_2}{Y_1 - Y_2}\right) \\
&= \frac{1}{1+t}(Y_1 + Y_2).
\end{aligned}$$

Step 2. We have the following identity

$$eY_{n-r+1}\cdots Y_n e = \boldsymbol{t}^{-k(n-k)/2} e\omega^r e \in \widehat{H}_n.$$

This can be proven quickly by diagrammatics.

$Y_4Y_5 =$  $=$ 

$=$  $=$  $= \omega^2 H^{-1}_{\cdots}$

This can also be proven by definition. For example, when $n = 3$

$$\begin{aligned} eY_2Y_3e &= e(H_2\omega H_1^{-1})(\omega H_1^{-1}H_2^{-1})e \\ &= e(H_2\omega^2 H_2^{-1}H_1^{-1}H_2^{-1}))e. \end{aligned}$$

We thus have

$$e_r(Y)\cdot e = e\cdot e_r(Y)\cdot e = \boldsymbol{t}^{-r(n-r)/2}\frac{\mathcal{S}_n(\boldsymbol{t})}{\mathcal{S}_r(\boldsymbol{t})\mathcal{S}_{n-r}(\boldsymbol{t})} e\omega^r e \in \widehat{H}_n.$$

In general, such simplification can be done for any minuscule weight.

Let us consider the Cherednik representation. Let $f \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}[x_1,\dots,x_n]^{\mathcal{S}_n}$. Then

$$\begin{aligned} w\omega^r f &= wf(\boldsymbol{q}x_{n-r+1},\cdots,\boldsymbol{q}x_n, x_1, x_2, \dots) \\ &= wf(x_1, x_2, \dots, x_{n-r}, \boldsymbol{q}x_{n-r+1},\cdots,\boldsymbol{q}x_n) \\ &= f(x_{w(1)}, x_{w(2)}, \dots, x_{w(n-r)}, \boldsymbol{q}x_{w(n-r+1)},\cdots,\boldsymbol{q}x_{w(n)}) \\ &= f(\boldsymbol{q}^{\theta_1}x_1,\dots,\boldsymbol{q}^{\theta_n}x_n) \end{aligned}$$

where $\theta_i = 1$ if $i \in wI_0 = w\{n\text{-}r\text{+}1, \dots, n\}$ and $\theta_i = 0$ otherwise. Now we can compute

$$\begin{aligned}
e\omega^r f &= \frac{1}{\mathcal{S}_n(t)} \sum_{w\in\mathcal{S}_n} w\left(\omega^r f \prod_{i<j} \frac{x_i/x_j - t}{x_i/x_j - 1}\right)\\
&= \frac{\mathcal{S}_r(t)\mathcal{S}_{n-r}(t)}{\mathcal{S}_n(t)} \sum_{w\in\mathcal{S}_n/(\mathcal{S}_{n-r}\times\mathcal{S}_r)} f|_{x_i\mapsto qx_i,\forall i\in wI_0} \prod_{\substack{1\le i\le n\text{-}r\\ n\text{-}r<j\le n}} \frac{x_{w(i)}/x_{w(j)} - t}{x_{w(i)}/x_{w(j)} - 1}\\
&= \frac{\mathcal{S}_r(t)\mathcal{S}_{n-r}(t)}{\mathcal{S}_n(t)} \sum_{I\in\binom{[n]}{r}} \left(\prod_{\substack{i\notin I\\ j\in I}} \frac{x_i - tx_j}{x_i - x_j}\right) f|_{x_i\mapsto qx_i,\forall i\in I}\\
&= \frac{\mathcal{S}_r(t)\mathcal{S}_{n-r}(t)}{\mathcal{S}_n(t)} \sum_{I\in\binom{[n]}{r}} \left(\prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\right) f|_{x_i\mapsto qx_i,\forall i\in I}.
\end{aligned}$$

We can finally conclude that

$$\begin{aligned}
e\cdot e_r(Y)\cdot e = t^{-r(n-r)/2} &\sum_{I\in\binom{[n]}{r}} \left(\prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\right) f|_{x_i\mapsto qx_i,\forall i\in I}\\
= &\sum_{I\in\binom{[n]}{r}} \left(\prod_{\substack{i\in I\\ j\notin I}} \frac{t^{1/2}x_i - t^{-1/2}x_j}{x_i - x_j}\right) f|_{x_i\mapsto qx_i,\forall i\in I}.
\end{aligned}$$

Thus this action gives the action of $e_r(Y)$ on $\mathbb{Q}_{q,t}[x_1,\dots,x_n]^{\mathcal{S}_n}$.

7.3. **Compatibility.** To check

$$P_\lambda(x_1,\dots,x_n,0,\dots)$$

is the symmetric Macdonald polynomial for the root system A_{n-1}, it suffices to check

$$D^r : f \mapsto \sum_{I \in \binom{[n]}{r}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{\boldsymbol{t} x_i - x_j}{x_i - x_j} \right) f|_{x_i \mapsto \boldsymbol{q} x_i, \forall i \in I}$$

is unitary with respect to the truncation of inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{q},\boldsymbol{t}}$. It suffices to show for $X_n = x_1 + \cdots + x_n$ and $Y_n = x_n + \cdots + y_n$ that

$$D_x^r \cdot \Omega\left[X_n Y_n \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right] = D_y^r \cdot \Omega\left[X_n Y_n \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right].$$

Note that

$$\Omega\left[X_n Y_n \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right] = \prod_{\substack{1 \le i,j \le n \\ 0 \le k}} \frac{1 - \boldsymbol{t}\boldsymbol{q}^k x_i y_j}{1 - \boldsymbol{q}^k x_i y_j}.$$

So

$$\Omega\left[X_n Y_n \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right]|_{x_i \mapsto \boldsymbol{q} x_i, i \in I} = \Omega\left[X_n Y_n \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right] \prod_{\substack{i \in I \\ 1 \le j \le n}} \frac{1 - x_i y}{1 - \boldsymbol{t} x_i y}.$$

So

$$LHS = \Omega\left[X_n Y \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right] (\text{expression only depend on } \boldsymbol{t}).$$

Similarly,

$$RHS = \Omega\left[X_n Y \tfrac{1-\boldsymbol{t}}{1-\boldsymbol{q}}\right] (\text{expression only depend on } \boldsymbol{t}).$$

Thus it suffices to show when $\boldsymbol{q} = \boldsymbol{t}$, i.e. $\langle \cdot, \cdot \rangle_{\boldsymbol{q},\boldsymbol{t}} = \langle \cdot, \cdot \rangle$. Then if we denote $\Delta = \prod_{i<j} (1 - x_j / x_i)$, we can rewrite

$$\sum_{I \in \binom{[n]}{r}} \frac{1}{\Delta} (\Delta f)_{x_i \mapsto \boldsymbol{q} x_i, i \in I} = n! \frac{1}{\Delta} \sum_{w \in W} (-1)^{\ell(w)} (\Delta f)_{x_i \mapsto \boldsymbol{q} x_{w(i)}}.$$

When $f = s_\lambda$, then each term Δs_λ gives a multiple of s_λ.

8. Origin of Plethysm

8.1. **K-theory.** Note that the topological K-group

$$\begin{aligned} K(X) &= \mathbb{Z}\{\text{vector bundles over } X\}/_{\cong}, \oplus = + \\ &= \pi_0(X, BGL_\infty \times \mathbb{Z}). \end{aligned}$$

But let us use the more classic definition. That is, $K(X)$ is the Grothendieck group of

$$K^+(X) = \{\text{vector bundles over } X\}/_{\cong}.$$

That is, the element in $K(X)$ is a formal difference $U - V$ for $U, V \in K^+(X)$ with

$$U - V = U' - V' \iff \begin{cases} U \oplus Y \cong U' \oplus Z \\ V \oplus Z \cong V' \oplus Y \end{cases} \text{for some } Y, Z \in K^+(X).$$

We can take Y and Z to be trivial bundles.

We would like to consider

$$\mathrm{End}_{\mathsf{set}}(K(-)) \xlongequal{\text{Yoneda Lemma}} K(BGL_\infty \times \mathbb{Z}) \supset \Lambda_{\mathbb{Z}}.$$

Plethysm is the composition of this endomorphism ring.

Let us state it in a more concrete way. For a vector bundle V, we define

$$\begin{aligned} \mathsf{S}_t V &= \textstyle\sum_{k=0}^\infty t^k \mathsf{S}^k V \in K(X)[\![t]\!], \\ \Lambda_t V &= \textstyle\sum_{k=0}^\infty (-t)^k \Lambda^k V \in K(X)[\![t]\!]. \end{aligned}$$

Note that

$$\mathsf{S}_t(U \oplus V) = (\mathsf{S}_t U)(\mathsf{S}_t V), \qquad \Lambda_t(U \oplus V) = (\Lambda_t U)(\Lambda_t V).$$

This extends to an operator over $K(X)$. That is

$$\mathsf{S}_t(U - V) := \tfrac{\mathsf{S}_t(U)}{\mathsf{S}_t(V)}, \qquad \Lambda_t(U - V) := \tfrac{\Lambda_t(U)}{\Lambda_t(V)}.$$

Note that these operators are not additive.

Now let us make it additive. To do this, we have to work over $K(X)_{\mathbb{Q}}$. We have

$$\begin{aligned} \ln(\mathsf{S}_t(U \oplus V)) &= \ln(\mathsf{S}_t U) + \ln(\mathsf{S}_t V), \\ \ln(\Lambda_t(U \oplus V)) &= \ln(\Lambda_t U) + \ln(\Lambda_t V). \end{aligned}$$

Now we define *Adams operation* $\psi_r : K(X) \to K(X)_{\mathbb{Q}}$ by the coefficients of $\ln(\mathsf{S}_t V)$:

$$\ln(\mathsf{S}_t V) = t\psi_1(V) + \frac{t^2}{2}\psi_2(V) + \frac{t^3}{3}\psi_3(V) + \cdots.$$

Actually, we have $-\ln(\mathsf{\Lambda}_{t}V)=\ln(\mathsf{S}_{t}V)$, but we will not use it. Note that

$$\psi_r(U\oplus V)=\psi_r(U)+\psi(V).$$

We extend $\psi_r: K(X)_{\mathbb{Q}}\to K(X)_{\mathbb{Q}}$ by linearity. Actually, if we expand ψ_r in terms of coefficients of S_{t}, it is not hard to see Adams operation is defined over $K(X)$.

Let us compute for line bundle L

$$\mathsf{S}_{t}L=1+Lt+L^{\otimes 2}t^2+\cdots=\frac{1}{1-Lt}.$$

$$\ln(\mathsf{S}_{t}L)=-\ln(1-Lt)=Lt+L^{\otimes 2}\frac{t^2}{2}+L^{\otimes 3}\frac{t^3}{3}+\cdots$$

This shows

$$\psi_r(L)=L^{\otimes r}.$$

Now let us compute $\psi_r(U\otimes V)$. By splitting principle, we can assume U and V are both direct sums of line bundles. Then immediately, we have

$$\psi_r(U\otimes V)=\psi_r(U)\psi_r(V).$$

As a result, ψ_r is not only additive but also multiplicative. Similarly, using the splitting principle again, we have

$$\psi_r(\psi_k(V))=\psi_{rk}(V).$$

Note that we can use $\mathsf{\Lambda}_{t}V=\sum_{k=0}^{\infty}(-t)^k\mathsf{\Lambda}^kV$. Then

$$\ln(\mathsf{\Lambda}_{t}V)=-t\phi_1(V)-\frac{t^2}{2}\phi_2(V)-\frac{t^3}{3}\phi_3(V)+\cdots.$$

8.2. **Character.** Let us find the Adams operation in terms of characters. That is,

$$K_G(\mathsf{pt})=\mathrm{Rep}(G)\xrightarrow{\chi}\mathsf{Fun}(G,\mathbb{C}^\times),$$

where $(*)$ is given by $V\mapsto\chi(V)=[g\mapsto\mathrm{Tr}(g;V)]$. Actually, the case $G=GL_n$ and $V=\mathbb{C}^n$ is the universal case. The restriction to a maximal torus is enough.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, phantom, "\in"] & \mathrm{Rep}(T) \arrow[r, "\chi"] & \mathsf{Fun}(T, \mathbb{C}^\times) \\
\mathbb{C}^n \arrow[u, mapsto] \arrow[d, mapsto] \arrow[r, phantom, "\in"] & \mathrm{Rep}(GL_n) \arrow[u, "\mathrm{res}"] \arrow[d, "\mathrm{res}"'] \arrow[r, "\chi"] & \mathsf{Fun}(GL_n, \mathbb{C}^\times) \arrow[u, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \\
V \arrow[r, phantom, "\in"] & \mathrm{Rep}(G) \arrow[r, "\chi"] & \mathsf{Fun}(G, \mathbb{C}^\times)
\end{tikzcd}$$

Thus finally, it suffices to deal with $G = GL_1$ and $V = \mathbb{C}$ whose character is $\mathrm{id} = [z \mapsto z]$. Then direct computation shows

$$\chi(\psi_r(V)) = [z \mapsto z^r].$$

As a result, if we define for $\chi \in \mathsf{Fun}(G, \mathbb{C}^\times)$

$$\psi_r(\chi) : [z \mapsto \chi(z^r)]$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(G)_{\mathbb{Q}} \arrow[r, "\chi"] \arrow[d, "\psi_r"'] & \mathsf{Fun}(G, \mathbb{C}^\times) \arrow[d, "\psi_r"'] \\
\mathrm{Rep}(G)_{\mathbb{Q}} \arrow[r, "\chi"] & \mathsf{Fun}(G, \mathbb{C}^\times).
\end{tikzcd}$$

8.3. **Lambda-ring.** A *lambda ring* is a commutative ring R with a family of operators λ^r for $r \in \mathbb{Z}_{\geq 0}$ with certain properties. Let R be a commutative algebra containing $\mathbb{Q}$. Then lambda-ring can be equivalently defined by a family of ring homomorphisms $p_r : R \to R$ for $r \in \mathbb{Z}_{>0}$ with $p_1 = \mathrm{id}$ and $p_r \circ p_k = p_{rk}$. We say $\varphi : R_1 \to R_2$ a lambda-ring homomorphism if φ is a ring homomorphism and $\varphi \circ p_r = p_r \circ \varphi$.

For a lambda-ring R, we have a ring homomorphism

$$\Lambda \longrightarrow \mathrm{End}_{\mathsf{set}}(R), \qquad \text{by} \quad p_r \longmapsto p_r.$$

Namely, it is extended to Λ by

$$(cf + gh)(x) = cf(x) + g(x)h(x).$$

Note that

- if $\varphi \in \mathrm{Hom}_{\lambda\text{-}\mathsf{Ring}}(R_1, R_2)$, then $\varphi(f(x)) = f(\varphi(x))$ for any f, since we assume φ is a ring homomorphism.
- since $p_r \circ p_k = p_k \circ p_r$, we have $p_r \in \mathrm{End}_{\lambda\text{-}\mathsf{Ring}}(R)$ so that $p_r \circ f = f \circ p_r$ for any $f \in \Lambda$.

I claim that

$$f \circ g = f[g] : K(X) \longrightarrow K(X).$$

Firstly, by construction,

$$f \mapsto \begin{cases} \mathsf{LHS} = f \circ g \in \mathrm{End}(R) \\ \mathsf{RHS} = f[g] \in \Lambda \end{cases}$$

are both algebra homomorphisms. Thus it suffices to check when $f = p_r$. In this case,

$$g \mapsto \begin{cases} \mathsf{LHS} = \psi_r \circ g = g \circ \psi_r \in \mathrm{End}(K(X)) \\ \mathsf{RHS} = p_k[g] = g[p_k] \in \Lambda \end{cases}$$

is also an algebra homomorphism.

Note that Λ itself is a lambda-ring with $p_k : A \mapsto p_k[A]$. We claim that

$$(\Lambda, Z = p_1)$$

is the universal lambda-ring in the following sense.

For any lambda-ring $R \supseteq \mathbb{Q}$ and any $x \in R$, there exists a unique lambda ring homomorphism $\varphi : \Lambda \to R$ such that $\varphi(Z) = x$.

$$\begin{tikzcd} Z \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & x \arrow[d, phantom, sloped, "\in"] \\ \Lambda \arrow[r, "\varphi"'] & R \end{tikzcd}$$

That is, for any $x \in R$, we define $\varphi : \Lambda \to R$ be $f \mapsto f(x)$. Since

$$(p_r \circ \varphi)(f) = p_r(f(x)) = (p_r[f])(x) = (\varphi \circ p_r)[f],$$

this is a lambda-ring homomorphism. Conversely, for any lambda-ring homomorphism $\varphi : \Lambda \to R$, we take $x = \varphi(Z) \in R$. Then $\varphi(f) = \varphi(f[p_1]) = f\varphi(p_1) = f(x)$.

For two lambda rings R_1, R_2, their tensor product is naturally a lambda ring by

$$p_k(x \otimes y) = p_k(x) \otimes p_k(y).$$

Since $p_k(1) = 1$, the natural map $R_i \to R_1 \otimes R_2$ is lambda-ring homomorphism for $i = 1, 2$. It has the universal property

$$\begin{tikzcd} R_1 \arrow[dr] \arrow[drr, bend left] & & \\ & R_1 \otimes R_2 \arrow[r] & R \\ R_2 \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd} \qquad \begin{aligned} &\mathrm{Hom}_{\lambda\text{-}\mathsf{Ring}}(R_1, R) \\ \times\, &\mathrm{Hom}_{\lambda\text{-}\mathsf{Ring}}(R_2, R) \\ =\, &\mathrm{Hom}_{\lambda\text{-}\mathsf{Ring}}(R_1 \otimes R_2, R). \end{aligned}$$

We claim if $f[X+Y] = \sum f_1[X] f_2[Y]$, then

$$f(a+b) = \sum f_1(a) f_2(b).$$

This follows directly from the universal property — we can replace X by a and Y by b. Namely, we have the following diagram

$$\begin{tikzcd} \Lambda \arrow[r, "Z \mapsto X+Y"] \arrow[dr, "Z \mapsto a+b"'] & \Lambda \otimes \Lambda \arrow[d, shift right, "Z \mapsto a"'] \arrow[d, shift left, "Z \mapsto b"] \\ & R \end{tikzcd}$$

Similarly, if $f[XY] = \sum f_1[X] f_2[Y]$, then $f(ab) = \sum f_1(a) f_2(b)$.

8.4. **Return to K-theory.** Now, for $V \in K^+(X)$ or $V \in \mathrm{Rep}(G)$,

$$h_r(V) = \mathsf{S}^r V, \qquad e_r(V) = \mathsf{\Lambda}^r V$$

from the construction:

$$\begin{aligned} &1 + h_1(V)\boldsymbol{t} + h_2(V)\boldsymbol{t}^2 + \cdots \\ &= (1 + h_1\boldsymbol{t} + h_2\boldsymbol{t}^2 \cdots)(V) \\ &= \exp\left(p_1\boldsymbol{t} + p_2\frac{\boldsymbol{t}^2}{2} + p_3\frac{\boldsymbol{t}^3}{3} + \cdots\right)(V) \\ &= \exp\left(p_1(V)\boldsymbol{t} + p_2(V)\frac{\boldsymbol{t}^2}{2} + p_3(V)\frac{\boldsymbol{t}^3}{3} + \cdots\right) \\ &= 1 + \mathsf{S}^1(V)\boldsymbol{t} + \mathsf{S}^2(V)\boldsymbol{t}^2 + \cdots . \end{aligned}$$

Similar computation for $e_r(V)$. We also have $h_r(-V) = -(-1)^r e_r(V)$.

Recall that for any partition $\lambda \vdash n$, there is an idempotent $e_\lambda \in \mathbb{Q}[\mathcal{S}_n]$, such that the irreducible representation of GL_m of highest weight λ is

$$\mathbb{V}(\lambda) = e_\lambda(\mathbb{C}^m)^{\otimes n}.$$

Here, if λ cannot be viewed as a weight of GL_m, i.e. the length of λ is more than m, we take the convention that $\mathbb{V}(\lambda) = 0$. We claim that for any $V \in K(X)$ or $\mathrm{Rep}(G)$

$$s_\lambda(V) = e_\lambda V^{\otimes n}.$$

This is known as *Schur functor*. For example,

- when $\lambda = (1^r)$, $e_\lambda = \sum_{w \in \mathcal{S}_n} (-1)^{\ell(w)} w$ then $e_\lambda V^{\otimes n} = \mathsf{\Lambda}^r V$;
- when $\lambda = (r)$, $e_\lambda = \sum_{w \in \mathcal{S}_n} w$ then $e_\lambda V^{\otimes n} = \mathsf{S}^r V$.

For any $V \in \mathrm{Rep}(G)$ of dimension m, we define

$$\varphi : \mathrm{Rep}(GL_m) \longrightarrow \mathrm{Rep}(G)$$

by restriction. For any $V \in K(X)$ of rank m, we define

$$\varphi(U) = \mathcal{F}_X(V) \times_{GL_m} U,$$

where

$$\mathcal{F}_X(V) = \{(x, v_1, \dots, v_m) : x \in X, \mathrm{span}(v_1, \dots, v_m) = V_x\}.$$

Since both construction is a functor, we have

$$\varphi((\mathbb{C}^m)^{\otimes n}) = V^{\otimes m}, \qquad \varphi(e_\lambda(\mathbb{C}^m)^{\otimes n}) = e_\lambda V^{\otimes m}.$$

In particular, φ commutes with $\mathsf{\Lambda}^k$ thus is a lambda ring homomorphism.

So it reduces to check the universal case, i.e. when $V = \mathbb{C}^m \in \mathrm{Rep}(GL_m)$, this follows from the fact that the ring homomorphism $\Lambda \to \mathrm{Rep}(GL_m)$ sending $e_r \mapsto \mathsf{\Lambda}^r\mathbb{C}^m$ sends s_λ to $\mathbb{V}(\lambda)$.

Remark. Note that it is not obvious that Λ^r extends to $K(X)_{\mathbb{Q}}$, the existence of the extension follows from the construction of Adams operators. Moreover, there is no direct meaning of e_r for any elements. For example,

$$\begin{aligned} e_2(\tfrac{1}{2}V) &= \left(\frac{p_1^2 - p_2}{2}\right)(\tfrac{1}{2}V) = \frac{p_1(\frac{1}{2}V)^2 - p_2(\frac{1}{2}V)}{2} \\ &= \frac{\frac{1}{4}p_1(V)^2 - \frac{1}{2}p_2(V)}{2} = -\frac{1}{8}p_1(V)^2 + \frac{1}{2}\frac{p_1(V)^2 - p_2(V)}{2} \\ &= -\frac{1}{8}V^{\otimes 2} + \frac{1}{2}\Lambda^2 V. \end{aligned}$$

But if $V = 2U$, then

$$-\frac{1}{8}V^{\otimes 2} + \frac{1}{2}\Lambda^2 V = -\frac{1}{2}U^{\otimes 2} + \frac{1}{2}(\Lambda^2 U + U \otimes U + \Lambda^2 U) = \Lambda^2 U.$$

References.

[Mac1] Affine Hecke algebras and orthogonal polynomials (Bourbaki seminar), by I.G. Macdonald.

[Mac2] Affine Hecke algebras and orthogonal polynomials (Cambridge University Press), by I.G. Macdonald.

[Mac3] Symmetric functions and Hall polynomials (Oxford Science Publication), by I.G. Macdonald.

[Kir] Lectures on the affine Hecke algebras and Macdonald conjectures, by A.A. Kirillov. Jr

[Hai] Cherednik algebras, Macdonald polynomials and combinatorics, by M. Haiman

9. Haiman Theory

9.1. Springer theory. Let $\mathcal{B} = G/B$ be the flag variety. Let

$$\pi : T^*\mathcal{B} = \tilde{\mathcal{N}} \longrightarrow \mathcal{N} \subset \mathfrak{gl}_n$$

be the Springer resolution of type A. For a nilpotent matrix of Jordan type $\lambda \vdash n$, we denote the Springer fibre by $\mathcal{B}_\lambda$. Let us consider

$$\begin{tikzcd} \mathcal{B}_\lambda \arrow[r] \arrow[d] & \text{fibre} \arrow[r] \arrow[d] & \tilde{\mathcal{N}} \arrow[d] \\ \text{any point} \arrow[r] & \mathbb{O}_\lambda \arrow[r] & \mathcal{N}. \end{tikzcd}$$

It was computed

$$\mathrm{End}_{\mathbf{Perv}(\mathcal{N})}(\pi_* \mathbf{1}_{\tilde{\mathcal{N}}}) = \mathbb{Q}[\mathcal{S}_n]$$

thus the (co)homology of $\mathcal{B}_\lambda$ has an $\mathcal{S}_n$ action. Note that all representation of $\mathcal{S}_n$ are isomorphic to its dual, thus

$$H^\bullet(\mathcal{B}_\lambda) \simeq H_\bullet(\mathcal{B}_\lambda) \text{ as } \mathcal{S}_n\text{-reprsentations.}$$

We will study the cohomology of $\mathcal{B}_\lambda$. We have (up to graded shifting) at the level of K-group

$$\pi_* \mathbf{1}_{\tilde{\mathcal{N}}} = \sum_{\lambda \vdash n} \boldsymbol{t}^? \, H^\bullet(\mathcal{B}_\lambda) \otimes \mathbf{1}_{\mathbb{O}_\lambda} \in K(\mathcal{S}_n\text{-Rep}) \otimes K(\mathcal{N})[\boldsymbol{t}^{\pm 1}].$$

Here $\mathbf{1}_{\mathbb{O}_\lambda} = i_! \mathbf{1}_{\mathbb{O}_\lambda}$. By decomposition theorem, we also have

$$\pi_* \mathbf{1}_{\tilde{\mathcal{N}}} = \bigoplus_{\lambda \vdash n} H_{top}(\mathcal{B}_\lambda) \otimes \mathbf{IC}_{\mathbb{O}_\lambda}.$$

Here the top degree can be compute explicitly, it is

$$n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i = \sum_{j \geq 1} \binom{\lambda'_j}{2} = \langle \rho, w_0 \lambda \rangle.$$

For example, for $n = 3$,

λ	$\mathcal{B}_\lambda$	dim	$H_\bullet$
(3)	a point	0	**tri**
$(2,1)$	union of two $\mathbb{P}^1$'s	1	**tri** $\oplus$ **std**
(1^3)	full flag variety	3	**tri** $\oplus$ **std** $\oplus$ **std** $\oplus$ **alt**

It was known that

$$H^{n(\lambda)}(\mathcal{B}_\lambda) \simeq \mathbf{irr}_\lambda, \quad \text{as an } \mathcal{S}_n\text{-representation.}$$

Later, we will see

$$H^\bullet(\mathcal{B}_\lambda) \simeq \mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \mathbf{tri}, \quad \text{as an } \mathcal{S}_n\text{-representation.}$$

9.2. **Lusztig embedding.** Recall affine Grassmannian for GL_n is

$$Gr = \left\{ \mathcal{O}\text{-lattices in } \mathcal{K}^{\oplus n} \right\} = G_{\mathcal{K}}/G_{\mathcal{O}}.$$

Denote Λ_0 the standard lattice $\mathcal{O}^{\oplus n}$. Recall the Schubert cell

$$\Sigma_\lambda^\circ = \left\{ \Lambda \subseteq \Lambda_0 : \Lambda_0/\Lambda \text{ has type } \lambda \right\} = G_{\mathcal{O}} t^\lambda \cdot \Lambda_0.$$

A torsion $\mathcal{O}$-module has type λ means it is isomorphic to $\mathcal{O}/t^{\lambda_1}\mathcal{O} \oplus \mathcal{O}/t^{\lambda_2}\mathcal{O} \oplus \cdots$ for t the generator of the maximal ideal. Note that

$$\Sigma_{(n)} = \overline{\Sigma_{(n)}^\circ} = \left\{ \Lambda \subseteq \Lambda_0 : \dim \Lambda/\Lambda_0 = n \right\}.$$

Let us take $\mathcal{O} = \mathbb{C}[[t]]$ and $\mathcal{K} = \mathbb{C}((t))$. Let us define

$$\iota : \mathcal{N} \longrightarrow \Sigma_{(n)}, \qquad A \longmapsto (t - A)\Lambda_0.$$

Then we have

$$(\Lambda_0/(t-A)\Lambda_0, t) \simeq (\mathbb{C}^n, A).$$

This shows $\mathbb{O}_\lambda$ is mapped into Σ_λ°. Not hard to show it is an embedding, and by dimension reason, it is open. See [Zhu, Example 2.1.8.] In this case, we get a linear map

$$K(\Sigma_{(n)}) \xrightarrow{\iota^! = \iota^*} K(\mathcal{N}), \qquad \begin{cases} \mathbf{1}_{\Sigma_\lambda^\circ} \longmapsto \mathbf{1}_{\mathbb{O}_\lambda} \\ \mathbf{IC}_{\Sigma_\lambda^\circ} \longmapsto \mathbf{IC}_{\mathbb{O}_\lambda}. \end{cases}$$

Recall that the character

$$\begin{cases} \mathbf{1}_{\Sigma_\lambda^\circ} \longmapsto P_\lambda|_{\boldsymbol{t} \mapsto \boldsymbol{t}^{-1}} & \text{up to some power of } \boldsymbol{t} \\ \mathbf{IC}_{\Sigma_\lambda^\circ} \longmapsto \chi_\lambda = s_\lambda \end{cases}$$

Here P_λ is the Hall–Littlewood polynomial (i.e. Macdonald polynomial at $\boldsymbol{q} = 0$) in n variables. It is not hard to see the the expansion of P_λ

to s_λ does not change if we understand them as symmetric functions. Then the identity

$$\sum_{\lambda\vdash n} \boldsymbol{t}^{?} H^\bullet(\mathcal{B}_\lambda) \otimes \mathbf{1}_{\mathbb{O}_\lambda} = \sum_{\lambda\vdash n} H_{top}(\mathcal{B}_\lambda) \otimes \mathbf{IC}_{\mathbb{O}_\lambda} \in K(\mathcal{S}_n\text{-Rep}) \otimes K(\mathcal{N})$$

reduces to

$$\sum_{\lambda\vdash n} \boldsymbol{t}^{?} H^\bullet(\mathcal{B}_\lambda) P_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}} = \sum_{\lambda\vdash n} H_{top}(\mathcal{B}_\lambda) \otimes s_\lambda \in K(\mathcal{S}_n\text{-Rep}) \otimes \Lambda_{\boldsymbol{t}}.$$

By applying the Frobenius character, $\mathbf{irr}_\lambda \mapsto s_\lambda$, we get

$$\sum_{\lambda\vdash n} \boldsymbol{t}^{?} \begin{pmatrix} \text{F-char of} \\ H^\bullet(\mathcal{B}_\lambda) \end{pmatrix} \otimes P_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}} = \sum_{\lambda\vdash n} s_\lambda \otimes s_\lambda \in \Lambda_{\boldsymbol{t}} \otimes \Lambda_{\boldsymbol{t}},$$

That is

$$\sum_{\lambda\vdash n} \boldsymbol{t}^{?} \begin{pmatrix} \text{F-char of} \\ H^\bullet(\mathcal{B}_\lambda) \end{pmatrix} [X]\, P_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}}[Y] = \sum_{\lambda\vdash n} s_\lambda[X] s_\lambda[Y].$$

The right hand side is $\Omega[XY]^{\deg = n}$. This implies, under the Frobenius character,

$$H^\bullet(\mathcal{B}_\lambda) \mapsto \boldsymbol{t}^{n(\lambda)} \begin{pmatrix} \text{dual basis of } P_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}} \\ \text{under the Hall pairing} \end{pmatrix}.$$

Let us have a quick look at the case $\boldsymbol{t} = 1$, i.e. if forgetting the grading. Recall that $P_\lambda|_{\boldsymbol{t}=1} = m_\lambda$. This tells the Frobenuis character of $H^\bullet(\mathcal{B}_\lambda)$ is $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$, the same as $\mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \mathbf{tri}$. So

$$H^{n(\lambda)}(\mathcal{B}_\lambda) \simeq \mathbf{irr}_\lambda, \quad \text{as an } \mathcal{S}_n\text{-representation.}$$

Now return to the graded version. Denote

$$\begin{aligned} Q_\lambda &= \frac{1}{\langle P_\lambda, P_\lambda\rangle_{\boldsymbol{t}}} P_\lambda && \text{dual HL polynomials} \\ H_\lambda &= Q_\lambda[Z \tfrac{1}{1-\boldsymbol{t}}] && \text{transformed HL polynomials} \\ \tilde{H}_\lambda &= \boldsymbol{t}^{n(\lambda)} H_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}} && \text{cocharge variant of THLP} \end{aligned}$$

Since Q_λ is the dual basis of P_λ under $\langle\cdot,\cdot\rangle_{\boldsymbol{t}}$, H_λ is the dual basis of P_λ under $\langle\cdot,\cdot\rangle$. As a result, $H_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}}$ is the dual basis of $P_\lambda|_{\boldsymbol{t}\mapsto \boldsymbol{t}^{-1}}$ under

$\langle\cdot,\cdot\rangle$. This implies

$$H^\bullet(\mathcal{B}_\lambda) \longmapsto \begin{pmatrix} \text{F-char of} \\ H^\bullet(\mathcal{B}_\lambda) \end{pmatrix} = \tilde{H}_\lambda.$$

Note that $\tilde{H}_\lambda$ can be characterized by

(1) $\langle s_{(n)}, H_\mu\rangle = 1$ for $n = |\mu|$.
(2) $\tilde{H}_\mu[Z] \in \mathrm{span}(s_\lambda : \lambda \geq \mu)$
(3) $\tilde{H}_\mu[Z(1-\boldsymbol{t})] \in \mathrm{span}(s_\lambda : \lambda \geq \mu')$.

The proof goes as follows:

(1) Since $H^\bullet(\mathcal{B}_\mu) \simeq \mathrm{Ind}_{\mathcal{S}_\mu}^{\mathcal{S}_n}\mathbf{tri}$, the invariant $H^\bullet(\mathcal{B}_\lambda)^{\mathcal{S}_n}$ has to be one-dimensional, thus it is $H^0(\mathcal{B}_\lambda)$.

(2) This is a standard fact about $\mathrm{Ind}_{\mathcal{S}_\mu}^{\mathcal{S}_n}\mathbf{tri}$.

(3) comes from

$$H_\mu[Z(1-\boldsymbol{t})] = Q_\mu = \frac{1}{\langle P_\mu, P\mu\rangle_{\boldsymbol{t}}} P_\mu \in \mathrm{span}(s_\lambda, \lambda \leq \mu).$$

So

$$\begin{aligned}\tilde{H}_\mu[Z(1-\boldsymbol{t})] &= \tilde{H}_\mu[\boldsymbol{t}Z(\boldsymbol{t}^{-1}-1)] = \boldsymbol{t}^{|\lambda|}\tilde{H}_\mu[-Z(1-\boldsymbol{t}^{-1})] \\ &= \frac{\boldsymbol{t}^{|\lambda|}\boldsymbol{t}^{n(\lambda)}}{\langle P_\mu, P\mu\rangle_{\boldsymbol{t}}} P_\mu[-Z]\big|_{\boldsymbol{t}\mapsto\boldsymbol{t}^{-1}} \in \mathrm{span}(s_{\lambda'} : \lambda \leq \mu).\end{aligned}$$

References.

[Lus] Green polynomials and singularities of unipotent classes, by G. Lusztig.

[Zhu] An introduction to affine Grassmannians and geometric Satake equivalence, by X. Zhu.

[Hai] Combinatorics, symmetric functions and Hilbert schemes, by M. Haiman.

9.3. **Computation of Springer resolution.** Let us describe the cohomology of $\mathcal{B}_\lambda$ more explicitly. We can assume $\mathcal{B}_\lambda$ has an action by torus T_λ. For example,

$$\lambda = (2,1) \rightsquigarrow \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \rightsquigarrow T_\lambda = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} : x, y \in \mathbb{C}^* \right\}.$$

Note that the rank of the torus has rank $\ell(\lambda)$ the number of parts of λ. Consider the following closed variety

$$X_\lambda = \left\{ (\mathbf{t}, \mathbf{x}) : \begin{array}{l} \text{the coordinate of } \mathbf{x} \\ \text{has } \lambda_i \text{ many copies of } t_i \end{array} \right\} \subseteq \mathbb{C}^{\ell(\lambda)} \times \mathbb{C}^n.$$

Then $\mathcal{O}(X_\lambda) = H^\bullet_{T_\lambda}(X_\lambda)_{\mathbb{C}}$ with

$$\begin{cases} \text{functions } t_1, \dots, t_{\ell(\lambda)} \text{ are equivariant parameters of } T_\lambda, \\ \text{functions } x_1, \dots, x_n \text{ are restricted from full flag variety,} \\ \text{grading is from the } \mathbb{C}^\times\text{-action on } t_j, x_i \\ \mathcal{S}_n \text{ acts by permuting } x_i\text{'s.} \end{cases}$$

We will view X_λ as a variety over $\mathbb{C}^r = \operatorname{Spec} H^\bullet_{T_\lambda}(\mathsf{pt})$. Then the generic fibre at $\mathbf{t} = (t_1, \dots, t_{\ell(\lambda)})$ is

$$\mathcal{S}_n\text{-orbit of } (\cdots, \underbrace{t_i, \cdots, t_i}_{\lambda_i}, \cdots) \qquad \subseteq \mathbb{C}^n.$$

For example,

λ	generic fibre of X_λ	$\mathcal{O}(X_\lambda)$
(3)	$\{(t,t,t)\}$	$\mathbb{Q}[x_1,x_2,x_3]/\langle x_1 = x_2 = x_3 = t\rangle$
$(2,1)$	$\{(t_1,t_1,t_2)\}$ $\cup\{(t_1,t_2,t_1)\}$ $\cup\{(t_2,t_1,t_1)\}$	$\mathbb{C}[x_1,x_2,x_3]/\langle\text{complicated}\rangle$
(1^3)	$\bigcup_{\sigma\in\mathcal{S}_3}\{(t_{\sigma(1)},t_{\sigma(2)},t_{\sigma(3)})\}$	$\mathbb{C}[x_1,x_2,x_3]/\langle e_i(x) = e_i(t), i = 1,2,3\rangle$

Finally, $H^\bullet(\mathcal{B}_\lambda) = H^\bullet_{T_\lambda}(\mathcal{B}_\lambda)/\langle t_i\rangle$. That is,

$$H^\bullet(\mathcal{B}_\lambda) = \mathcal{O}(X^0_\lambda),$$

where X^0_λ is the zero fibre (scheme theoretic, not reduced). Geometrically,

$$\begin{tikzcd} X^0_\lambda \arrow[r] \arrow[d] & X_\lambda \arrow[r] \arrow[d] & (\text{fibre})_{red} \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C}^{\ell(\lambda)} \arrow[r] & \overline{\text{stratum of } \lambda} \arrow[r] & \mathbb{C}^n/\mathcal{S}_n. \end{tikzcd}$$

Let us denote graded ring with $\mathcal{S}_n$-action

$$R_\lambda(\mathbf{x}) = \mathcal{O}(X^0_\lambda), \text{ i.e. } \operatorname{Spec} R_\lambda(\mathbf{x}) = X^0_\lambda.$$

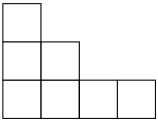
Then

$$\text{Frobenius character of } R_\mu(\mathbf{x}) \in \operatorname{span}(s_\lambda : \lambda \geq \mu).$$

9.4. **Haiman Theory.** Now we consider the two dimensional analogy. We need Hilbert schemes. Let H_n be the Hilbert schemes of n points over $\mathbb{C}^2$. That is

$$H_n = \{\text{ideal } I \subset \mathbb{C}[x,y] : \dim \mathbb{C}[x,y]/I = n\}.$$

For $\mu \vdash n$, and generic $(\mathbf{a}, \mathbf{b}) \in \mathbb{C}^{\ell(\lambda)} \times \mathbb{C}^{\ell(\lambda')}$ (i.e. a_i's and b_i's are distinct), we construct n different points in $\mathbb{C}^2$. We illustrate the definition by an example.



$$\begin{array}{llll} (a_3, b_1) & & & \\ (a_2, b_1) & (a_2, b_2) & & \\ (a_1, b_1) & (a_1, b_2) & (a_1, b_2) & (a_1, b_2) \end{array}$$

The ideal for these n points defines an ideal, i.e. defines a point of H_n. Let us consider $C_\mu \subset H_n$ the closure of all points constructed in this way. Note that the monomial ideal I_μ defined by the diagram of μ is in C_μ. For example, I_μ is given by

$$\begin{ytableau} y^2 \\ y & xy \\ 1 & x & x^2 & x^3 \end{ytableau} = \langle x^4, x^3y^2, xy^3, y^3 \rangle.$$

Let us consider

$$\begin{tikzcd} \text{fibre}_\mu \arrow[r] \arrow[d] & \text{fibre} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "\rho"'] & \mathbb{C}^{2n} \arrow[d] \\ \{I_\mu\} \arrow[r] & C_\mu \arrow[r] & H_n \arrow[r] & \mathbb{C}^{2n}/\mathcal{S}_n. \end{tikzcd}$$

The notations will be explained one by one:

- We view

$$\mathbb{C}^{2n} = \{n\text{-tuples of points over } \mathbb{C}^2\};$$

$$\mathbb{C}^{2n}/\mathcal{S}_n = \{n\text{-multi-sets of points over } \mathbb{C}^2\}.$$

We write the $\mathcal{S}_n$-orbit of $(P_1, \ldots, P_n) \in \mathbb{C}^{2n}$ by

$$[P_1] + \cdots + [P_n] \in \mathbb{C}^{2n}/\mathcal{S}_n.$$

- Here $H_n \to \mathbb{C}^{2n}/\mathcal{S}_n$ is given by

$$
\begin{aligned}
I \longmapsto & \text{ the 0-cycle defined by } I \\
& = \sum_{P \in \mathbb{C}^2} \operatorname{mult}_P(\mathbb{C}[x,y]/I) \in \mathbb{C}^{2n}/\mathcal{S}_n.
\end{aligned}
$$

  For example, if I is the ideal for n-distinct points $P_1, \ldots, P_n$, then $I \mapsto [P_1] + \cdots + [P_n]$. If $I = I_\mu$ defined by a partition, $I_\mu \longmapsto n[\mathbf{0}]$.
- Here X_n is the *reduced* fibre product, called *isospectral Hilbert scheme*. Say,

$$
X_\mu = \left\{ (I, P_1, \ldots, P_n) : \begin{array}{l} \text{the 0-cycle defined by } I \\ \text{over } \mathbb{C}^2 \text{ is } [P_1] + \cdots + [P_n] \end{array} \right\}.
$$

  Note that at each point, the fibre is a closed subscheme of $\mathbb{C}^2$. At the generic points, i.e. ideals $I \in H_n$ defined by n distinct points, the fibre is reduced and is $\mathcal{S}_n$-orbit of those n-tuple in $\mathbb{C}^2$.

Let us define

$$
R_\mu(\mathbf{x}, \mathbf{y}) = \mathcal{O}(\text{fibre}_\mu), \text{ i.e. } \operatorname{Spec} R_\mu(\mathbf{x}, \mathbf{y}) = \text{fibre}_\mu.
$$

Since I_μ is a $(\mathbb{C}^*)^2$-fixed point, $R_\mu(\mathbf{x}, \mathbf{y})$ is a bigraded ring with an $\mathcal{S}_n$-action. The ring $R_\mu(\mathbf{x}, \mathbf{y})$ is the two dimensional analogy of $R_\lambda(\mathbf{x})$ above.

Denote the bigraded Frobenius character by $\chi : \bigoplus_{n \geq 0} K(\mathcal{S}_n\text{-Rep})_{\boldsymbol{q},\boldsymbol{t}} \to \Lambda_{\boldsymbol{q},\boldsymbol{t}}$. We are going to show $\chi_{R_\mu}[Z] \in \Lambda_{\boldsymbol{q},\boldsymbol{t}}$ satisfies the following characterization of transformed Macdonald polynomials $\tilde{H}_\mu$

(1) $\langle s_{(n)}, \tilde{H}_\mu \rangle = 1$, where $n = |\mu|$;
(2) $\tilde{H}_\mu[Z(1 - \boldsymbol{q})] \in \operatorname{span}(s_\lambda : \lambda \geq \mu)$;
(3) $\tilde{H}_\mu[Z(1 - \boldsymbol{t})] \in \operatorname{span}(s_\lambda : \lambda \geq \mu')$.

Haiman proved the map $\rho : X_n \to H_n$ is flat, i.e.

$$
\boxed{\rho_* \mathcal{O}_{X_n} \text{ is a vector bundle of rank } n!.} \tag{$*$}
$$

As a result,

$$
R_\mu(\mathbf{x}, \mathbf{y}) = \text{fibre of } \rho_* \mathcal{O}_{X_n} \text{ at } I_\mu.
$$

Now we can prove (1). Since generically the fibre is an $\mathcal{S}_n$-orbit of n distinct points, thus the fibre of $\mathcal{O}_{X_n}$ is the regular representation $\mathbb{C}[\mathcal{S}_n]$ at generic points. By $(*)$, all the fibre of $\mathcal{O}_{X_n}$ is the regular representation. Say, the multiplicity sheaf

$$\mathcal{H}om(\mathbf{irr}_\lambda, \rho_* \mathcal{O}_{X_n})$$

is a vector bundle. In particular, we have $R_\mu \simeq \mathbb{C}[\mathcal{S}_n]$ the regular representation. Thus $R_\mu^{\mathcal{S}_n} = R_\mu^{\deg=0}$, so $\langle s_{(n)}, \chi_{R_\mu} \rangle = 1$.

Let us prove (2). The proof of (3) is similar. We need to notice that the first projection of points reduces to dimension 1 case. For example:

$$\begin{array}{llllcllll} (a_3,b_1) & & & & & a_3 & & & \\ (a_2,b_1) & (a_2,b_2) & & & \mapsto & a_2 & a_2 & & \\ (a_1,b_1) & (a_1,b_2) & (a_1,b_2) & (a_1,b_2) & & a_1 & a_1 & a_1 & a_1 \end{array}$$

Thus

$$R_\mu(\mathbf{x},\mathbf{y})/\langle \mathbf{y} \rangle = R_\mu(\mathbf{x},\mathbf{y})/\langle y_1, \dots, y_n \rangle = R_\mu(\mathbf{x}).$$

Denote Q_μ the only point over I_μ, i.e. $Q_\mu = (I_\mu, \mathbf{0}, \cdots, \mathbf{0}) \in X_n$. Let us consider the diagram

$$\begin{array}{ccccccc} S & = & \mathcal{O}_{X_n,Q_\mu} & \xrightarrow{R/\mathfrak{m}} & R_\mu(\mathbf{x},\mathbf{y}) & & \\ & & \big\downarrow & & \big\downarrow & & \\ R & = & \mathcal{O}_{H_n,I_\mu} & \xrightarrow{R/\mathfrak{m}} & R/\mathfrak{m} & = & \mathbb{C} \end{array}$$

Here S and R are just local rings at Q_μ and I_μ respectively, and $\mathfrak{m}$ is the maximal ideal of $I_\mu \in H_n$. By $(*)$, the ring S is free over R, from the diagram, we have

$$\chi_S[Z] = \chi_{R_\mu}[Z] \cdot \chi_R.$$

Since R has trivial $\mathcal{S}_n$ action, $\chi_R[Z] \in \mathbb{Q}_{\boldsymbol{q},\boldsymbol{t}}$, so we denote it just by χ_R. It was proved by Haiman that $y_1, \dots, y_n$ form a regular sequence of $\mathcal{O}_{X_n,Q\mu}$. By Koszul complex (see below), we get

$$\chi_{S/\langle \mathbf{y} \rangle}[Z] = \chi_S[Z(1-\boldsymbol{q})] = \chi_{R_\mu}[Z(1-\boldsymbol{q})] \cdot \chi_R.$$

Now we have

$$S/\langle \mathbf{y} \rangle \xrightarrow{R/\mathfrak{m}} R_\mu(\mathbf{x},\mathbf{y})/\langle \mathbf{y} \rangle = R_\mu(\mathbf{x}).$$

Since $\mathfrak{m}$ has trivial $\mathcal{S}_n$-action, by Nakayama lemma,

$$\chi_{R_\mu(\mathbf{x})} \in \operatorname{span}(s_\lambda : \lambda \geq \mu) \Longrightarrow \chi_{S/\langle \mathbf{y} \rangle}[Z] \in \operatorname{span}(s_\lambda : \lambda \geq \mu).$$

Thus

$$\chi_{R_\mu}[Z(1-\boldsymbol{q})] \in \operatorname{span}(s_\lambda : \lambda \geq \mu).$$

This proves (2).

Appendix: Koszul complex. Let V be a $\mathcal{S}_n \times \mathbb{C}^*$-representation. Then

$$\chi_V[Z(1-\boldsymbol{q})] = \sum_k (-\boldsymbol{q})^k \chi_{V \otimes \Lambda^k \mathbb{C}^n}[Z].$$

Actually it suffices to check the right hand side is a ring homomorphism (i.e. well-behaved under induction), and it is true for **tri**.

References.

[Gor] Haiman's work on the $n!$ theorem, and beyond by Iain Gordon.
[Hai1] Macdoanlad Polynomials and Geoemtry, by M. Haiman.
[Hai2] Combinatorics, symmetric functions and Hilbert schemes, by M. Haiman.