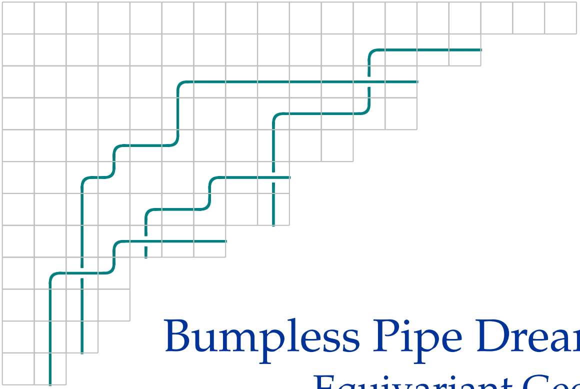


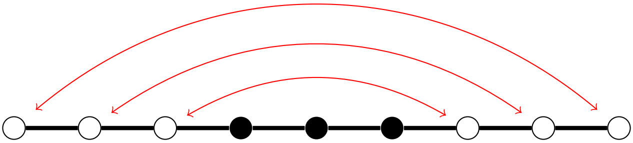
Bumpless Pipe Dream Fragments

—Equivariant Geometry of Clans

(joint with Yiming Chen, Neil J.Y. Fan and Ming Yao)

Rui Xiong

Background



(Satake diagram of type AIII)

Symmetric Pairs

A **symmetric pair** is a pair (K, G) where $K = G^\theta$ for some involution θ of G. When $G = GL_n$, we have three types

type	involution	$K \subset G$
AI	$g \mapsto (g^t)^{-1}$	$O_n \subset GL_n$
AII	$g \mapsto J_{2m}(g^t)^{-1}J_{2m}$	$Sp_{2m} \subset GL_{2m}$
AIII	$g \mapsto I_{p,q} \cdot g \cdot I_{p,q}^{-1}$	$GL_p \times GL_q \subset GL_{p+q}$

where $J_{2m} = \begin{bmatrix} & I_m \\ -I_m & \end{bmatrix}$ and $I_{p,q} = \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix}$.

Real Groups

Symmetric pair arises from the study of real groups.

type	$K \subset G$	real form
AI	$O_n \subset GL_n$	$GL_n(\mathbb{R})$
AII	$Sp_{2m} \subset GL_{2m}$	$GL_m(\mathbb{H})$
AIII	$GL_p \times GL_q \subset GL_{p+q}$	$U(p, q)$

This will allow us translate real question to complex questions. For example, the geometry of K-orbit closure over flag variety is closely related to the representation of real group, thanks to the serial work of Vogan.

Spherical Subgroups

A subgroup H of G is called **spherical** if the set

$$H\backslash G/B = \{H\text{-orbits of } gB \in G/B\}$$

is finite, where G/B is the flag variety.

The Borel subgroup B is spherical. We have **Bruhat decomposition**

$$B\backslash G/B \overset{1:1}{=\!=} \text{Weyl group.}$$

Each orbit is known as a **Schubert cell**.

The group K in a symmetric pair is spherical (a version of **Iwasawa decomposition**).

$$K\backslash G/B \overset{1:1}{=\!=} \text{a finite set.}$$

We are interested in the K-orbits of G/B.

Index set of K-orbits

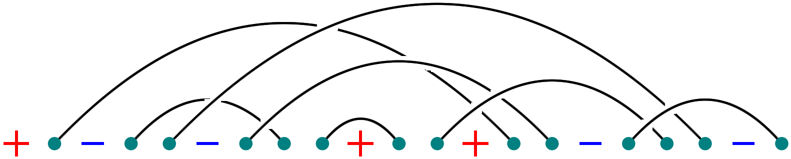
When $G = GL_n$, the flag variety

$$G/B = \{0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n\}.$$

type	K	index set of $K\backslash G/B$
AI	O_n	$\{w \in S_n : w^2 = 1\}$
AII	Sp_{2m}	$\{w \in S_{2m} : w^2 = 1, w(i) \neq i\}$
AIII	$GL_p \times GL_q$	$\{(p, q)\text{-clans}\}$

Today we will focus on type AIII.

Combinatorics of Clans



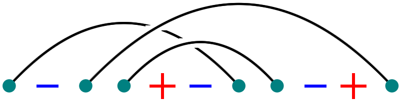
(an example of (10,11)-clan)

Clans

A (p, q)-**clan** is a partial matching of $p + q$ nodes with unmatched nodes colored by $+$ or $-$ with

$$\#\{+\} - \#\{-\} = p - q.$$

For example, the following is a $(5, 6)$-clan

$\gamma =$ 

$$\boldsymbol{v}_i^{\gamma} = \begin{matrix} e_1 \\ + \\ e_8 \end{matrix} \; e_6 \; \begin{matrix} e_2 \\ + \\ e_{11} \end{matrix} \; \begin{matrix} e_3 \\ + \\ e_9 \end{matrix} \; e_4 \; e_7 \; e_8 \; e_9 \; e_{10} \; e_5 \; e_{11},$$

$$\text{the orbit} = K \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} B/B.$$

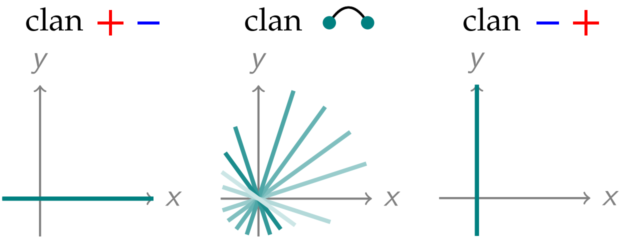
Example

Consider the case

$$GL_1 \times GL_1 \subset GL_2 \circlearrowright \mathbb{P}^1 = \{0 \subset L \subset \mathbb{C}^2\} = \mathbb{C} \sqcup \{\infty\}.$$

There are three orbits

clan + − | clan | clan − +



$\{0\} = \mathbb{C}e_1$ | $\mathbb{C}^\times$ | $\{\infty\} = \mathbb{C}e_2$

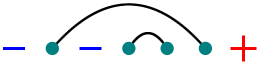
Special types of clans

matchless

$- + - + + - -$

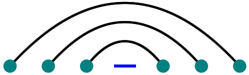
closed K-orbits
(there are $\binom{p+q}{p}$ many)

non-crossing



Richardson varieties (indexed by two inverse Grassmannian permutations)

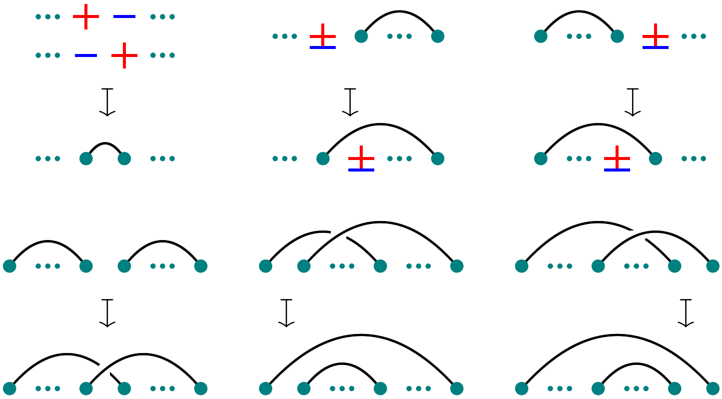
rainbow



open orbit
(open orbit is unique)

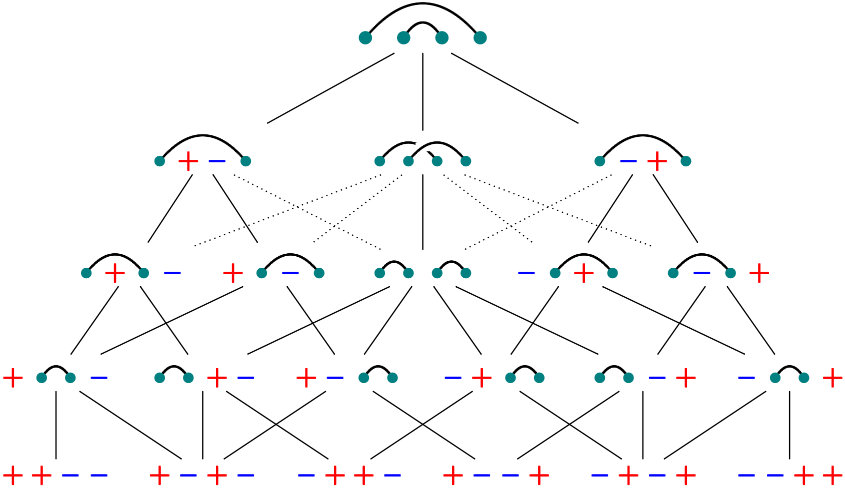
Orders on Clans

The weak order of clans are generated by

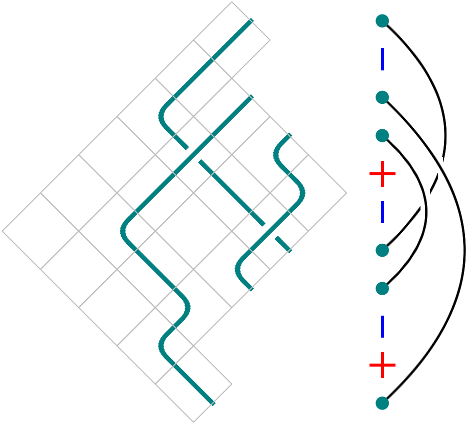


There is also a strong order induced by orbital closure.

Example



Bumpless Pipe Dreams



(BPD meets clans)

Schubert Calculus

In the work of Wyser and Yong [WY14], many fundamental framework was built.

- **polynomial representations**
- **non-equivariant Schubert expansion**
- **commutative algebra interpolations** and **etc.**

However, there remain some questions

- **combinatorial models of the polynomials**
- **equivariant Schubert expansion** and **etc.**

We will answer the second question.

B. J. Wyser and A. Yong. Polynomials for $GL_p \times GL_q$ orbit closures in the flag variety. Sel. Math. New Ser. (2014) 20:1083-1110.

Polynomial representatives

The polynomials of [WY14] are characterized by the following two properties:

- When γ is non-crossing,

$$\Upsilon_\gamma(x;y) = \mathfrak{S}_{u_\gamma}(x;y) \cdot \mathfrak{S}_{v_\gamma}(x;\overleftarrow{y}).$$

(Since the closed K-orbit is a Richardson variety)

- When $s_i * \gamma \neq \gamma$ (weak order)

$$\Upsilon_{s_i*\gamma} = \frac{\Upsilon_\gamma - \Upsilon_\gamma|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}.$$

(Follow from the geometry of weak order)

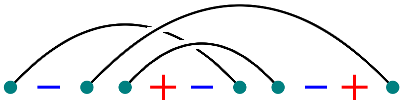
Main result — Localization

Let us denote the localization

$$\Upsilon_\gamma|_w = \Upsilon_\gamma(y_{w(1)}, \ldots, y_{w(n)}; y_1, \ldots, y_n).$$

Our main result is a combinatorial formula of the localization at the minimal $w = v_\gamma$ such that $\Upsilon_\gamma|_w \neq 0$:

$\gamma =$

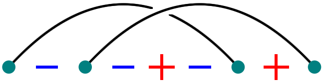


$v_\gamma = \; 1 \;\; 6 \;\; 2 \;\; 3 \;\; 4 \;\; 7 \;\; 8 \;\; 9 \;\; 10 \;\; 5 \;\; 11$

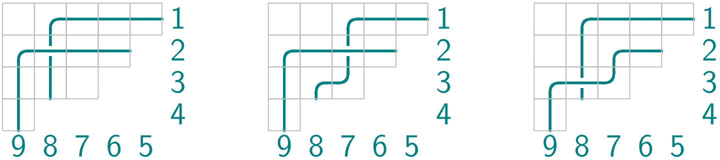
Theorem (Chen–Fan–X.–Yao, 2025+)

The localization $\Upsilon_\gamma|_{v_\gamma} \in \mathbb{Q}[y_1, \ldots, y_n]$ *is given by* **bumpless pipe dream fragments**.

Example

$\gamma =$ 

$v_\gamma = 1 \quad 5 \quad 2 \quad 6 \quad 3 \quad 7 \quad 8 \quad 4 \quad 9$

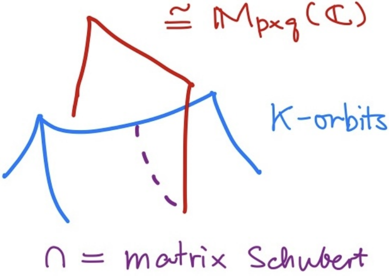


$$\begin{matrix} & & & \\ & & & y_5-y_2 \\ & & y_6-y_3 & y_5-y_3 \\ y_8-y_4 & y_7-y_4 & y_6-y_4 & y_5-y_4 \end{matrix} \cdot \left(\begin{matrix} y_1-y_9 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & y_3-y_7 & & \\ 1 & & & & \end{matrix} + \begin{matrix} y_1-y_9 & y_1-y_8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & & & \\ 1 & & & & & \end{matrix} + \begin{matrix} y_1-y_9 & 1 & 1 & 1 & 1 \\ y_2-y_9 & 1 & 1 & 1 & \\ 1 & 1 & 1 & & \\ 1 & & & & \end{matrix} \right)$$

Idea of the Proof

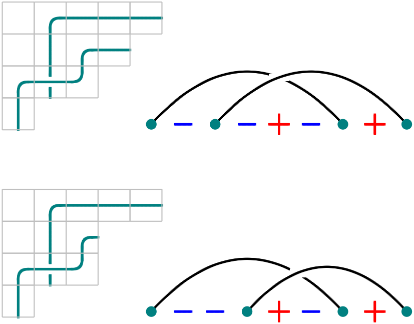
Geometric part of the proof

The initial case is by study the local structure of closed K-orbits at id.



Algebraic part of the proof

The inductive step is by study the local change of a clan.

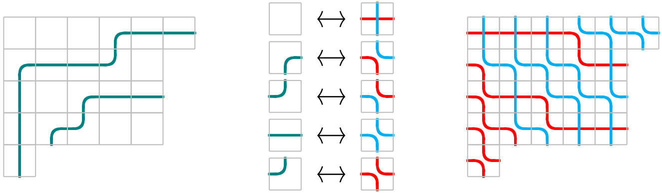


Relation to Billey formula

When γ is matchless, the K-orbit closure is a Richardson variety. In this case,

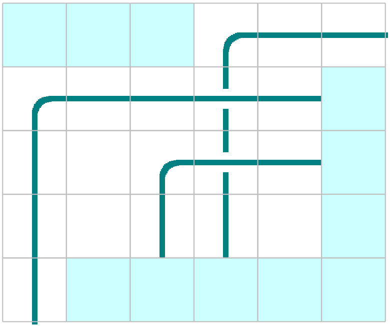
$$\Upsilon_\gamma|_{v_\gamma} = \mathfrak{S}_{v_\gamma}(v_\gamma y, y).\mathfrak{S}_{u_\gamma}(v_\gamma y, \overleftarrow{y}).$$

After some work, one can see our formula agrees with Billey formula. Actually, the key step is a classical bijection of five-vertex model:



Note that  does not appear since γ is matchless.

Applications



(normal T-stable curves at v_γ)

Application — equivariant Schubert expansion

We give a complete answer of [WY14, Question 2].
Assume

$$\Upsilon_\gamma(x;y) = \sum_{w \in S_n} c_{\gamma w}(y) \mathfrak{S}_w(x;y).$$

Theorem (Chen–Fan–X.–Yao, 2025+)

We have

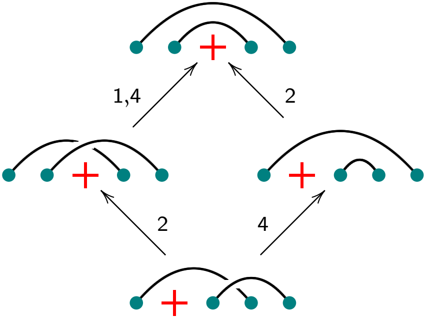
$$c_{\gamma w}(y) = \begin{matrix} \textit{Schubert polynomial} \\ \textit{of a partial permutation} \end{matrix} \ \textit{or } 0$$

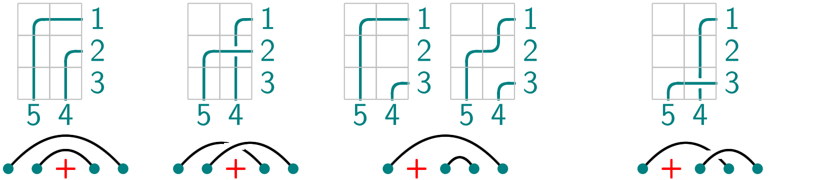
where the partial permutation is determined by $w * \gamma$.

Example

$\Upsilon_{\gamma=}$  (x,y)

$$= (y_1 - y_5)(y_2 - y_5)\mathfrak{S}_{\mathrm{id}}(x,y)$$
$$+ (y_1 - y_5)\mathfrak{S}_{s_2}(x,y)$$
$$+ (y_2 - y_4 + y_1 - y_5)\mathfrak{S}_{s_4}(x,y)$$
$$+ \mathfrak{S}_{s_1 s_2}(x,y)$$
$$+ \mathfrak{S}_{s_2 s_4 = s_4 s_2}(x,y).$$





1 | $y_1 - y_5$ | $y_2 - y_4 + y_1 - y_5$ | $(y_1 - y_5)(y_2 - y_5)$

Application — structure constants

Since when γ is non-crossing, the K-orbit closure is a Richardson variety, we can obtain the following result.

Corollary (Chen–Fan–X.–Yao, 2025+)

For any non-crossing clan γ and any $v \in S_n$, assume that

$$\mathfrak{S}_{v_\gamma}(x;y)\mathfrak{S}_v(x;y) = \sum \mathbf{d}^w_{u_\gamma,v}(y)\mathfrak{S}_w(x;y).$$

Then the coefficient of $\mathfrak{S}_{w_0 v_\gamma}(x;y)$ is given by

$$\mathbf{d}^{w_0 v_\gamma}_{u_\gamma,v}(y) = c_{\gamma,w_0 v}(\overleftarrow{y}).$$

Note that this is a formula of Graham-positivity [Gr].

William Graham. Positivity in equivariant Schubert calculus. Duke Math. J., 109(3):599–614, 2001.

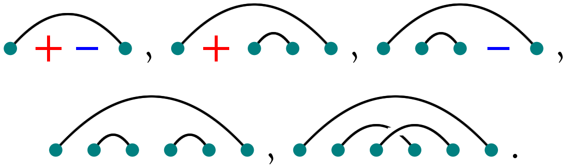
Application — smoothness

Localization can be used to detect smoothness.

Corollary (Chen–Fan–X.–Yao, 2025+)

For a clan γ, the following statements are equivalent:

- *the orbit closure Y_γ is smooth at v_γ;*
- *the clan γ has exactly one BPD fragment;*
- *the clan γ avoids the following 5 patterns:*



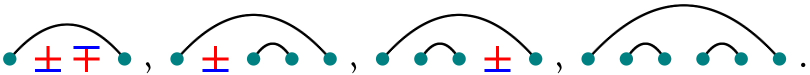
The proof is by comparing the localization and the T-stable curves.

Application — smoothness

We can recover the smoothness criterion of McGovern [1] and Woo and Wyser [2].

Theorem (McGovern, Woo–Wyser)

For a non-crossing γ, the orbit closure Y_γ is smooth if and only if γ avoids the following patterns:



W. McGovern, Closures of K-orbits in the flag variety for $U(p, q)$, J. Alg. 322 (2009), 2709–2712.

A. Woo and B. J. Wyser. Combinatorial results on (1, 2, 1, 2)-avoiding $GL(p, C) \times GL(q, C)$-orbit closures on $GL(p + q, C)/B$, Int. Math. Research Notices 24 (2015), 13148-13193.

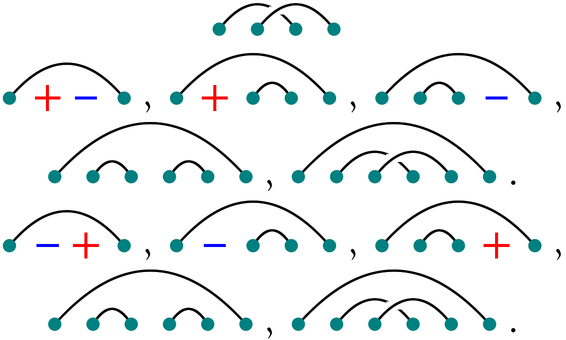
The proof

By [1], a Richardson variety is smooth if and only if it is smooth at the maximal and the minimal permutation.

Richardson

smooth at v_γ (minimal point)

smooth at $w_0 u_\gamma$ (maximal point)



A. Knutson, A. Woo, and A. Yong. Singularities of Richardson varieties. Mathematical Research Letters.

Thank You