Combinatorics, Elliptic Cohomology and Representation Theory

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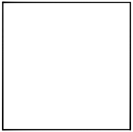
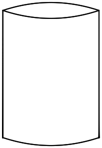
Classification of one-dimensional groups

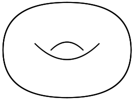
There are three kinds of **one-dimensional algebraic groups**.



$\mathbb{G}_a = (\mathbb{C}, +)$
additive group

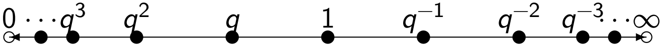


$\mathbb{G}_m = (\mathbb{C}^\times, \cdot) \simeq \mathbb{C}/\mathbb{Z}$
multiplicative group



$\mathbb{C}^\times / q^{\mathbb{Z}} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$
elliptic curves

$q^{\mathbb{Z}} \quad =$

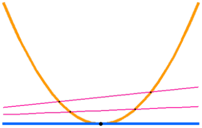


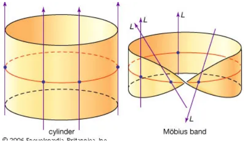
Recall the genus of an elliptic curve is 1.

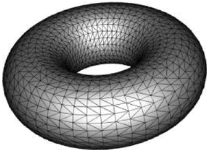
Elliptic cohomology

By **Quillen**, generalized cohomology theory corresponds to some **formal group law**. For example,

$\mathbb{G}_a = (\mathbb{C}, +)$	$\mathbb{G}_m = (\mathbb{C}^\times, \cdot)$	$\mathbb{C}^\times / q^{\mathbb{Z}}$
cohomology Chow ring	topological/algebraic K-theory	**elliptic cohomology**







$$CH(X) = \mathbb{Q}[\text{algebraic cycles}]/\text{rational equivalence};$$
$$K(X) = \mathbb{Q}[\text{vector bundles}]/\text{exact sequences}.$$

Up to now, we still do not know what mathematical objects elliptic cohomology parameters.

Flag varieties

Let us consider

flag variety G/B $\quad$ G = a reductive group
B = a Borel subgroup

For example, when $G = GL_n$,

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n : \dim V_i = i\} = \mathcal{F}\ell(n)$$

is the classical flag variety. We want to do **Schubert calculus** in

$$E_T(G/B) = \begin{array}{c}\text{equivariant elliptic cohomology} \\ \text{of the flag variety } G/B\end{array}$$

where $T \subset B$ is the maximal torus.

Theta functions

We will use the **Jacobi theta function**

$$\theta(u) = (x^{1/2} - x^{-1/2}) \prod_{n>0} (1 - q^n x)(1 - q^n/x)$$

where $x = e^{2\pi i u}$. This is not a function over $E = \mathbb{C}^\times / q^{\mathbb{Z}}$, but a global section of a degree-one line bundle.

From the construction, typical elements in elliptic cohomology are like

$$\theta(\text{a vector bundle}) \quad \text{or} \quad \theta(\text{a manifold}).$$

Actually, elliptic cohomology is constructed such that these symbols make sense. For example, we could take

$$E_T(X) = K_T(X)[[q]].$$

Schubert classes

There are two sources of elliptic Schubert classes.

- Rimányi and Weber [RW20] introduced the **elliptic characteristic** of Schubert varieties twisted by a rational line bundle.
- Aganagic and Okounkov [AO] defined **elliptic stable envelopes** for general conic symplectic resolutions, including Springer resolution T^*G/B.

It is known that the two families of Schubert classes are equivalent.

R. Rimányi, A. Weber, *Elliptic classes of Schubert varieties via Bott-Samelson resolution,*

M. Aganagic, A. Okounkov. *Elliptic stable envelopes.*

Schubert classes

Let $\lambda \in \mathrm{Pic}(G/B)_{\mathbb{Q}}$ be a rational divisor.

For an element of Weyl group $w \in W$, Rimányi and Weber [RW20] defined a Schubert class (depend on λ)

$$\mathfrak{E}(X_w) \in E_{T \times \mathbb{G}_m}(G/B).$$

We will study its **dual basis** under the Poincaré pairing

$$E_w \in E_{T \times \mathbb{G}_m}(G/B)_{\mathrm{loc}} \overset{\text{localization}}{\hookrightarrow} \mathsf{Map}(W, E_{T \times \mathbb{G}_m}(\mathsf{pt}))_{\mathrm{loc}}.$$

We denote

$$z_\alpha = \text{equivariant parameter for } \alpha \in X_*(T), \qquad \alpha \text{ is a root}$$
$$\lambda_{\alpha^\vee} = \text{dynamical parameter for } \langle \lambda, \alpha^\vee \rangle \in \mathbb{Q}, \quad \alpha^\vee \text{ is a coroot}.$$

Billey-type formula

The following two functions are useful

$$\mathsf{P}(x,y)=\frac{\theta(x-y)\theta(\hbar)}{\theta(y+\hbar)\theta(x)},\qquad \mathsf{Q}(x,y)=\frac{\theta(x+\hbar)\theta(y)}{\theta(y+\hbar)\theta(x)}.$$

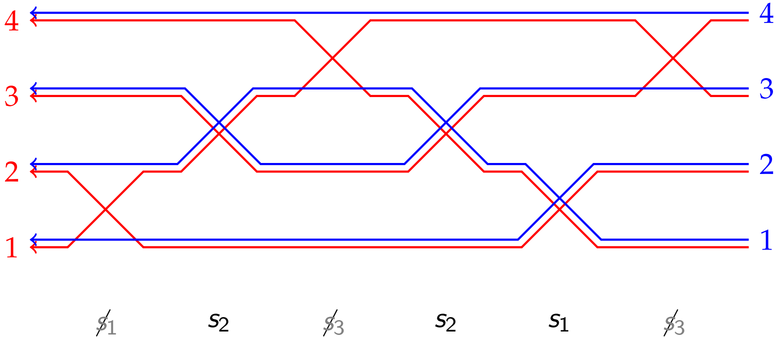
Theorem (Lenart–X.–Zhong)

Let $u,w\in W$. For a decomposition $u=s_{i_1}\cdots s_{i_\ell}$, the localization of elliptic Schubert class admits the following combinatorial formula

$$E_w(u)=\sum_J\prod_{j=1}^{\ell}\begin{cases}\mathsf{Q}(\lambda_{\check{\gamma}_j^J},z_{\beta_j}), & j\in J,\\ \mathsf{P}(\lambda_{\check{\gamma}_j^J},z_{\beta_j}), & j\notin J,\end{cases}\qquad \begin{cases}\beta_j=s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j},\\ \check{\gamma}_j^J=s_{i_\ell}^{\epsilon_\ell}\cdots s_{i_{j+1}}^{\epsilon_{j+1}}\alpha_{i_j}^\vee,\\ \epsilon_j=\delta_{j\in J}^{\mathrm{Kr}}\end{cases}$$

with the sum over $J\subset\{1,\ldots,n\}$ such that $w=s_{i_1}^{\epsilon_1}\cdots s_{i_\ell}^{\epsilon_\ell}$.

Example



$P(\lambda_2 - \lambda_1, z_1 - z_2)$ $P(\lambda_1 - \lambda_4, z_1 - z_4)$ $Q(\lambda_1 - \lambda_2, z_2 - z_4)$

$Q(\lambda_3 - \lambda_1, z_2 - z_3)$ $Q(\lambda_1 - \lambda_3, z_3 - z_2)$ $P(\lambda_3 - \lambda_4, z_3 - z_1)$

3D mirror symmetry

The new feature of elliptic Schubert calculus is the appearance of **dynamical parameters**. 3D mirror symmetry, also known as S(ymplectic) duality, predicts a close relation between

$$E_w \text{ and } E_w^L \text{ for the Langlands dual group } G^L.$$

The following is a shadow of 3D mirror symmetry

Theorem

We have

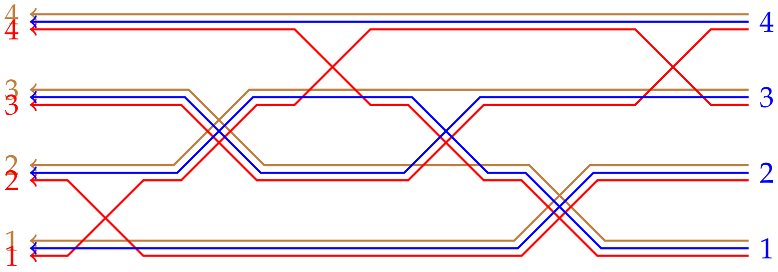
$$\left(E_w(u)\right)^{-1}_{u,w\in W} = \left(E^L_{w^{-1}}(u^{-1})\right)_{u,w\in W}$$

under the identification $z^L_{\alpha^\vee} = \lambda_{\alpha^\vee}$ *and* $\lambda^L_\alpha = z_\alpha$.

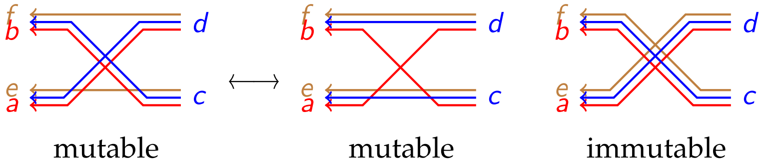
How it was proved

From *The Farnsworth Parabox*, Futurama.

Proof by diagrams



Mutation of the leftmost mutable cross gives a cancellation



Pipe dreams

Now let us restrict to type A.

Theorem (Lernat–X.–Zhong)

For any $w \in S_n$, the class

$$\mathcal{E}_w = \text{ an element specialized from } E_{w\times \mathrm{id}}(u_0)$$

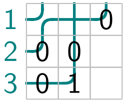
gives a polynomial representative of E_w where $u_0 \in S_{2n}$ satisfies

$$u_0 : i \overset{\text{transposition}}{\longleftrightarrow} n+i, \qquad i = 1, \ldots, n$$

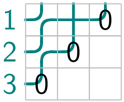
and $w \times \mathrm{id}$ is viewed as an element of S_{2n}.

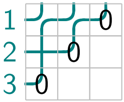
By Billey formula above, $\mathfrak{E}_w$ admits a combinatorial formula.

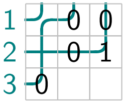
Example

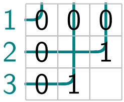


$P(\lambda_1-\lambda_2, y_1-x_1)$	$Q(\lambda_2-\lambda_3, y_2-x_1)$	$P(\lambda_2, y_3-x_1)$
$P(\lambda_2, y_1-x_2)$	$Q(-\lambda_3, y_2-x_2)$	1
$Q(\lambda_3, y_1-x_3)$	$P(\lambda_3-\hbar, y_2-x_3)$	1









$P(\lambda_1, y_1-x_1)$	$Q(-\lambda_3, y_2-x_1)$	$Q(\lambda_2, y_3-x_1)$
$Q(\lambda_2, y_1-x_2)$	$Q(\lambda_2-\lambda_3, y_2-x_2)$	$P(\lambda_1-\hbar, y_3-x_2)$
$Q(\lambda_3, y_1-x_3)$	$P(\lambda_3-\hbar, y_2-x_3)$	1

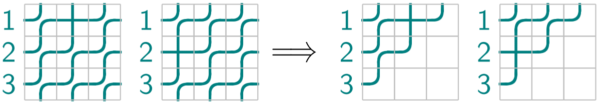
Schubert polynomials

We are inspired by the following well-known trick for Schubert polynomials. It also motivates the definition of matrix Schubert varietiesi.

For $w \in S_n$, we know the **Schubert polynomial**

$$\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

So the specialization $x_i \mapsto y_{n+i}$ does not lose any information, and at the same time can be computed by Billey formula, $\mathfrak{S}_{w \times \mathrm{id}}|_{u_0}$. This coincides with the pipe dream model of Schubert polynomials.



Limit to K-theory

If we (1) substitute $\lambda_{\alpha^\vee} = -s\tau$ for all simple roots α and $0 < s \ll 1$ ($e^{2\pi i} = q$), (2) take limit $q \to 0$, and (3) twist the class by a power of $y = e^{2\pi i \hbar}$, we will get the **Segre motivic class**

$$\mathrm{SMC}_y(\mathring{X}^w) = y^{-\ell(w)} \lim_{q \to 0} E_w \Big|_{\substack{\lambda_{\alpha^\vee} = -s\tau \\ \forall \alpha \in \Sigma}} \in K_T(G/B)[y]_{\mathrm{loc}}.$$

If we further set $y = 0$, we will get the **structure sheaf** (**Grothendieck polynomial**)

$$\mathcal{O}_{X^w} = \mathfrak{G}_w = \mathrm{SMC}_0(\mathring{X}^w) \in K_T(G/B).$$

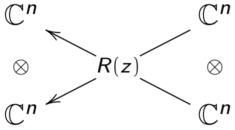
The combinatorial formula behaves well under the limit.

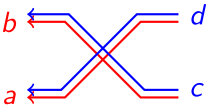
Representation theory

From the three kinds of solutions of Yang–Baxter equations, corresponding to three kinds of quantum groups.

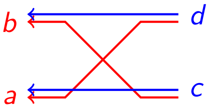
$\mathbb{G}_a = (\mathbb{C}, +)$	$\mathbb{G}_m = (\mathbb{C}^\times, \cdot)$	$\mathbb{C}^\times / q^{\mathbb{Z}}$
rational	trigonometric	elliptic
Yangian	**quantized loop group**	**elliptic quantum group**

Actually, Billey formula in type A (including all parabolic subgroups) could be viewed as an application of R-matrices between two standard representations.





or



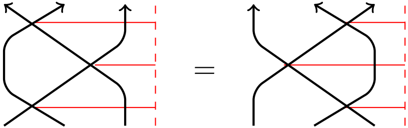
Yang–Baxter equation

That is, we define

$$e_i \otimes e_j \overset{R(z,\lambda)}{\longmapsto} \begin{cases} \mathsf{P}(\lambda_i - \lambda_j, z) e_i \otimes e_j + \mathsf{Q}(\lambda_i - \lambda_j, z) e_j \otimes e_i, & i \neq j, \\ 1, & i = j. \end{cases}$$

They satisfy the **dynamical Yang–Baxter equation**

$$R^{12}(z_1 - z_2, \lambda - \lambda^{(3)} \hbar) R^{13}(z_1 - z_3, \lambda) R^{23}(z_2 - z_3, \lambda - \lambda^{(1)} \hbar)$$
$$= R^{23}(z_2 - z_3, \lambda) R^{13}(z_1 - z_3, \lambda - \lambda^{(2)} \hbar) R^{12}(z_1 - z_2, \lambda).$$



Nakajima quiver varieties

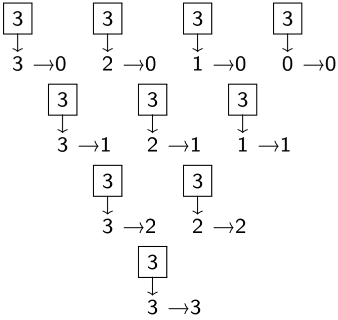
The best explanation might be **Nakajima quiver varieties**. For $G = GL_n$, we have

$$T^* \begin{pmatrix} G/P, \text{ a partial} \\ \text{flag varieties} \end{pmatrix} = \mathfrak{M} \left(\begin{array}{l} \boxed{n} \\ \ \downarrow \\ \bigcirc \to \bigcirc \to \cdots \to \bigcirc \to \bigcirc \end{array} \right).$$

The variety corresponds to a weight space of the GL_m-representation $(\mathbb{C}^m)^{\otimes n}$. Via stable envelopes, we have a geometric construction of R-matrices.

In particular, we can package all parabolic cases together to get a quantum group action, which is a philosophy from Schur–Weyl duality.

Example



$\{0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^3\}$
with $\dim(V_1, V_2) =$

$(0,3) \quad (1,3) \quad (2,3) \quad (3,3)$

$(0,2) \quad (1,2) \quad (2,2)$

$(0,1) \quad (1,1)$

$(0,0)$

THANKS

(modified from a painting in Tianjin Museum)