Chern-Schwartz-MacPherson Classes over Flag Varieties: Pieri Rules, Conjectures, and More arXiv:2211.06802

(Joint work with Neil J.Y. Fan and Peter L.Guo)

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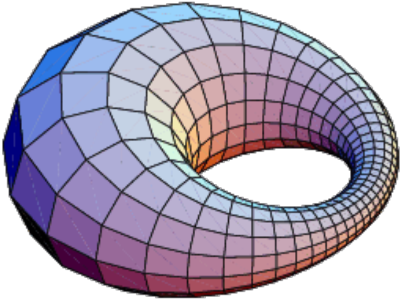
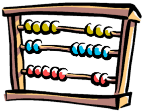
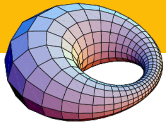
Geometric Background



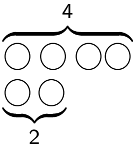
Euler characteristic



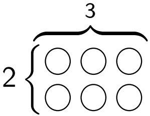


Euler Chatacteristic

Enumerating



$$\begin{array}{r} 4 \\ +\ 2 \\ \hline 6 \end{array}$$



$$\begin{array}{r} 3 \\ \times\ 2 \\ \hline 6 \end{array}$$

$$0 \to H_n(X) \to H_n(U) \to H_{n\text{-}1}(X\backslash U) \to H_{n\text{-}1}(X) \to H_{n\text{-}1}(U) \to \cdots$$

$$\chi(X) = \chi(U) + \chi(X\backslash U)$$

$$\begin{array}{cccc} E^2_{02} & E^2_{12} & E^2_{22} & E^2_{32} \\ E^2_{01} & E^2_{11} & E^2_{21} & E^2_{31} \\ E^2_{00} & E^2_{10} & E^2_{20} & E^2_{30} \end{array}$$

$$\chi(X) = \chi(B) \cdot \chi(F)$$

Functors

There are two **functors** between

$$\mathsf{Variety}_{/\mathbb{C}} \longrightarrow \mathsf{LinearSpace}_{/\mathbb{Q}}.$$

For a complex variety X, denote the space of **constructible functions** by

$$\mathrm{Fun}(X) = \sum_{\substack{A \subset X \\ \text{closed}}} \mathbb{Q} \cdot \mathbb{1}_A.$$

For a proper morphism $f : X \to Y$, we define **push forward**

$$f_*(\mathbb{1}_A)(y) = \chi(A_y).$$

"counting χ" along fibres.

For a complex variety X, the **Borel–Moore homology**

$$H_\bullet(X) \supseteq \sum_{\substack{A \subset X \\ \text{closed}}} \mathbb{Q} \cdot [A].$$

For a proper morphism $f : X \to Y$, the **push forward**

$$f_*(\omega) = \int_f \omega$$

"integral" along fibres.

Chern–Schwartz–MacPherson Classes

For smooth projective variety X, a classical result by Chern relates Euler characteristic and **Chern classes** $\chi(X) = \int_X c(\mathscr{T}_X)$. So we have the following diagram

$$\begin{array}{ccccccc}
\mathbb{1}_X & \in & \mathrm{Fun}(X) & & H_\bullet(X) & \ni & c(\mathscr{T}_X) \frown [X] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\chi(X) & \in & \mathrm{Fun}(\mathrm{pt}) & = & H_\bullet(\mathrm{pt}) & \ni & \int_X c(\mathscr{T}_X)
\end{array}$$

where $\chi(X) = \int_X c(\mathscr{T}_X)$.

Theorem (MacPherson[7], conj. by Grothendieck and Delign)

There is a natural transformation (commuting with push forward)

$$c_{\mathrm{SM}} : \mathrm{Fun} \longrightarrow H_\bullet$$

mapping $\mathbb{1}_X$ to $c(\mathscr{T}_X) \frown [X]$ when X is smooth.

Example

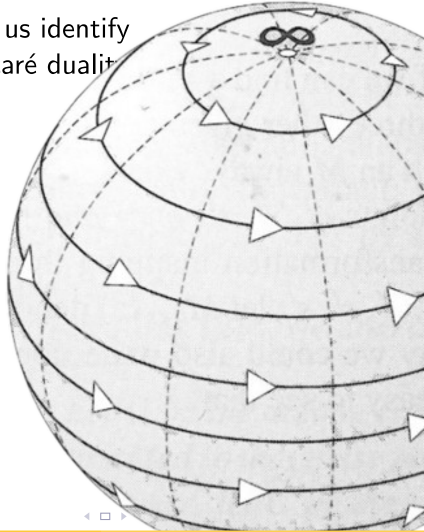
For $W \subset X$ constructible, let us denote $c_{\mathrm{SM}}(W) = c_{\mathrm{SM}}(\mathbb{1}_W)$.

Let us consider the projective line $\mathbb{P}^1$. Let us identify (BM-)homology and cohomology by Poincaré duality

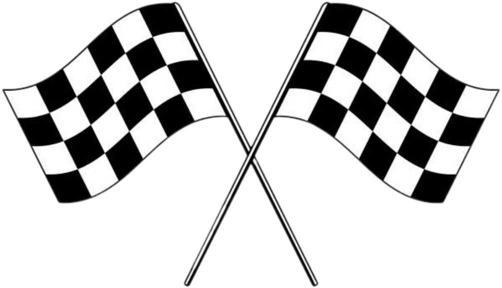
$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$$

$$H^\bullet(\mathbb{P}^1) = \mathbb{Q}[x]/\langle x^2 \rangle.$$

$$\begin{aligned} c_{\mathrm{SM}}(\mathbb{P}^1) &= c(\mathscr{T}_{\mathbb{P}^1}) = 1 + 2x \\ c_{\mathrm{SM}}(\{\infty\}) &= [\infty] = x \\ c_{\mathrm{SM}}(\mathbb{A}^1) &= c_{\mathrm{SM}}(\mathbb{P}^1) - c_{\mathrm{SM}}(\{\infty\}) \\ &= 1 + x. \end{aligned}$$



Flag Varieties



Flag Varieties — Our Best Friends

We will concentrate on the **classical flag variety**

$$\mathcal{F}\ell(n) = \left\{ 0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n \,\middle|\, \dim V_i = i \right\}.$$

We can decompose $\mathcal{F}\ell(n)$ into disjoint union of **Schubert cells**

$$\mathcal{F}\ell(n) = \bigcup_{w \in \mathfrak{S}_n} Y(w)^\circ, \qquad Y(w)^\circ \cong \mathbb{A}^{\dim \mathcal{F}\ell(n) - \ell(w)}.$$

The CSM classes of Schubert cells over flag variety are computed by Aluffi and Mihalcea [1] using the Bott–Samelson resolution. They showed CSM classes can be computed by Demazure–Lusztig operators.

Pieri Rules

The following is our first main result. Recall that the Chern classes of the dual of k-th tautological bundle $c_r(\mathcal{V}_k^\vee) = e_r(x_1, \dots, x_k)$.

Theorem (Fan, Guo and Xiong)

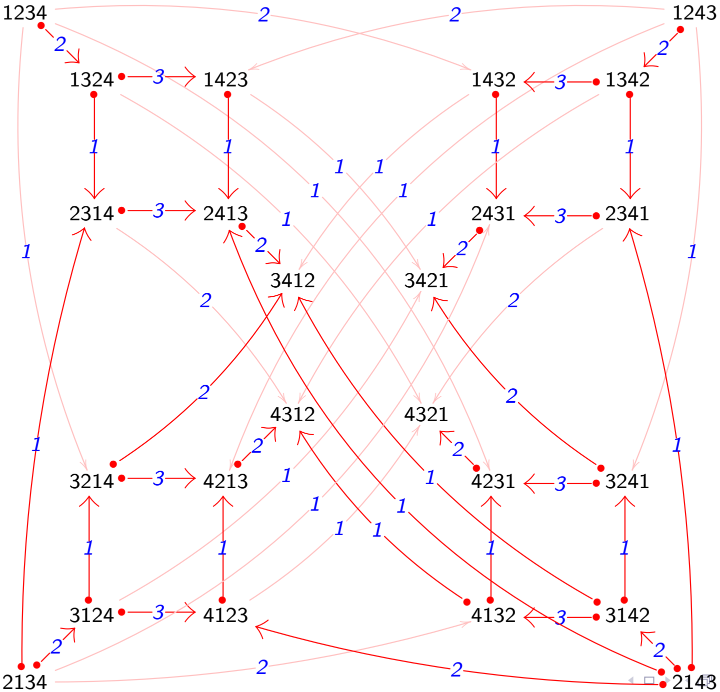
For any permutation $u \in \mathfrak{S}_n$, $1 \le k \le n$, and $r \ge 0$

$$c_{\mathrm{SM}}(Y(u)^\circ) \cdot c_r(\mathcal{V}_k^\vee) = \sum\nolimits_\gamma c_{\mathrm{SM}}\big(Y(\mathrm{end}(\gamma))^\circ\big)$$

with sum over **decreasing path** *γ of length r starting from u in the following diagram:*

$$u \xrightarrow{\tau} w \iff w = ut_{ab} \text{ for some } a \le k < b, \text{ and } \ell(w) \ge \ell(u) + 1; \text{ moreover, } \tau = u(a).$$

Compare with Sottile [16].

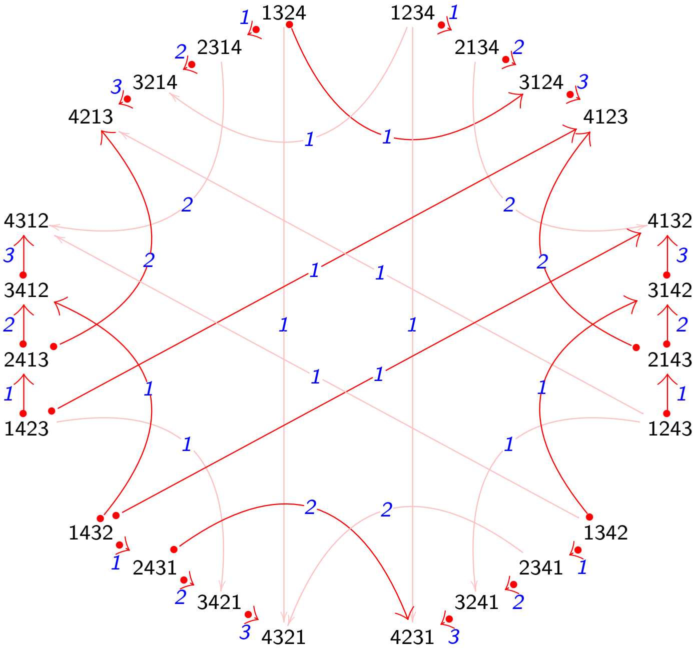


$u \xrightarrow{\tau} w$, $\ell(w)=\ell(u)+1$

$u \xrightarrow{\tau} w$, $\ell(w)>\ell(u)+1$

n=4

k=2

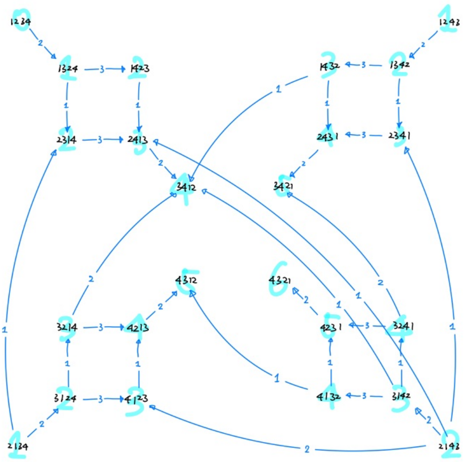


$u \xrightarrow{\tau} w$ (solid red arrow with dot)
$\ell(w)=\ell(u)+1$

$u \xrightarrow{\tau} w$ (light arrow)
$\ell(w)>\ell(u)+1$

$n=4$

$k=1$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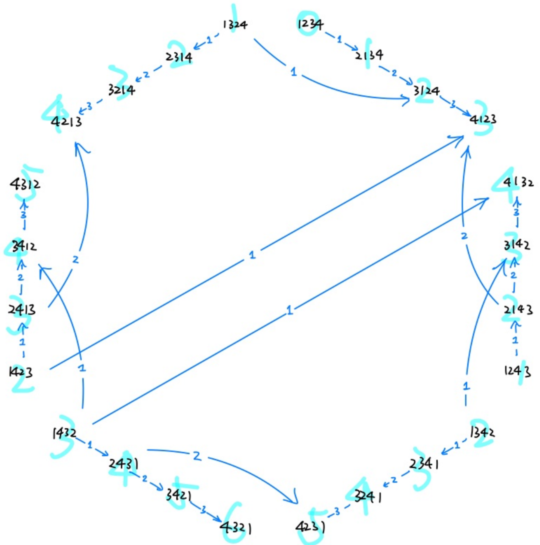
n=4

k=2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n=4

k=1

Equivariant Pieri Rules

We prove a "**rigidity theorem**" which states that the equivariant coefficients are controlled by non-equivariant coefficients. As an application, we achieve the following equivariant Pieri rule.

Theorem (Fan, Guo and Xiong)

For any permutation $u \in \mathfrak{S}_n$, $1 \le k \le n$, and $r \ge 0$

$$c^T_{\mathrm{SM}}(Y(u)^\circ) \cdot c_r(\mathcal{V}_k^\vee) = \sum_\gamma e_{r-\mathsf{length}(\gamma)}(t_{\Delta_k(u,\mathrm{end}(\gamma))}) \cdot c^T_{\mathrm{SM}}\big(Y(\mathrm{end}(\gamma))^\circ\big),$$

with the sum over **decreasing path** *from u. Here*

$$\Delta_k(u,w) = \{u(i) \colon i \in [k]\} \setminus \{u(i) \colon u(i) \neq w(i)\}.$$

Equivariant Murnaghan–Nakayama Rules

We also obtain an equivariant **Murnaghan–Nakayama rule** for CSM classes. Recall

$$p_r(x_{[k]}) = x_1^r + \cdots + x_k^r.$$

Theorem (Fan, Guo and Xiong)

For any permutation $u \in \mathfrak{S}_n$, $0 \le k \le n$ and $r \ge 1$,

$$c_{\mathrm{SM}}^T(Y(u)^\circ) \cdot p_r(x_{[k]}) = p_r(t_{u[k]}) \cdot c_{\mathrm{SM}}^T(Y(u)^\circ) + \sum_w h_{r-r'}(t_{uM(u,w)}) \cdot c_{\mathrm{SM}}^T(Y(w)^\circ)$$

with the sum over all w which can be written as $u\eta$ for an $(r'+1)$-cycle ($r' \le r$) and admit a path from u to $u\eta$. Here

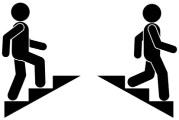
$$M(u,w) = \{i \in [n] : u(i) \neq w(i)\}.$$

Generalization

We proved a more general rule for **Schur polynomials of hook shapes** including Segre classes of tautological bundle.

Chern classes of $\mathcal{V}_k^\vee$

Pieri rule for "e"



Pieri rule for "hook"

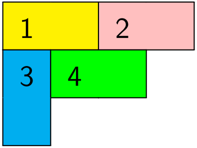
Serge classes of $\mathcal{V}_k$

Pieri rule for "h"

The formulas we obtained are generalization of

- The **Chevalley formulas** for CSM classes by Aluffi, Mihalcea, Schürmann and Su [2]
- The **Schubert Pieri rules** by Sottile [16].
- The **Equivariant Schubert Pieri formula** by Robinson [15], see also Li, Ravikumar, Sottile and Yang [8].
- The **Schubert MN** rule due to Morrison and Sottile [9].

Hook Formulas



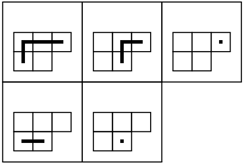
Hook formulas

An application of our formulas is **hook formula**. Classically, the number of **standard Young tableaux** (= dimension of irreducible representation of $\mathfrak{S}_n$) is given by hook length formula

$$\#\left\{ \begin{array}{ccc} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} & \end{array} \right\} = \frac{5!}{4\times 3\times 1\times 2\times 1},$$

where the denominator is the product of all **hook length**.

$$\begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2&1\\\cline{1-2}\end{array} =$$



Naruse Method

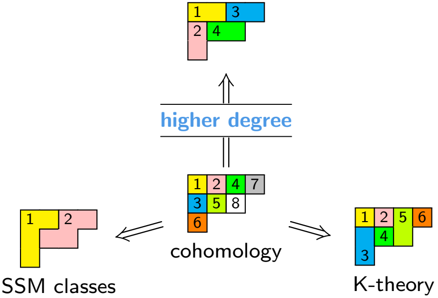
Ikeda and Naruse [14] observed that the classical hook formula is a shadow of **equivariant** Chevalley formula over Grassmannian

$$D \cdot [Y(\lambda)]_T = D|_\lambda \cdot [Y(\lambda)]_T + \sum_{\mu=\lambda+\square} [Y(\mu)]_T.$$

It can be generalized to

- **Classical Types** [13],
- **K-theory** [12],
- **SSM classes** [11].

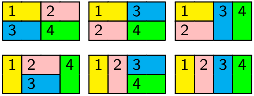
Our generalization goes to another dimension — we replace a divisor D by a higher degree classes.

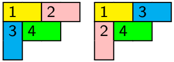


Domino Tableaux

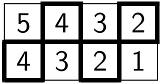
There is a hook formula for **domino tableaux** by Fomin and Lulov [3], which we will illustrate by two examples.

The number of **domino tableaux** (relating to representation theory of $\mathfrak{BC}_n = (\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$) is given by

$\#\{$  $\}$ $= \dfrac{8!!}{4 \times 2 \times 4 \times 2}$.

$\#\{$  $\}$ $= \dfrac{8!!}{6 \times 4 \times 4 \times 2}$.

where the denominator is the product of all **even** hook lengths.



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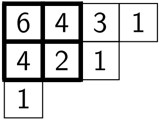

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Our formulation

Actually, we deal with the general cases of **r-rim hook tableaux** (donimo case is when $r = 2$) using equivariant MN rule.

Theorem (Fan, Guo and Xiong)

For a skew shape Λ/λ of size dr, we have the following **Laurant expansion**

$$\left.\frac{[Y(\lambda)]_T|_\Lambda}{[Y(\Lambda)]_T|_\Lambda}\right|_{t_i=z^i} = \frac{1}{(z^r-1)^d}\left(\pm\frac{\#\mathrm{RHT}^r(\Lambda/\lambda)}{r^d d!}+o(1)\right),$$

near a primitive r-th root of unity.

When $\lambda = \varnothing$, by using **Galois theory**, we recover Fomin and Lulov [3, Corollary 2.2].

Positivity Conjectures

Schürmann–Simpson–Wang Theorem

Several authors formulated the following conjecture, Mihalcea [10], Knutson and Paul-Justin [6], Kumar [5].

Conjecture

For any permutations $u, v \in \mathfrak{S}_n$,

$$c_{\mathrm{SM}}(Y(u)^\circ) \cdot c_{\mathrm{SM}}(Y(v)^\circ) = \sum_w \mathbb{Z}_{\geq 0} \cdot c_{\mathrm{SM}}(Y(w)^\circ).$$

Just recently, this conjecture is proved by Schürmann Simpson and Wang [17] using the theory of perverse sheaves.

Precisely, the conjecture should be stated in terms of SSM classes. Over $\mathcal{F}\ell_n$*, non-equivariant CSM and SSM differ by certain sign modification.*

Aluffi–Mihalcea–Schürmann–Su Theorem

From the recrusion formula, we have

$$c_{\mathrm{SM}}(Y(w)^\circ) = [Y(w)] + \sum_{u>w} \mathbb{Z} \cdot [Y(u)].$$

Actually, the coefficients are **non-negative**.

Theorem (P. Aluffi, L. Mihalcea, J. Schürmann and C. Su [2])

For each $w \in \mathfrak{S}_n$, the CSM class $c_{\mathrm{SM}}(Y(w)^\circ)$ is effective, i.e.

$$c_{\mathrm{SM}}(Y(w)^\circ) = [Y(w)] + \sum_{u>w} \mathbb{Z}_{\geq 0} \cdot [Y(u)].$$

The proof uses the theory of $\mathcal{D}$-modules to relate CSM classes with Verma $\mathcal{D}$-modules.

Kumar's Conjecture

It is natural to ask

Conjecture (Kumar [5])

The CSM class of any Richardson cell $c_{SM}(X(u)^\circ \cap Y(v)^\circ)$ is effective, i.e.

$$c_{SM}(X(u) \cap Y(v)^\circ) = \sum_w \mathbb{Z}_{\geq 0} \cdot [Y(w)].$$

We proved a **weaker form** of it (in type A):

Theorem (Fan, Guo and Xiong)

The class $c_{SM}(X(u)^\circ \cap Y(v)^\circ)$ is monomial-positive i.e.

$$c_{SM}(X(u) \cap Y(v)^\circ) = \sum_{\delta \leq (n-1,\cdots,1,0)} \mathbb{Z}_{\geq 0} \cdot x^\delta.$$

Our Conjecture

We actually prove that Kumar's conjecture is equivalent to the following

Conjecture (Fan, Guo and Xiong)

For any permutations $u, v \in \mathfrak{S}_n$,

$$c_{\mathrm{SM}}(Y(u)^\circ) \cdot [Y(v)] = \sum_w \mathbb{Z}_{\geq 0} \cdot c_{\mathrm{SM}}(Y(w)^\circ).$$

We also formulate the equivariant version of this conjecture.

Conjecture (Fan, Guo and Xiong)

For any permutations $u, v \in \mathfrak{S}_n$,

$$c^T_{\mathrm{SM}}(Y(u)^\circ) \cdot [Y(v)]_T = \sum_w \mathbb{Z}_{\geq 0}[\alpha]_{\alpha > 0} \cdot c^T_{\mathrm{SM}}(Y(w)^\circ).$$

Our Pieri rule can be used to check for special v's.

Thanks

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