Automorphisms of the Quantum Cohomology of the Springer Resolution and Applications

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1 Introduction

In this paper, we introduce **quantum Demazure–Lusztig operators** acting by **ring automorphisms** on the equivariant quantum cohomology of the **Springer resolution**. Our main application is a presentation of the torus-equivariant quantum cohomology in terms of generators and relations. We provide explicit descriptions for the classical types. We also recover Kim's earlier results for the complete flag varieties by taking the **Toda limit**.

2 Cohomology of Flag Vareties

Let us restrict to classic **flag varieties**, i.e. $\mathcal{F}\ell_n$ is the variety parameterize **flags** in $\mathbb{C}^n$:

$$0 = \phi_0 \subset \phi_1 \subset \phi_2 \subset \cdots \subset \phi_{n-1} \subset \phi_n = \mathbb{C}^n.$$

A classic result by Borel [1] claims

$$H^\star(\mathcal{F}\ell_n) = \mathbb{Q}[x_1, \ldots, x_n] / \langle e_1(x), \ldots, e_n(x) \rangle$$

where

$$\begin{cases} e_1(x) = x_1 + \cdots + x_n \\ e_2(x) = \sum_{i<j} x_i x_j \\ \quad \cdots = \cdots \\ e_n(x) = x_1 \cdots x_n. \end{cases}$$

3 Quantum cohomology of Flag Varieties

Quantum cohomology is a deformation of usual cohomology encodes the enumeration of curves over algebraic varieties. By Givental and Kim [3],

$$QH^\star(\mathcal{F}\ell_n) = \mathbb{Q}[q][x_1, \ldots, x_n] / \langle e_1^q(x), \ldots, e_n^q(x) \rangle$$

where $e_i(x)$ is the coefficients of the characteristic polynomial of the tridiagonal matrix

$$\begin{bmatrix} x_1 & -1 & 0 & \cdots & 0 \\ \frac{q_1}{q_2} & x_2 & -1 & \cdots & 0 \\ 0 & \frac{q_2}{q_3} & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}.$$

4 Cohomology of Springer Resolutions

The cotangent bundle $T^*\mathcal{F}\ell_n$ is known as the **Springer resolution**. It can be identified as the variety of pairs $(\phi_\bullet, A)$ of flags in $\mathbb{C}^n$ and an n by n nilpotent matrix such that $\phi_\bullet$ is A-invariant.

The cohomology ring of $T^*\mathcal{F}\ell_n$ is nothing different from that of $\mathcal{F}\ell_n$. But we prefer to take the **dilation $\mathbb{C}^*$-action** along the cotangent fibres into consideration. So we consider $\mathbb{C}^*$-equvariant cohomology

$$H^\star_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n) = H^\star(\mathcal{F}\ell_n)[\hbar].$$

5 Quantum Cohomology of Springer resolutions

It turns out the quantum cohomology of $T^*\mathcal{F}\ell_n$ behavior much different from that of $\mathcal{F}\ell_n$. Our main result is the computation of it.

$$QH^\star_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n) = \mathbb{Q}(q)[\hbar, x_1, \ldots, x_n] / \langle \mathcal{E}_1^q(\chi), \ldots, \mathcal{E}_n^q(\chi) \rangle$$

where $\mathcal{E}_i(\chi)$ is the coefficients of the characteristic polynomial of the

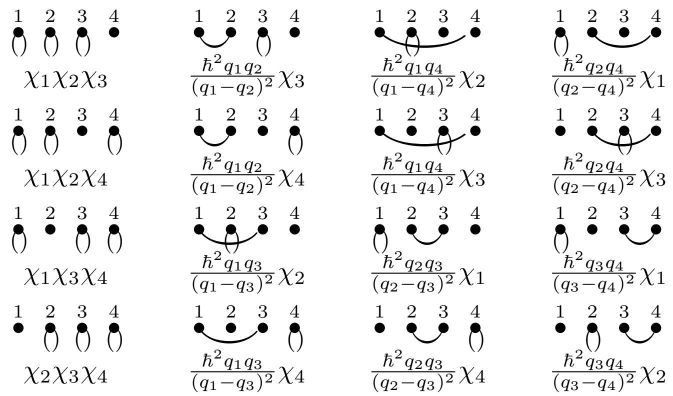
$$\begin{bmatrix} \chi_1 & \frac{\hbar}{1-q_1/q_2} & \frac{\hbar}{1-q_1/q_3} & \cdots & \frac{\hbar}{1-q_1/q_n} \\ \frac{\hbar}{1-q_2/q_1} & \chi_2 & \frac{\hbar}{1-q_2/q_3} & \cdots & \frac{\hbar}{1-q_2/q_n} \\ \frac{\hbar}{1-q_3/q_1} & \frac{\hbar}{1-q_3/q_2} & \chi_3 & \cdots & \frac{\hbar}{1-q_3/q_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\hbar}{1-q_n/q_1} & \frac{\hbar}{1-q_n/q_2} & \frac{\hbar}{1-q_n/q_3} & \cdots & \chi_n \end{bmatrix}$$

where

$$\chi_i = x_i + \hbar \sum_{a<i} \frac{q_a/q_i}{1-q_a/q_i} - \hbar \sum_{i<b} \frac{q_i/q_b}{1-q_i/q_b}$$

6 Combinatorial formula

We give a **combinatorial formula** of $\mathcal{E}_k(\chi)$. For example, if $k = 3$ and $n = 4$



7 Toda Limit

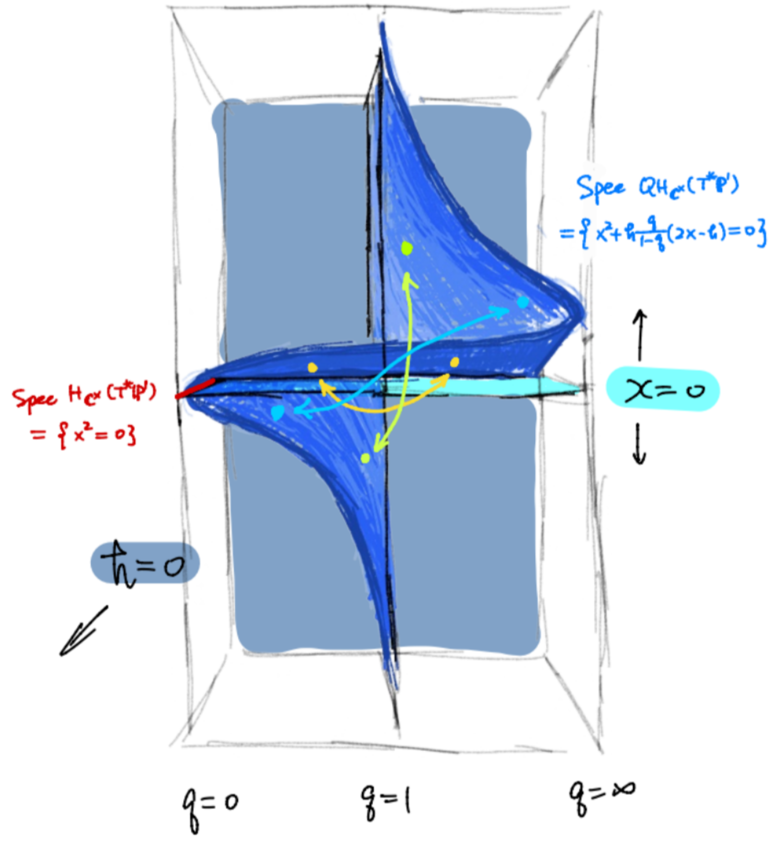
Geometrically, we can recover $QH^\bullet(\mathcal{F}\ell_n)$ by taking the **Toda limit** [2], roughly speaking,

$$QH^\star_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n) \xrightarrow{\hbar \to \infty} QH^\bullet(\mathcal{F}\ell_n).$$

By taking entry-wise limit, we can recover the tridiagonal matrix for $QH^\star(\mathcal{F}\ell_n)$.

8 Quantum Demazure–Lusztig Operators

Our result is proved by finding a new family of **symmetry** over $QH^\star(T^*\mathcal{F}\ell_n)$ which turns out to be **ring automorphisms**. Here is an illustration



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