GEOMETRY OF CHROMATIC SYMMETRIC FUNCTIONS

RUI XIONG

1. Chromatic symmetric functions

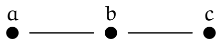
1.1. **Chromatic symmetric functions.** Let G be a graph. We can construct a symmetric function

$$X_G = \sum_{\kappa} \prod_{v \in G} x_{\kappa(v)} \in \Lambda.$$

where the sum goes over proper coloring $\kappa : G \to \{1, 2, 3, \dots\} = [\infty]$, i.e.

$$v\text{—}w \Longrightarrow \kappa(v) \neq \kappa(w).$$

Example. when $G = C_3$



Then

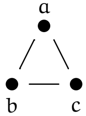
$$X_G = \sum_{a \neq b \neq c} x_a x_b x_c = \left(\sum_{a,b,c \text{distinct}} + \sum_{a=c\neq b} \right) x_a x_b x_c = 6m_{111} + m_{12}.$$

In general, we always have

$$X_G = n! \cdot m_{1^n} + \cdots \qquad n = |G|.$$

Example. When $G = n\{*\}$, we have $X_G = p_1^n$.

Example. When $G = K_n$ a complete graph, we have $X_G = n!e_n$. For example, when $n = 3$,



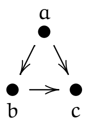
$$X_G = \sum_{a,b,c \text{ distinct}} x_a x_b x_c = 6 \sum_{a<b<c} x_a x_b x_c = 6e_3.$$

Theorem (Stanley). Using the theory of quasi symmetric functions, we have

$$\omega X_G = \sum_{\kappa:G\to[\infty]} \# \left\{ \begin{array}{c} \text{acyclic orientation } O \\ \text{for any} a\text{—}b \\ \kappa(a) < \kappa(b) \Rightarrow a \to b \end{array} \right\} \prod_{\nu\in G} x^{\kappa(\nu)}.$$

We will use this in our second proof.

Example. when $G = K_3$, there are 8 orientations, and 6 of them are acyclic:



and its permutation, $\omega X_G = \sum_{a\leq b\leq c} x_a x_b x_c + (\cdots) = 6h_3.$

1.2. **Frobenius character.** We have Frobenius character

$$\mathbf{Frob} : \bigoplus_{n=0}^{\infty} [\mathrm{Rep}(S_n)] \xrightarrow{\sim} \Lambda.$$

The isomorphism is described in many ways.

In p *basis.* For a character χ of S_n,

$$\mathbf{Frob}(\chi) = \frac{1}{n!} \sum_{w\in S_n} \chi(w) p_{\mathrm{type}(w)} = \sum_{w\in S_n} \frac{1}{z_\lambda} \chi(\lambda) p_\lambda.$$

The notations here:

- For $w \in S$, if the cycle type of w is $1^{m_1} 2^{m_2} \cdots$, then $\mathrm{type}(w)$ is the partition with m_1 many 1's, m_2 many 2's etc.
- For λ with m_1 many 1's, m_2 many 2's etc., $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ which is the number of $w \in S_n$ with $\mathrm{type}(w) = \lambda$.

In h *basis.* The induced representation

$$\mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{tri} \xrightarrow{\mathbf{Frob}} h_\lambda$$

where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \subset S_n$ is the Young subgroup.

In e *basis.* Similarly,

$$\mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{sgn} \xrightarrow{\mathbf{Frob}} e_{\lambda'}.$$

In s *basis.* For the irreducible representation V_λ of S_n, we have

$$\mathbf{Frob}(V_\lambda) = s_\lambda.$$

In m *basis.* For a representation V, we have

$$\mathbf{Frob}(V) = \sum_{\lambda \vdash n} \dim(V^{S_\lambda}) m_\lambda.$$

This formula is very important when computing the Frobenius character.

From the description above,

$$\mathbf{Frob}\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \boxtimes U\right) = \mathbf{Frob}(V)\,\mathbf{Frob}(U).$$

$$\mathbf{Frob}\left(\mathrm{Res}_{S_n \times S_m}^{S_{n+m}} W\right) = \Delta\,\mathbf{Frob}(W) \in \Lambda \otimes \Lambda.$$

$$\dim \mathrm{Hom}_{S_n}(V, U) = \langle \mathbf{Frob}(V), \mathbf{Frob}(U)\rangle$$

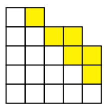
$$\mathbf{Frob}(V \otimes \mathbf{sgn}) = \omega\,\mathbf{Frob}(V).$$

1.3. **Hessenberg variety.** Let $h : [n] \to [n]$ be a function such that

$$i \le h(i), \qquad i \le j \Rightarrow h(i) \le h(j).$$

Such a function is called a **Hessenberg function**. For example,

$$h = (2, 4, 5, 5, 5)$$



Note that Hessenberg functions are in bijection with Dyck path of length n, so the number of them is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $S \in \mathfrak{gl}_n$ be an $n \times n$ matrix. We define **the Hessenberg variety** to be

$$\mathbf{Hess}(S, h) = \{0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n : SV_i \subseteq V_{h(i)}\}.$$

When S is regular semisimple (i.e. with distinct eigenvalues), we can assume S is diagonal, so $\mathbf{Hess}(S, h)$ admits an action of entire torus T, the subgroup of diagonal matrices in GL_n. In this case, $\mathbf{Hess}(S, h)$ is smooth.

Example. When $h(i) = i$, the condition $SV_i \subseteq V_i$ means each V_i is an eigensubspace of S. But the eigensubapces of S are all one-dimensional, so we have $\mathbf{Hess}(S,h) = S_n = n! \cdot \{*\}$.

Example. When $h(i) = n$, we have

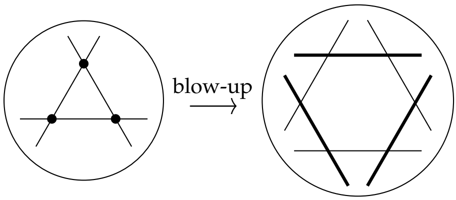
$$\mathbf{Hess}(S,h) = \mathcal{F}\ell_n = \text{full flag variety}.$$

Example. When $h(i) = \min(i+1, n)$, we have

$$\mathbf{Hess}(S,h) = \mathbf{Perm}_n = \text{permutohedral variety}.$$

It is a toric variety whose fan is the Weyl chambers.

Example. In particular, $\mathbf{Perm}_3$ is a del Pezzo surface of degree 6, obtained by blow-up three torus fixed points over $\mathbb{P}^2$.



The moment graph of $\mathbf{Hess}(S,h)$ will be a subgraph of that of flag variety:

$$u \xrightarrow{t_{u(i)}-t_{u(j)}} w \iff w = ut_{ij} \quad i < j \leq h(i).$$

As a result, its cohomology admits the following discription

$$H_T^*(\mathbf{Hess}(S,h)) = \left\{ (\alpha_w)_{w\in W} : \begin{array}{l} \text{each } \alpha_w \in \mathbb{Q}[t_1,\dots,t_n] \\ t_{u(i)} - t_{u(j)} \mid \alpha_w - \alpha_u \text{ for} \\ w = ut_{ij} \text{ with } i < j \leq h(i) \end{array} \right\}.$$

The symmetric group S_n action on it by

$$(\nu \cdot \alpha)_w = \nu\alpha_{\nu^{-1}w}, \qquad \nu \in S_n, \alpha \in H_T^*(\mathbf{Hess}(S,h)).$$

This is called the **Tymoczko left action**. This action reduces to non-equivariant cohomology.

Example. When $h(i) = i$, the action is by permuting the points.

Example. When $h(i) = n$, the action coincides with the induced action from the natural action $S_n \subset GL_n$ on $\mathcal{F}\ell_n$, since GL_n is connected, the action on equivariant cohomology is trivial.

Example. When $h(i) = \min(i+1, n)$, the action coincides the S_n-action induced by the symmetric group action on Weyl chambers.

1.4. **Frobenius character of** $H^*(\mathbf{Hess}_h(S))$**.** For a Hessenberg function h, we define a graph G whose vertices set of $[n]$ and for $i < j$,

$$i \text{—} j \iff j \le h(i).$$

Example.

$h = (2, 4, 5, 5, 5)$ 

Theorem (Brosnan, Chow). We have

$$\omega\, \mathbf{Frob}(H^*(\mathbf{Hess}(S, h))) = X_{G(h)}.$$

For later reference, let us denote $H(h) = H^*(\mathbf{Hess}(S, h))$ for S regular semisimple.

This note is devoted to review the proof of this theorem.

Example. For $h(i) = i$, $G(h) = n\{*\}$. We know from the previous examples that

$$\mathbf{Hess}(S, h) = n!\{*\}, \qquad H^*(n!\{*\}) \cong \mathbb{Q}[S_n].$$

Recall $G(h) = n\{*\}$. We have

$$X_{G(h)} = p_1^n = \omega\, \mathbf{Frob}(H^*(\mathbf{Hess}(S, h))).$$

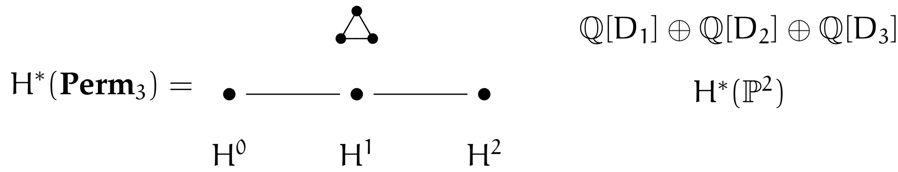
Example. For $h(i) = n$, $G(h) = K_n$. We know from the previous examples that

$$\mathbf{Hess}(S, h) = \mathbf{F}\ell_n, \qquad H^*(\mathbf{F}\ell_n) \cong n! \cdot \mathbf{tri}.$$

Recall $G(h) = K_n$. We have

$$X_{G(h)} = n! \cdot h_n = \omega\, \mathbf{Frob}(H^*(\mathbf{Hess}(S, h))).$$

Example. For $h(i) = \min(i, n)$, $G(h) = C_n$. When $n = 3$, we know



for three exceptional divisors D_1, D_2, D_3. The symmetric group S_3 acts trivially on $H^*(\mathbb{P}^2)$ and permuting exceptional divisors. So

$$H^*(\mathbf{Perm}_3) = 3\mathbf{tri} + \mathbf{tri} \uparrow_{S_1 \times S_2}^{S_3} .$$

That is,

$$\mathbf{Frob}(H^*(\mathbf{Perm}_3)) = 3h_3 + h_2 h_1.$$

Then

$$\omega\, \mathbf{Frob}(H^*(\mathbf{Perm}_3)) = 3e_3 + e_2 e_1 = 6m_{111} + m_{12}.$$

Remark. There is a q-analogy of chromatic symmetric functions, computing the graded Frobenius character.

2. GEOMETRIC PREPARATION

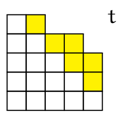
2.1. **Monodromy.** Let us consider the **universal Hessenberg variety**

$$\mathfrak{H}(h) = \{(S, V_\bullet) : S \in \mathfrak{gl}_n,\ V_\bullet \in \mathbf{Hess}(S, h)\}.$$

Then we have

$$\begin{array}{ccc} \mathfrak{H}(h) & \xrightarrow{\rho} & \mathcal{F}\ell_n \\ \downarrow{\scriptstyle \pi} & & \\ \mathfrak{gl}_n & & \end{array}$$

Note that ρ is a GL_n-equivariant vector bundle over F_n with fibre $H = \bigoplus_{1 \le j \le h(i)} \mathbb{C} \cdot E_{ji}$ at the base point. For example,

$h = (2, 4, 5, 5, 5)$ $H =$  $\subset \mathfrak{gl}_n$.

Note that

$$\begin{array}{ccc} \pi^{-1}(\mathfrak{gl}_n^{rs}) & \subset & \mathfrak{H}(h) \\ \downarrow & & \downarrow \pi \\ \mathfrak{gl}_n^{rs} & \subset & \mathfrak{gl}_n \end{array}$$

is smooth, so there is a fundamental group action on cohomology of fibre, i.e. monodromy.

When $h(i) = i$, we usually denote it by $\mathfrak{H}(h) = \widetilde{\mathfrak{gl}}_n$, and $\widetilde{\mathfrak{gl}}_n^{rs} \to \mathfrak{gl}_n$ forms an S_n-Galois covering. We have

$$\pi^{-1}(S) = \{\text{flag of eigen-subspaces of } S\}.$$

When $S \in \mathfrak{gl}_n^{rs}$, the fiber is just an ordering of one-dimensional eigen-subspaces of S, and the monodromy action factors through S_n and it coincides with the permutation action. In particular, for diagonal $S \in \mathfrak{gl}_n^{rs}$, the coordinate subspaces are eigen-subspaces, so the fibre is naturally identified with S_n.

More general, at each $S \in \mathfrak{gl}_n^{rs}$, we have a maximal torus

$$T_S = C_G(S).$$

We have

$$\begin{array}{ccc} \mathbf{Hess}(S, h)^{T_S} = \pi^{-1}(S)^{T_S} & \longrightarrow & \pi^{-1}(\mathfrak{gl}_n^{rs}) \\ & \searrow & \downarrow \\ & & \mathfrak{gl}_n^{rs} \end{array}$$

is the S_n-Galois covering just mentioned. Since

$$H^*_{T_S}(\mathbf{Hess}(S, h)) \longrightarrow H^*_{T_S}(\mathbf{Hess}(S, h)^{T_S})$$

is injective, the monodromy action factor through S_n, and is given by the Tymoczko left action.

2.2. **Springer theory.** Let $\mathcal{N}$ be the nilpotent cone

$$\mathcal{N}_n = \{S \in \mathfrak{gl}_n : S^n = 0\}$$

and $\tilde{\mathcal{N}}_n$ the **Springer resolution**

$$\tilde{\mathcal{N}}_n = \{(V_\bullet, S) \in \mathcal{F}\ell_n \times \mathcal{N} : SV_i \subseteq V_{i-1}\}.$$

We have the following diagram

$$\begin{array}{ccccccc} & & \mathcal{F}\ell_n \times \mathfrak{gl}_n & \overset{\text{Fourier}}{\longleftrightarrow} & \mathcal{F}\ell_n \times \mathfrak{gl}_n & & \\ \tilde{\mathcal{N}}_n & \hookrightarrow & \downarrow \pi & & \downarrow \pi & \hookleftarrow & \widetilde{\mathfrak{gl}}_n \\ & & \mathfrak{gl}_n & \overset{\text{Fourier}}{\longleftrightarrow} & \mathfrak{gl}_n & & \end{array}$$

Here we identify $\mathfrak{gl}_n^*$ and $\mathfrak{gl}_n$ by the trace pairing. Since $\tilde{\mathcal{N}}_n = \widetilde{\mathfrak{gl}}_n^{\perp}$, we have

$$\begin{array}{ccc} \mathbb{Q}_{\tilde{\mathcal{N}}_n} & \overset{\mathscr{F}}{\longmapsto} & \mathbb{Q}_{\widetilde{\mathfrak{gl}}_n} \\ \pi_* \downarrow & & \downarrow \pi_* \\ \pi_*\mathbb{Q}_{\tilde{\mathcal{N}}_n} & \overset{\mathscr{F}}{\longmapsto} & \mathbf{IC}(\mathfrak{gl}_n^{rs}, \mathbb{Q}[S_n]). \end{array}$$

We have isomorphisms

$$S_n \cong \operatorname{End}(\pi_*\mathbb{Q}_{\tilde{\mathbf{N}}_n}) \overset{\mathscr{F}}{\cong} \operatorname{End}(\mathbf{IC}(\mathfrak{gl}_n^{rs}, \mathbb{Q}[S_n])) \cong S_n$$

which twists by a sign $w \mapsto (-1)^{\ell(w)} w$. Recall V_λ is the irreducible S_n-representation. Then $V_\lambda \mapsto V_{\lambda'}$ under this twisting. By decomposition theorem

- On the left-hand-side, we have

$$\pi_*\mathbb{Q}_{\tilde{\mathcal{N}}_n} = \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathbf{IC}(\mathbb{O}_\lambda),$$

  where $\mathbb{O}_\lambda$ is a nilpotent orbit of Jordan type λ.
- On the right-hand-side

$$\pi_*\mathbb{Q}_{\widetilde{\mathfrak{gl}}_n} = \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathbf{IC}(\mathfrak{gl}_n^{rs}, V_\lambda),$$

  where V_λ are viewed as a local system over $\mathfrak{gl}_n^{rm}$.

This proves

$$\mathbf{IC}(\mathbb{O}_\lambda) \overset{\mathscr{F}}{\longmapsto} \mathbf{IC}(\mathfrak{gl}_n^{rs}, \mathbf{sgn} \otimes V_\lambda).$$

2.3. Sheaf theoretic formulation. Let us turn to our $\mathfrak{H}(h)$. We similarly have

$$\begin{array}{ccccccc} \tilde{\mathcal{N}}(h) & \hookrightarrow & \mathcal{F}\ell_n \times \mathfrak{gl}_n & \overset{\text{Fourier}}{\longleftrightarrow} & \mathcal{F}\ell_n \times \mathfrak{gl}_n & \hookleftarrow & \mathfrak{H}(h) \\ & & \downarrow \pi & & \downarrow \pi & & \\ & & \mathfrak{gl}_n & \overset{\text{Fourier}}{\longleftrightarrow} & \mathfrak{gl}_n & & \end{array}$$

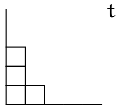
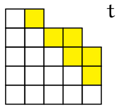
where

$$\tilde{\mathcal{N}}(h) = \{(V_\bullet, S) : SV_j \subseteq V_{g(j)}\},$$

where $g(i) = \max\{j < i : h(j) < i\}$. That is,

$$h(j) < i \iff j \leq g(i).$$

Note that $\tilde{\mathcal{N}}(h)$ is a GL_n-vector bundle over $\mathcal{F}\ell_n$ with fibre $H^\perp = \bigoplus_{j \leq g(i)} \mathbb{C}E_{ji}$ at the base point. For example,

$H^\perp =$  $\subset H =$  $\subset \mathfrak{gl}_n.$

Using Fourier transformation, we have

$$\begin{array}{ccc} \mathbb{Q}_{\tilde{\mathcal{N}}(h)} & \overset{\mathscr{F}}{\longmapsto} & \mathbb{Q}_{\mathfrak{H}(h)} \\ \pi_* \downarrow & & \downarrow \pi_* \\ \pi_* \mathbb{Q}_{\tilde{\mathcal{N}}(h)} & \overset{\mathscr{F}}{\longmapsto} & \mathbf{IC}(\mathfrak{gl}_n^{rs}, H(h)). \end{array}$$

Recall that $H(h) = H^*(\mathbf{Hess}(S, h))$.

Note that a priori, $\pi_* \mathbb{Q}_{\mathfrak{H}(h)}$ would have summand of IC sheaves supported on lower stratum. Since we have in type A, the left-hand side $\pi_* \mathbb{Q}_{\tilde{\mathcal{N}}(h)}$ contains only $\mathbf{IC}(\mathbb{O}_\lambda)$. So by Fourier transform, no other IC sheaves appears in $\pi_* \mathbb{Q}_{\mathfrak{H}(h)}$. It was shown that this is also true for all types.

Remark. The variety $\mathcal{N}(h)$ is used to give a geometric definition of the Catalan function in MacDonald theory. It is not clear to the author what is the precise relation between these two pictures.

3. Proof from the Springer side

3.1. **Topological part.** For $\mu \in \mathbb{O}_\mu$,

$$\pi_* \mathbb{Q}_{\tilde{\mathbf{N}}_n}|_\lambda = H^*(\text{Springer fiber}) \cong \operatorname{Ind}_{S_\mu}^{S_n} \mathbf{tri}.$$

As a result,

$$\mathbf{IC}(\mathbb{O}_\lambda)|_\mu = \operatorname{Hom}_{S_n}(V_\lambda, \operatorname{Ind}_{S_\mu}^{S_n} \mathbf{tri}) = V_\lambda^{S_\mu}.$$

Thus we have

$$\begin{aligned}
\mathbf{IC}(\mathfrak{gl}_n^{rs}, H(h)) &= \sum_{\lambda \vdash n} \operatorname{Hom}_{S_n}(V_\lambda, H(h)) \otimes \mathbf{IC}(\mathfrak{gl}_n^{rs}, V_\lambda) \\
&\overset{\mathscr{F}}{\longmapsto} \sum_{\lambda \vdash n} \operatorname{Hom}_{S_n}(V_\lambda, \mathbf{sign} \otimes H(h)) \otimes \mathbf{IC}(\mathbb{O}_\lambda) \\
&\overset{-|_\mu}{\longmapsto} \sum_{\lambda \vdash n} \operatorname{Hom}_{S_n}(V_\lambda, \mathbf{sign} \otimes H(h)) \otimes V_\lambda^{S_\mu} = (\mathbf{sign} \otimes H_h)^{S_\mu}.
\end{aligned}$$

This proves

$$(\mathbf{sign} \otimes H(h))^{S_\mu} = \pi_* \mathbb{Q}_{\tilde{\mathcal{N}}(h)}|_\mu$$

which is the cohomology group of the fiber $\tilde{\mathcal{N}}(\mu, h)$ of $\tilde{\mathcal{N}}(h)$ at μ.

Let us choose the standard form of μ.

$$\mu = \operatorname{diag}(J_{\mu_1}, J_{\mu_2}, \cdots), \qquad J_k = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdots\cdots & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}_k.$$

Let us denote the torus

$$\mathbb{C}_\rho^\times = \{\rho(t) : t \in \mathbb{C}^\times\} \subset GL_n, \qquad \rho(t) = \operatorname{diag}(t^{n-1}, t^{n-2}, \cdots, t, 1).$$

It is not hard to check

$$\rho(t) \cdot \mu \cdot \rho(t)^{-1} = t \cdot \mu,$$

so $\mathbb{C}_\rho^\times$ acts on $\tilde{\mathcal{N}}(\mu, h)$. Since $\mathcal{F}\ell_n^{\mathbb{C}_\rho^\times} = S_n$, we can conclude

$$\tilde{\mathcal{N}}(\mu, h)^{\mathbb{C}_\rho^\times} = S_n \cap \tilde{\mathcal{N}}(\mu, h) = \{w \in S_n : w \in \tilde{\mathcal{N}}(\mu, h)\} =: S_n(\mu, h).$$

In particular,

$$\dim H^*(\tilde{\mathcal{N}}(\mu, h)) = \# S_n(\mu, h).$$

3.2. **Combinatorial part.** Let us enumerate $S_n(\mu, h)$. Assume $\mu = (\mu_1, \dots, \mu_k)$ has k parts. Then μ defines a partition of $[n]$ into k parts

$$[n] = A_1 \sqcup \cdots \sqcup A_k, \qquad \# A_i = \mu_i, \qquad A_1 < \cdots < A_k.$$

For $i \in [n]$, we denote $\mathbf{ind}(i)$ to be the index j such that $i \in A_j$. For a permutation $w \in S_n$, we view it as a color on G. We shall view μ as a map

$$\mu : [n] \to [n] \cup \{0\}, \qquad i \mapsto \begin{cases} 0, & i = \mu_1, \mu_1 + \mu_2, \dots, n \\ i+1, & \text{otherwise} \end{cases}$$

That is, it is mapped to $i+1$ if $\mathbf{ind}(i) = \mathbf{ind}(i+1)$ and to 0 otherwise.

Then for $w \in S_n(\mu, h)$, we need to require for any a, b,

$$\begin{matrix} w(b) = w(a) + 1 \\ \mathbf{ind}(w(a)) = \mathbf{ind}(w(b)) \end{matrix} \Rightarrow b \leq g(a),$$

i.e. $h(b) < a$ i.e.

$$b < a, \qquad b \not\leftrightarrow a.$$

Then

$$\kappa : G(h) \to [k], \qquad a \mapsto \mathbf{ind}(w(a))$$

gives an element in K, where

$$K = \left\{ \begin{matrix} \text{proper coloring } G(h) \to [k] \\ \text{with each color } i \text{ used } \mu_i \text{ times} \end{matrix} \right\}.$$

Conversely, every element $\kappa \in K$ gives a unique element $w \in S_n(\mu, h)$ such that

$$w(a) \in A_{\kappa(a)}, \qquad a < b, \kappa(a) = \kappa(b) \Rightarrow w(a) > w(b).$$

Note that

$$\#K = [x^\mu]X_G = [m_\mu]X_G = \langle h_\mu, X_G \rangle.$$

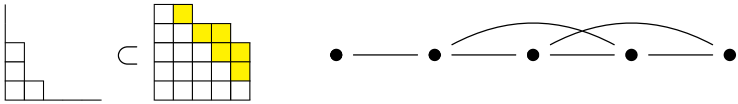
So,

$$\dim(\mathbf{sign} \otimes H_h)^{S_\mu} = [m_\mu]X_G.$$

This proves

$$\omega\, \mathbf{Frob}(H^*(\mathbf{Hess}(S, h))) = X_{G(h)}.$$

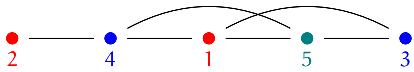
Example. Let us give an example. Take $h = (2,4,5,5,5)$:



and $\mu = (2,2,1)$:

$$1 \longmapsto 2 \qquad 3 \longmapsto 4 \qquad 5$$

Here is an example



V_i	$\mu \cdot V_i$	$V_{g(i)}$
2	∅	∅
2 4	∅ ∅	∅
2 4 1	∅ ∅ 2	2
2 4 1 5	∅ ∅ 2 ∅	2
2 4 1 5 3	∅ ∅ 2 ∅ 4	2 4

4. Proof from the Galois side

4.1. **Topological part.** Let us restrict to $\mathfrak{gl}_n^r$. Let us denote $\mathfrak{t}$ the subspace of diagonal matrices. We have the following diagram

$$\begin{array}{ccccc} \mathfrak{t} & \longleftarrow & \widetilde{\mathfrak{gl}}_n^r & \subset & \widetilde{\mathfrak{gl}}_n \\ {\scriptstyle p}\downarrow & \text{pull} & \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{t}/S_n & \overset{f}{\longleftarrow} & \mathfrak{gl}_n^r & \subset & \mathfrak{gl}_n. \end{array}$$

So

$$\pi_* \mathbb{Q}_{\mathfrak{gl}_n}|_{\mathfrak{gl}_n^r} = \pi_* \mathbb{Q}_{\mathfrak{gl}_n^r} = f^* p_* \mathbb{Q}_{\mathfrak{t}}.$$

For $x \in \mathfrak{t}$, the reduced fiber of p at $\bar{x}$ is naturally identifies with the S_n orbit of x. So for any regular element $x \in \mathfrak{gl}_n^r$ of type μ, we have

$$\pi_* \mathbb{Q}_{\mathfrak{gl}_n}|_x = \mathbb{C}[S_n \cdot x] = \mathrm{Ind}_{S_\mu}^{S_n} \mathbf{tri}.$$

Let us denote $-|_\mu$ for $-|_x$ for any regular element of type μ. Then

$$\mathbf{IC}(\mathfrak{gl}_n^{rs}, V_\lambda)|_\mu = \mathrm{Hom}_{S_n}(V_\lambda, \mathrm{Ind}_{S_\mu}^{S_n} \mathbf{tri}) = V_\lambda^{S_\mu}.$$

Just similar as the discussion on the Springer side, we have

$$H(h)^{S_\mu} = \pi_* \mathbb{Q}_{\mathfrak{H}(h)}|_\mu$$

which is the cohomology group of the fiber $\mathbf{Hess}(x, h)$ of $\mathfrak{H}(h)$ at $x \in \mathfrak{gl}_n^r$ of type μ. Let us expand the definition. We have

$$H^*(\mathbf{Hess}(S, h))^{S_\mu} = H^*(\mathbf{Hess}(x, h))$$

where S is a regular semisimple element and x is a regular element of type μ. In particular, $H^*(\mathbf{Hess}(x, h))$ satisfies Poincaré duality.

Now let us study $\mathbf{Hess}(x, h)$. We can take

$$x = \operatorname{diag}(s_1 I_{\mu_1} + J_{\mu_1}, s_2 I_{\mu_2} + J_{\mu_2}, \ldots)$$

for distinct $s_1, s_2, \ldots \neq 0$. Let us consider the torus generated by

$$\mathbb{C}_s^\times = \operatorname{diag}(s_1 I_{\mu_1}, s_2 I_{\mu_2}, \ldots).$$

Then

$$\mathbf{Hess}(x, h)^{\mathbb{C}_s^\times} = \mathbf{Hess}(\mu, h)^{\mathbb{C}_s^\times}.$$

Similar as the discussion in the Springer side, we have

$$\dim H^*(\mathbf{Hess}(x, h)) = \dim H^*(\mathbf{Hess}(\mu, h)) = \# S_n'(\mu, h),$$

where

$$S_n'(\mu, h) = \{w \in S_n : w \in \mathbf{Hess}(\mu, h)\}.$$

4.2. **Combinatorial part.** Let us enumerate $S_n'(\mu, h)$. Let us define

$$S_n \to \left\{ (O, \kappa) : \begin{array}{l} O \text{ is an acyclic orientation of } G(h) \\ \kappa \text{ is a weakly increasing } G(h) \to [k] \\ \text{with each color } i \text{ used } \mu_i \text{ times} \end{array} \right\} =: K'$$

by $w \mapsto (O, \kappa)$. Here O is the orientation

$$a \longrightarrow b \iff w(a) < w(b)$$

and κ is the coloring

$$\kappa : G(h) \to [k], \qquad a \mapsto \mathbf{ind}(w(a)).$$

From the construction, it is obvious that κ is weakly increasing

$$a \longrightarrow b \Rightarrow \kappa(a) \leq \kappa(b).$$

This map is many-to-one.

For a pair (O, κ) as above, we consider $G_c = (\kappa^{-1}(c), \leq_O)$ the subgraph colored by c. We find the maximal number c_1 among the minimal element $\min(G_c)$. Then we find the maximal number i_2 among the minimal element $\min(G_C \setminus \{c_1\})$ etc. We enumerate elements

$$G_c = \{c_1, c_2, \cdots\}.$$

We define $(O, \kappa) \mapsto w$ such that

$$\kappa(a) < \kappa(b) \Rightarrow w(a) < w(b), \qquad w(c_1) < w(c_2) < \cdots \text{ for any color } c.$$

This defines a section of the map, i.e. $(O, \kappa) \mapsto w \mapsto (O, \kappa)$. Note that $w \mapsto (O, \kappa) \mapsto w'$ is not the identity, and $w' = wv$ for some $v \in S_\mu$. The condition of the image, i.e. the condition for $w = w'$, can be described

$$\begin{matrix} w(b) = w(a) + 1 \\ \kappa(b) = \kappa(a) \end{matrix} \Rightarrow \text{either } b < a \text{ or } a \to b.$$

That is, $b \leq h(a)$. This is exactly the condition for $w \in S_n'(\mu, h)$. Similarly, we have

$$\#K' = [m_\mu](\omega X_G).$$

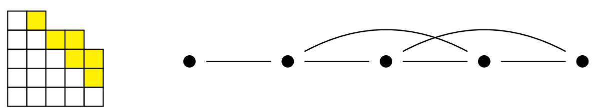
As a result,

$$\dim H(h)^{S_\mu} = \# S_n'(\mu, h) = |K'|.$$

This proves

$$\omega\, \mathbf{Frob}(H^*(\mathbf{Hess}(S, h))) = X_{G(h)}.$$

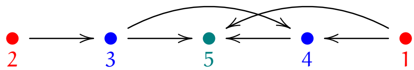
Example. Let us give an example. Take $h = (2, 4, 5, 5, 5)$:



and $\mu = (2, 2, 1)$:

$$1 \longmapsto 2 \qquad 3 \longmapsto 4 \qquad 5.$$

Here is an example



V_i	$\mu \cdot V_i$	$V_{h(i)}$
2	∅	2 3
2 3	∅ 4	2 3 5 4
2 3 5	∅ 4 ∅	2 3 5 4 1
2 3 5 4	∅ 4 ∅ ∅	2 3 5 4 1
2 3 5 4 1	∅ 4 ∅ ∅ 2	2 3 5 4 1

5. Appendix

5.1. **Symmetric functions.** Let $\Lambda = \varprojlim \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ be the ring of symmetric functions.

- Recall the **monomoical symmetric function** for a partition λ is
$$m_\lambda = \sum_{\alpha \in S_n \lambda} x^\alpha = \frac{1}{|S_\lambda|} \sum_{w \in S_n} x^{w\lambda}.$$
- Recall the **elementary symmetric function** for a partition λ
$$e_\lambda = e_{\lambda'_1} e_{\lambda'_2} \cdots, \qquad e_r = \sum_{1 \le i_1 < i_2 < \cdots} x_{i_1} x_{i_2} \cdots$$
where λ' is the conjugation of λ. Another way of determining e_r is
$$\sum_{r=0}^{\infty} t^r e_r = \prod_{i=0}^{\infty} (1 + t x_i).$$
- Recall the **homogeneous symmetric function** for a partition λ
$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots, \qquad h_r = \sum_{1 \le i_1 \le i_2 \le \cdots} x_{i_1} x_{i_2} \cdots .$$
Another way of determining h_r is
$$\sum_{r=0}^{\infty} t^r h_r = \prod_{i=0}^{\infty} \frac{1}{1 - t x_i}.$$

- Recall the **power symmetric function** for a partition λ of ℓ rows

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}, \qquad p_r = x_1^r + x_2^r + \cdots .$$

- We denote

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T$$

the **Schur function** for a partition λ.

We have a Hall inner product $\langle , \rangle$ whose kernel is

$$\Omega = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda \frac{1}{z_\lambda !} p_\lambda(x) p_\lambda(y).$$

We have an ω-involution, which is the ring automorphism

$$h_\lambda \leftrightarrow e_{\lambda'} \quad (e_r \leftrightarrow h_r), \qquad p_r \leftrightarrow (-1)^r p_r, \qquad s_\lambda \leftrightarrow s_{\lambda'}.$$

5.2. **Quasi-symmetric functions.** We say a polynomial

$$f \in \mathbb{Q}[x_1, \ldots, x_n]$$

is quasi-symmetric if for all $a_1, \ldots, a_k$ the coefficient in f of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ equals the coefficient of $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. We denote

$$\mathrm{QSym} = \varprojlim \mathbb{Q}[x_1, \ldots, x_n]^{\text{Quasi-sym}}.$$

In stead of using partition, we will use **strong composition**, i.e. $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ for positive integers $\alpha_i > 0$. We define $|\alpha| := \alpha_1 + \cdots \alpha_\ell$ and $\ell(\alpha) := \ell$. We write $\beta \models \alpha$ if β refines α. We can illustrate a strong composition by $|\alpha|$ balls and $\ell - 1$ bars:

$$(4,1,3,1) \qquad \overset{1}{\bigcirc}\overset{2}{\bigcirc}\overset{3}{\bigcirc}\overset{4}{\bigcirc}\,|\,\overset{5}{\bigcirc}\,|\,\overset{6}{\bigcirc}\overset{7}{\bigcirc}\overset{8}{\bigcirc}\,|\,\overset{9}{\bigcirc}$$

From this, we see the strong composition is in bijection with pairs (n, S) for $S \subset [n-1]$. For example,

$$(4,1,3,1) \qquad (9, \{4,5,8\}).$$

Monomial. We define the **monomial quasi-symmetric function** for a strong composition α

$$M_\alpha = \sum_{\beta^+ = \alpha} x^\beta$$

with the sum over all compositions β and β^+ is obtained by deleting 0's in β. It could be viewed as generating function of the filling of α, such that

$$\textcircled{a}\textcircled{b} \Rightarrow a = b, \qquad \textcircled{a}|\textcircled{b} \Rightarrow a < b.$$

For example,

$$M_{4131} = \sum_{a<b<c<d} x_a^4 x_b x_c^3 x_d.$$

There is an explicit rule for multiplying M_α and M_β, so in particular, QSym is a ring.

Fundamental. We define the **fundamental (Gessel) quasi-symmetric function** for a strong composition α

$$F_\alpha = \sum_{\beta \models \alpha} M_\beta$$

with the sum over strong compositions β and $\beta \models \alpha$ means β is a refine of α. It could be viewed as generating function of the filling of α, such that

$$\textcircled{a}\textcircled{b} \Rightarrow a \leq b, \qquad \textcircled{a}|\textcircled{b} \Rightarrow a < b.$$

For example

$$F_{4131} = \sum_{i_1 \leq i_2 \leq i_3 \leq i_4 < i_5 < i_6 \leq i_7 \leq i_8 < i_9} x_{i_1} \cdots x_{i_9}.$$

Moreover, we have

$$F_{1^n} = e_n, \qquad F_n = h_n.$$

We also denote

$$F_{n,S} = F_\alpha = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \\ a \in S \Rightarrow i_a < i_{a+1}}} x_{i_1} \cdots x_{i_n}$$

for (n, S) corresponding to α.

Coproduct and involution. We have coproduct

$$\Delta : \mathrm{QSym} \to \mathrm{QSym}$$

by using new alphabet

$$x_1 \otimes 1 < x_2 \otimes 1 < \cdots < 1 \otimes x_1 < 1 \otimes x_2 < \cdots .$$

We have an ω-involution by

$$\omega(F_\alpha) = F_{\alpha'}$$

where α' is the dual composition obtained by

◯|◯ ⟷ ◯◯.

For example $(4,1,3,1)' = (1,1,1,3,1,2)$. Compare:

$(4,1,3,1)$	◯ ◯ ◯ ◯|◯|◯ ◯ ◯|◯
$(1,1,1,3,1,2)$	◯|◯|◯|◯ ◯ ◯|◯|◯ ◯

In terms of monomial quasi-symmetric functions, we have

$$\Delta M_\alpha = \sum_{k=0}^{\ell(\alpha)} M_{\alpha_{\le k}} \otimes M_{>k}, \qquad \omega(M_\alpha) = (-1)^{\ell(\alpha)} \sum_{\alpha \models \beta} M_\beta .$$

We define antipode by $S(M_\alpha) = \omega(M_{\mathrm{rev}(\alpha)})$. These equip QSym a structure of a Hopf algebra. Actually the dual of QSym is the so-called **non-commutative symmetric functions**. Note that the coproduct is not commutative. The natural embedding is a Hopf algebras homomorphism

$$\Lambda \xrightarrow{\subseteq} \mathrm{QSym} .$$

That is, it commutes with coproduct and antipode. Since

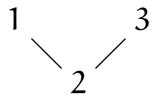
$$m_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} M_\alpha = \sum_{\mathrm{sort}(\alpha)=\lambda} M_{\mathrm{rev}(\alpha)},$$

the inclusion also commutes with the omega involution.

Example. Assume we have a partial order P on $[n]$. We call $T : P \to [\infty]$ a P-partition if

$$\begin{aligned} a <_P b, a > b \Longrightarrow \quad & T(a) < T(b), \\ a <_P b, a < b \Longrightarrow \quad & T(a) \leq T(b). \end{aligned} \tag{$*$}$$

For example,



$$T(1) > T(2) \leq T(3).$$

Let $\mathscr{A}(P)$ be the set of (P, ω)-partitions. For an abstract poset P, we need first find a bijection $P \to [n]$.

Here are more examples

- When the bijiection is increasing, P-partition is just strictly increasing map $P \to [\infty]$.
- When the bijection is decreasing, P-partition is just weakly increasing map $P \to [\infty]$.
- When P is a chain $\{w_1 < w_2 < \cdots < w_n\}$, then

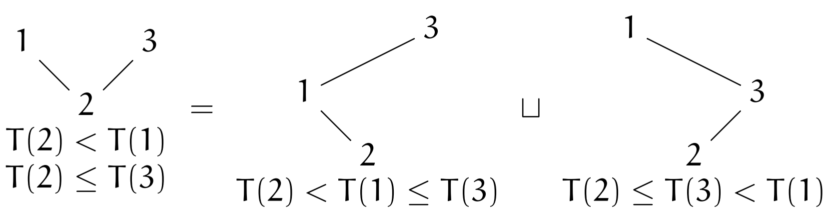
$$\mathscr{A}(P) = \left\{ [n] \xrightarrow{T} [\infty] : \begin{array}{ll} T(i) \leq T(i+1) & w_i < w_{i+1} \\ T(i) < T(i+1) & w_i > w_{i+1} \end{array} \right\}$$

the increasing sequence strictly at descent of w.

The fundamental theorem of P-partition is

$$\mathscr{A}(P) = \bigsqcup_{P'} \mathscr{A}(P')$$

for all linear extension P' of P. For example,



The bijection is given by "standardization". That is, for any P-partition, we can define a linear extension P' by the lexicographic order of $i \mapsto (T(i), i)$. Then T gives a P'-partition.

Let us define

$$F_P = \sum_{T\in\mathscr{A}(P)} x^T = \sum_{\substack{i_1,\dots,i_n \\ a<_P b, a<b \Rightarrow i_a\le i_b \\ a<_P b, a>b\Rightarrow i_a<i_b}} x_{i_1}\cdots x_{i_n} = \sum_{P'} F_{P'}.$$

When P' is a chain $\{w_1 < \cdots < w_n\}$, we have

$$F_{P'} = F_{n,\mathrm{des}(P')} \quad \text{where} \quad \mathrm{des}(P') = \{i\in[n-1] : w_i > w_{i+1}\}.$$

As a result, we have

$$\omega F_P = \sum_{\substack{i_1,\dots,i_n \\ a<_P b, a<b \Rightarrow i_a< i_b \\ a<_P b, a>b\Rightarrow i_a\le i_b}} x_{i_1}\cdots x_{i_n} = F_Q$$

for Q the order $i <_Q j$ iff $n-i <_P n-j$.

Example. Let us see how it can be used to compute ωs_λ. Let $n = |\lambda|$. Given a standard tableaux $S\in\mathrm{SYT}(\lambda)$, $\boxed{i+1}$ must be NE to $\boxed{\ i\ }$ (including its direct right) or SW to $\boxed{\ i\ }$ (including its direct down). We define a compositon $\alpha(S)$ of n between the i-th ball and the $(i+1)$-th ball if $\boxed{i+1}$ is SW to $\boxed{\ i\ }$. Then we have

$$s_\lambda = \sum_{S\in\mathrm{SYT}(\lambda)} M_{n,\mathrm{des}(S)}.$$

Actually, $\mathrm{SSYT}(\lambda)$ is a special case of P-partitions. Explicitly, for each $T\in\mathrm{SSYT}(\lambda)$, we can associate its standardization $S = \mathrm{std}(T)\in\mathrm{SYT}(\lambda)$ a standard tableaux such that

- if $T(\square_1) < T(\square_2)$, then $S(\square_1) < S(\square_2)$;
- if $T(\square_1) = T(\square_2)$, and $\square_1$ is left to the $\square_2$, then $S(\square_1) < S(\square_2)$.

For each $S\in\mathrm{SYT}(\lambda)$, we define its descent set to be

$$\mathrm{dec}(S) = \left\{ i\in[n-1] : \boxed{i+1} \text{ is lower than } \boxed{\ i\ } \right\}.$$

It is not hard to figure out the identity:

2	2	5	5	20
5	10	20		
10				

1	2	4	5	9
3	7	8		
6				

$$\underset{1}{\textcircled{2}}\ \underset{2}{\textcircled{2}} \mid \underset{3}{\textcircled{5}}\ \underset{4}{\textcircled{5}}\ \underset{5}{\textcircled{5}} \mid \underset{6}{\textcircled{10}}\ \underset{7}{\textcircled{10}}\ \underset{8}{\textcircled{20}}\ \underset{9}{\textcircled{20}}$$

Then

$$\omega s_\lambda = s_{\lambda'},$$

where the condition of $<$ and $\leq$ are switched.

Example. Let G be a graph. We have

$$\left\{ \begin{array}{c} \text{proper coloring} \\ \kappa : G \to [\infty] \end{array} \right\} = \bigsqcup_{O} \left\{ \begin{array}{c} \text{strictly increasing map} \\ \kappa : (G, \leq_O) \to [\infty] \end{array} \right\}$$

where O is an acyclic orientation, which equips G a partial order. This shows

$$X_G = \sum_{O} \sum_{\substack{\kappa: G \to [\infty] \\ a \to b \Rightarrow \kappa(a) < \kappa(b)}} \prod_{v \in G} x_{\kappa(v)}.$$

By the example above, we have

$$\omega X_G = \sum_{O} \sum_{\substack{\kappa: G \to [\infty] \\ a \to b \Rightarrow \kappa(a) \leq \kappa(b)}} \prod_{v \in G} x_{\kappa(v)}.$$

In other word,

$$\omega X_G = \sum_{\kappa: G \to [\infty]} \# \left\{ \begin{array}{c} \text{acyclic orientation } O \\ \kappa(a) < \kappa(b) \Rightarrow a \to b \end{array} \right\} \prod_{v \in G} x^{\kappa(v)}.$$

References.

- R.P. Stanley. A Symmetric Function Generalization of the Chromatic Polynomial of a Graph.
- P. Brosnan, Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. [arXiv]
- K. Vilonen, T. Xue. A note on Hessenberg varieties. [arXiv]
- M. Precup, E. Sommers. Perverse sheaves, nilpotent Hessenberg varieties, and the modular law. [arXiv]
- D. Grinberg, V. Reiner, Hopf Algebras in Combinatorics. [arXiv]