Pieri and Murnaghan–Nakayama Type Rules for Chern classes of Schubert Cells

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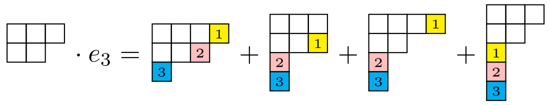
1 Introduction

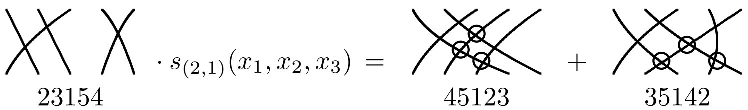
We develop **Pieri type** as well as **Murnaghan–Nakayama** (MN) type formulas for equivariant **Chern–Schwartz–MacPherson** classes of Schubert cells in the classical flag variety. These formulas include as special cases many previously known multiplication formulas for Chern–Schwartz–MacPherson classes or Schubert classes. We apply the equivariant Murnaghan–Nakayama formula to the enumeration of **rim hook tableaux**.

2 Classic Pieri Rules

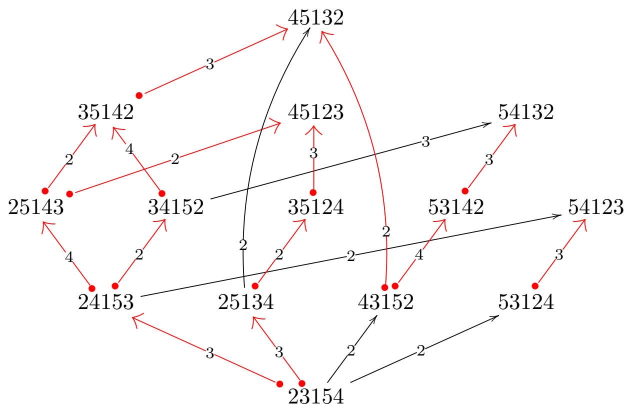
Historically, the classical Pieri formula provides a solution to the multiplication of a **Schur polynomial** by an **elementary** or a **complete homogeneous symmetric polynomial**. For example,



This formula is generalized to **Schubert polynomials** times hook shape Schur polynomial by Sottile [9]. The formula is given by chasing over **k-Bruhat graph**. For example,



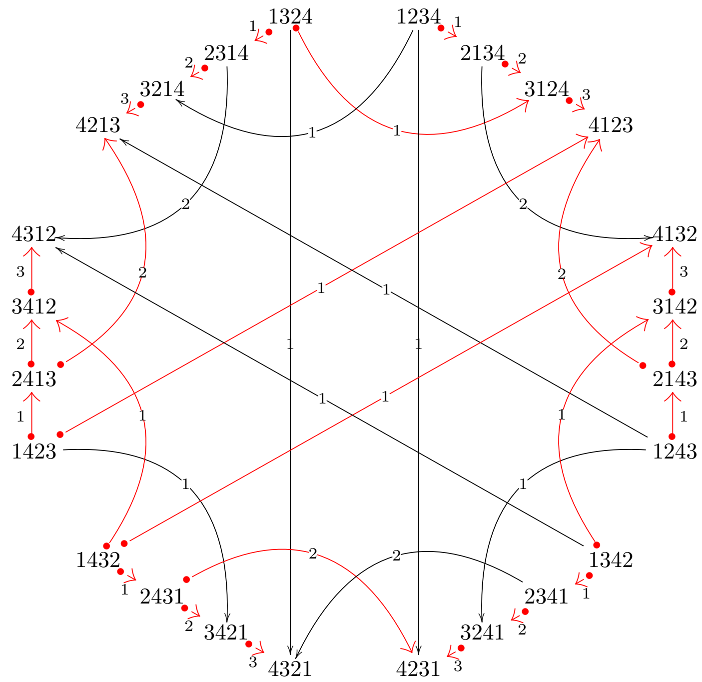
The graph is (black arrows)



Geometrically, elementary or complete homogeneous symmetric polynomial serves as not only Schubert classes but also the **Chern classes** or **Segre classes** of the tautological bundle over flag variety. We generalize the classic Pieri rule to Chern–Schwartz–MacPherson classes.

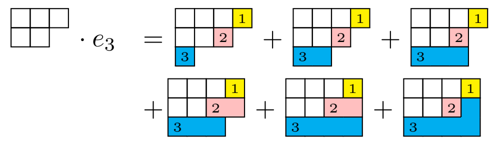
3 CSM Pieri rules

The **Chern–Schwartz–MacPherson (CSM)** classes are one way to extend the Chern classes of complex manifolds to complex varieties (possibly singular or noncompact). The structure of (equivariant) CSM classes for Schubert cells in flag varieties has received much attention in recent years and devolved into a rich area of research, see [1,2]. Our formula is given by chasing over the **extended k-Bruhat graph**. For example,



One of the key ingredients we developed is the **Rigidity Theorem**, which means that the structure constants for equivariant rules can be controlled by the structure constants in their nonequivariant situations.

When restricting to Grassmannian, we can get a description in terms of partition combinatorics. For example, in $H^\bullet(Gr(3,7))$, we have

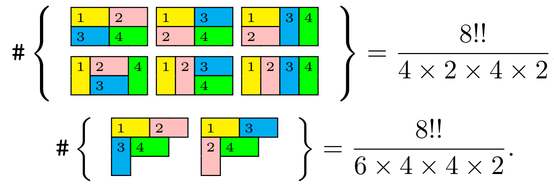


Note that all partitions cannot exceed the rectangle $(4,4,4)$.

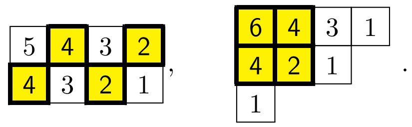
4 Hook formulas

Starting from the work of Ikeda and Naruse [5], it was gradually realized that **equivariant Chevalley type** formulas could be utilized to deduce **hook formulas**, see Naruse [6] and the recent work of Morales, Pak and Panova [8] and Mihalcea, Naruse and Su [7].

We generalize the idea of [5] to higher degrees by viewing the equivariant MN formula as a **higher degree analogue** of the equivariant Chevalley formula. It amazingly applies to the enumeration of **rim hook tableaux**. We reveal the enumeration formulas for standard rim hook tableaux due to Alexandersson, Pfannerer, Rubey and Uhlin [3] and Fomin and Lulov [4] in a relatively uniform manner.



where the denominator is the product of all **even hook lengths**.



References

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