Structure algebras, Hopf algebroids and oriented cohomology of a group

arXiv:2303.02409

(Joint work with Martina Lanini and Kirill Zainoulline)

Rui Xiong

(uOttawa)

April 27, 2023

A Classical Source of Hopf Algebras

Let G be a topological group, the **cohomology ring**

$$H^\bullet(G)$$

is a Hopf algebra with

ALGEBRA STRUCTURE	COALGEBRA STRUCTURE
cup product	from group multiplication
$H^\bullet(G) \otimes H^\bullet(G) \xrightarrow{\smile} H^\bullet(G)$	$H^\bullet(G) \xrightarrow{\Delta} H^\bullet(G) \otimes H^\bullet(G)$

This seems to be the **motivation** for the definition of Hopf algebras.

Generalization

There are numerous generalized cohomology theories other than usual cohomology, for example, **K-theory**, **cobordism**, etc. On the **algebraic geometry** side, parallel stories also exist. Here is a mini-dictionary:

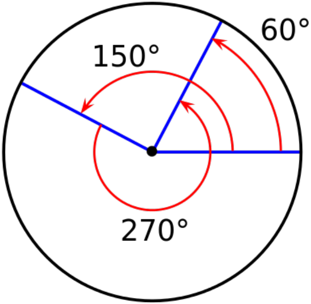
TOPOLOGY	ALGEBRA
topological Groups	algebraic Groups
cohomology	Chow ring intersection theory
topological K-theory	algebraic K-theory
topological cobordism	algebraic cobordism

Example I — $SO(2)$

$$SO_2 = \{\text{2-dimensional rotations}\} \simeq S^1.$$

$$\begin{array}{rcccccccc}
 & & 0 & & 1 & & 2 & & \\
H^\bullet(SO_2) & = & \mathbb{Z} & \oplus & \mathbb{Z}\xi & \oplus & 0 & \oplus & \cdots \\
\cup & & \cup & & \cup & & \cup & & \\
\mathrm{CH}^\bullet(SO_2) & = & \mathbb{Z} & \oplus & 0 & \oplus & 0 & \oplus & \cdots
\end{array}$$

$$\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1.$$



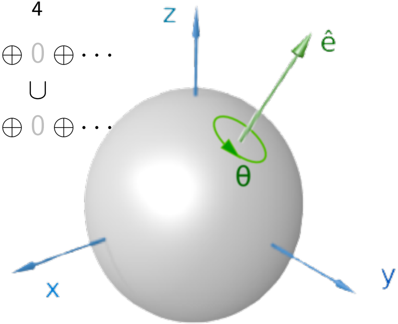
Example II — $SO(3)$

$$SO_3 = \{\text{3-dimensional rotations}\} \simeq \mathbb{R}P^3.$$

$$\begin{array}{rcccccccccccc}
 & & 0 & & 1 & & 2 & & 3 & & 4 & & \\
H^\bullet(SO_3) & = & \mathbb{Z} & \oplus & 0 & \oplus & \mathbb{F}_2 x & \oplus & \mathbb{Z}\xi & \oplus & 0 & \oplus & \cdots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
\mathrm{CH}^\bullet(SO_3) & = & \mathbb{Z} & \oplus & 0 & \oplus & \mathbb{F}_2 x & \oplus & 0 & \oplus & 0 & \oplus & \cdots
\end{array}$$

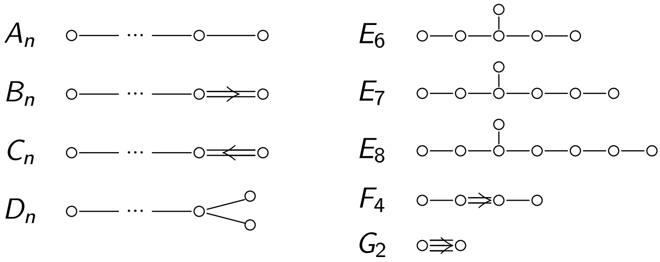
$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

$$\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1.$$



Dynkin diagrams

We shall focus on semisimple **Lie groups**, which is classified by **Dynkin diagrams**.



Root systems

Each Dynkin diagram has a corresponding **root system**.

For example, for $G = SL_4$

$$\begin{bmatrix} * & \square & * & * \\ * & * & \heartsuit & * \\ * & * & * & \diamondsuit \\ * & * & * & * \end{bmatrix} \qquad \begin{aligned} \square &= x_1 - x_2 \\ \heartsuit &= x_2 - x_3 \\ \diamondsuit &= x_3 - x_4 \end{aligned} \qquad \circ - \circ - \circ$$

Roughly speaking, each node $\circ$ stands for a vector and each edge means a non-orthogonal angle between vectors.

Weyl Groups

Each Dynkin diagram corresponds to a **Weyl group**.

For example, for $G = SL_4$,

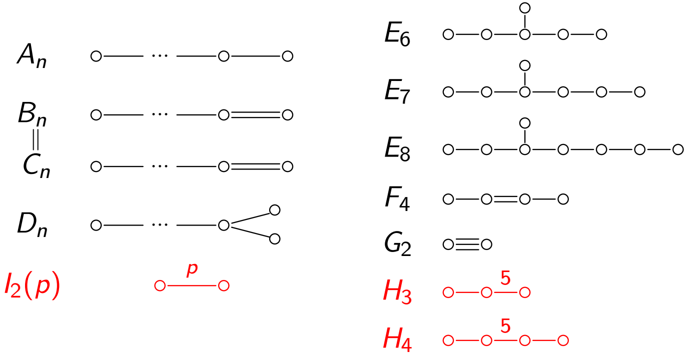
$$\left.\begin{array}{lcl} \square = x_1 - x_2 & \longleftrightarrow & \text{swap } x_1 \text{ and } x_2 \\ \heartsuit = x_2 - x_3 & \longleftrightarrow & \text{swap } x_2 \text{ and } x_3 \\ \diamondsuit = x_3 - x_4 & \longleftrightarrow & \text{swap } x_3 \text{ and } x_4 \end{array}\right\} \text{generate } \mathfrak{S}_4.$$

In general, the Weyl group is a discrete group generated by reflections (a **Coxeter group**).

However, not all Coxeter groups are Weyl groups.

Coxeter Diagrams

The following is the classification of finite **Coxeter groups**.



Structure Algebras

For any generalized cohomology theory (or more precisely formal group law $\mathtt{F}$), we can construct for each root system Λ a **structure algebra**

$$\mathcal{Z} = \left\{ (z_w) \in \mathrm{Sym}_{\mathtt{F}}(\Lambda)^{\Pi W} : x_\alpha \mid z_w - z_{ws_\alpha} \right\}.$$

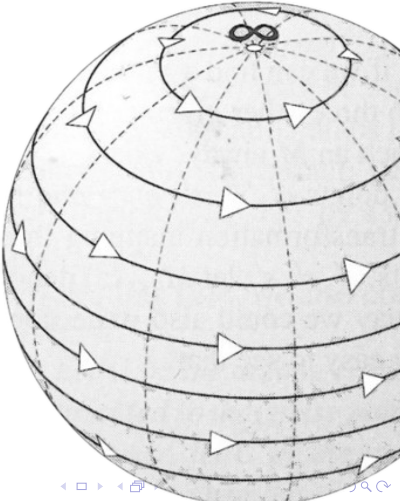
Geometrically, we have

$$\mathcal{Z} = \mathrm{h}_T(G/B) = \begin{pmatrix} \text{generalized } T\text{-equivariant} \\ \text{cohomology of flag varieties} \end{pmatrix}.$$

Example I — SL_2

$$
\begin{aligned}
G &= SL_2, \\
T &= \begin{bmatrix} * & \\ & * \end{bmatrix} \cong \mathbb{C}^\times \\
B &= \begin{bmatrix} * & * \\ & * \end{bmatrix} \\
G/B &\simeq \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}
\end{aligned}
$$

$$\mathcal{Z} = \left\{ (z_0, z_\infty) : x \mid z_0 - z_\infty \right\}.$$



Example II — SL_3

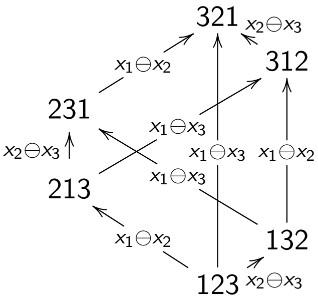
$$G = SL_3,$$

$$T = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$G/B \simeq \{0 \subset \ell \subset P \subset \mathbb{C}^3\}.$$

$$\mathcal{Z} = \left\{ (z_{123}, \cdots, z_{321}) : \cdots \right\}.$$



Duoidal Category

Denote $S = \mathrm{Sym}_{\mathbb{F}}(\Lambda)$. Actually, the structure algebra $\mathcal{Z}$ sits in the category of S-bimodules under **Hecke action** and **Weyl action**.

In the category of S-bimodules, there are two tensor structures

- $X \otimes Y$ with $sxr \otimes y = x \otimes syr$.
- $X \mathbin{\hat{\otimes}} Y$ with $xs \otimes y = x \otimes sy$.

They form a **duoidal category** under the natural **interchange**

$$(X_1 \mathbin{\hat{\otimes}} X_2) \otimes (Y_1 \mathbin{\hat{\otimes}} Y_2) \longrightarrow (X_1 \otimes Y_1) \mathbin{\hat{\otimes}} (X_2 \otimes Y_2).$$

Roughly, a duoidal category is a category with two compatible monoidal structures.

Bimonoid

Theorem (Lanini, Xiong, Zainouline)

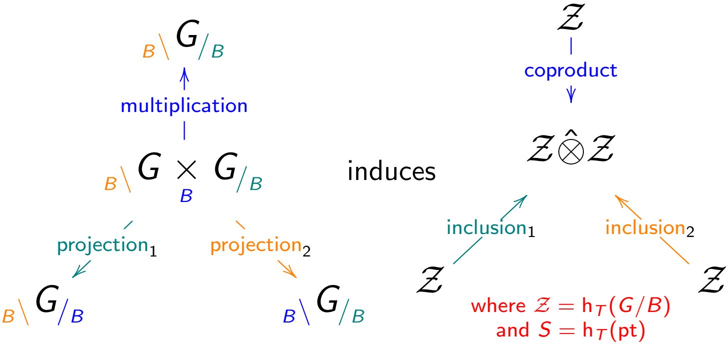
The structure algebra $\mathcal{Z}$ is a Hopf algebroid:

1. *$\mathcal{Z}$ is an algebra under $\otimes$;*
2. *$\mathcal{Z}$ is an coalgebra under $\hat{\otimes}$;*
3. *$\mathcal{Z}$ satisfies diagrams of compatibility of two structures. For example,*

$$
\begin{array}{ccccc}
\mathcal{Z} \otimes \mathcal{Z} & \xrightarrow{\ \text{prod}\ } & \mathcal{Z} & \xrightarrow{\ \text{coprod}\ } & \mathcal{Z} \hat{\otimes} \mathcal{Z} \\
\big\downarrow {\scriptstyle \text{coprod}\otimes\text{coprod}} & & & & \big\uparrow {\scriptstyle \text{prod}\hat{\otimes}\text{prod}} \\
(\mathcal{Z} \hat{\otimes} \mathcal{Z}) \otimes (\mathcal{Z} \hat{\otimes} \mathcal{Z}) & & \xrightarrow{\ \text{interchange}\ } & & (\mathcal{Z} \otimes \mathcal{Z}) \hat{\otimes} (\mathcal{Z} \otimes \mathcal{Z})
\end{array}
$$

Geometric meaning

Our construction of the coproduct is motivated by geometry.



Double Quotient

We have an augmented map

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathrm{Sym}_{\mathbb{F}}(\Lambda) \xrightarrow{\text{ring}} \binom{\text{base}}{\text{ring}} \longrightarrow 0.$$

Note that base change to the base ring makes two tensor structures $\otimes$ and $\hat{\otimes}$ coincide. Thus, the **double quotient**

$$\mathcal{Z}\big/_{\mathcal{I}\mathcal{Z}+\mathcal{Z}\mathcal{I}}, \qquad \otimes \equiv \hat{\otimes} \bmod \mathcal{I}$$

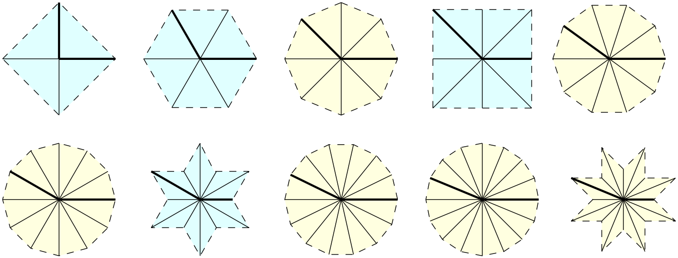
is a Hopf algebra. By a theorem of Grothendieck, its geometric meaning is $h(G)$ the generalized cohomology of a semisimple group.

Dihedral Combinatorics

We obtain a purely algebraic proof of the fact h(G) is a Hopf algebra. Moreover, it works for any finite **Coxeter groups**!

Let us illustrate when it is a **dihedral group** and **Chow ring/cohomology**

$$I_2(n) : \circ \overset{n}{\text{———}} \circ$$



Dihedral Combinatorics

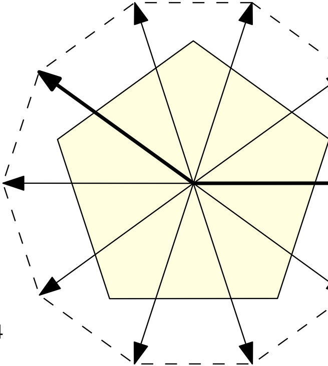
$$I_2(5) : \circ\overset{5}{\text{———}}\circ$$

$$W = \text{dihedral group of order } 10$$

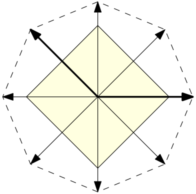
$$\mathrm{CH}^\bullet(\text{"}I_2(5)\text{"}) = {}^{\mathcal{Z}}\!/_{\mathcal{I}\mathcal{Z}+\mathcal{Z}\mathcal{I}}$$

$$\cong \mathbb{Z}\left[\tfrac{\sqrt{5}-1}{2}\right][x]/\langle x^5, \sqrt{5}\rangle$$

$$= \overset{0}{\mathbb{Z}\left[\tfrac{\sqrt{5}-1}{2}\right]} \oplus \overset{1}{\mathbb{F}_5 x} \oplus \overset{2}{\mathbb{F}_5 x^2} \oplus \overset{3}{\mathbb{F}_5 x^3} \oplus \overset{4}{\mathbb{F}_5 x^4}$$



Dihedral Combinatorics



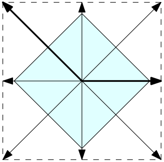
$I_2(4)$: ○—4—○

W = dihedral group of order 8

$$\mathrm{CH}^\bullet(``I_2(4)")_{\mathbb{F}_2} = {}^{\mathcal{Z}}/_{\mathcal{I}\mathcal{Z}+\mathcal{Z}\mathcal{I}}$$

$$\cong \mathbb{F}_2[x,y,z]/\langle x^2, y^2, z^2\rangle$$

$$= \overset{0}{\mathbb{F}_2} \oplus \overset{1}{\begin{matrix}\mathbb{F}_2 x\\ \mathbb{F}_2 y\end{matrix}} \oplus \overset{2}{\begin{matrix}\mathbb{F}_2 xy\\ \mathbb{F}_2 z\end{matrix}} \oplus \overset{3}{\begin{matrix}\mathbb{F}_2 xz\\ \mathbb{F}_2 yz\end{matrix}} \oplus \overset{4}{\mathbb{F}_2 xyz}$$



$B_2 = C_2$: ○⇐○

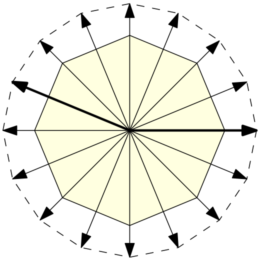
W = dihedral group of order 8

$$\mathrm{CH}^\bullet(SO_5)_{\mathbb{F}_2} = {}^{\mathcal{Z}}/_{\mathcal{I}\mathcal{Z}+\mathcal{Z}\mathcal{I}}$$

$$\cong \mathbb{F}_2[x]/\langle x^4\rangle$$

$$= \overset{0}{\mathbb{F}_2} \oplus \overset{1}{\mathbb{F}_2 x} \oplus \overset{2}{\mathbb{F}_2 x^2} \oplus \overset{3}{\mathbb{F}_2 x^3}$$

Dihedral Combinatorics

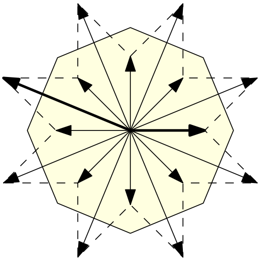


$I_2(8) : \circ \overset{8}{\text{———}} \circ$

$W =$ dihedral group of order 8

$$\mathrm{CH}^\bullet(\text{"}I_2(8)\text{"})_{\mathbb{F}_2} = {}^{\mathcal{Z}}\big/_{\mathcal{IZ}+\mathcal{ZI}}$$

$$\cong \mathbb{F}_2[y_1, x_1, x_2, x_4]/\langle y_1^2, x_1^2, x_2^2, x_4^2\rangle$$



$I_2(8) : \circ \overset{8}{\Leftarrow} \circ$

$W =$ dihedral group of order 8

$$\mathrm{CH}^\bullet(\text{"}I_2(8)\text{"})_{\mathbb{F}_2} = {}^{\mathcal{Z}}\big/_{\mathcal{IZ}+\mathcal{ZI}}$$

$$\cong \mathbb{F}_2[y, x_4]/\langle y_1^4, x_4^2\rangle$$

Thanks