Automorphisms of Quantum Cohomology of Springer Resolution arXiv:2304.07173

(Joint work with Changjian Su and Changzheng Li)

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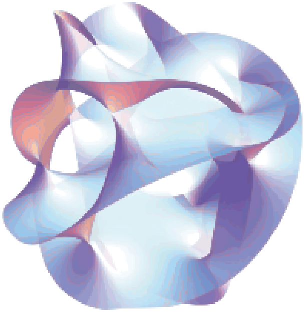
Quantum coHomology



Quantum Cohomology

Let X be an quasi-projective variety. The **quantum cohomology** is a deformation of usual cohomology ring $H^\bullet(X)$

$$QH^\bullet(X) = H^\bullet(X;\mathbb{Q})[[q^\beta]]_{\beta\in \mathrm{Eff}(X)}$$

with **quantum product** $*$ such that

$$\Big\langle [C_1]*[C_2],\,[C_3]\Big\rangle_{\text{poincaré}} = \sum_\beta \#\left\{\begin{array}{c}\text{rational curves}\\ \text{going through}\\ C_1, C_2, C_3 \text{ of class } \beta\end{array}\right\} q^\beta.$$

The formal definition uses the moduli space $\overline{M}_{0,3}(X,\beta)$.

Projective Line

Denote the class of a point

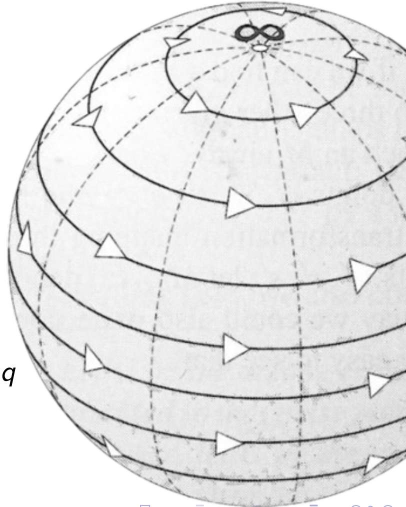
$$x = [\infty] = [0]$$

$$0 \cap \infty = \varnothing \quad \Rightarrow \quad x \cdot x = 0$$

$$H^\bullet(\mathbb{P}^1) = \mathbb{Q}[x]/\langle x^2 \rangle$$

$\exists$ rational curve through $0, \infty$ $\quad \Rightarrow \quad x \cdot x = q$

$$QH^\bullet(\mathbb{P}^1) = \mathbb{Q}[x, q]/\langle x^2 = q \rangle$$



Flag Varieties

Let $\mathcal{F}\ell_n$ be the classical flag variety

$$\mathcal{F}\ell_n = \left\{ 0 = \phi_0 \subset \phi_1 \subset \cdots \subset \phi_{n-1} \subset \phi_n = \mathbb{C}^n \;\middle|\; \dim \phi_i = i \right\}.$$

By Borel [1]

$$H^\bullet(\mathcal{F}\ell_n) = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle e_1(x), \ldots, e_n(x) \rangle},$$

where $e_i(x)$ is the i-th elementary symmetric polynomials.

$$\begin{aligned} e_1(x) &= x_1 + \cdots + x_n \\ e_2(x) &= \textstyle\sum_{i<j} x_i x_j \\ \cdots &= \cdots \\ e_n(x) &= x_1 \cdots x_n \end{aligned}$$

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By Givental and Kim [4],

$$QH^\bullet(\mathcal{F}\ell_n) = \frac{\mathcal{O}[x_1, \ldots, x_n]}{(E_1(x), \ldots, E_n(x))}$$

where $E_i(x)$ is the coefficient of the characteristic polynomial of the tridiagonal matrix

$$\begin{bmatrix} x_1 & -1 & 0 & \cdots & 0 \\ \frac{q_1}{q_2} & x_2 & -1 & \cdots & 0 \\ 0 & \frac{q_2}{q_3} & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}$$

co$\mathcal{T}$angent bundle of $\mathcal{F}$lag variety

Symplectic Resolutions

> *Symplectic resolutions are the Lie algebras of the 21st century*
>
> —*Andrei Okounkov*

Symplectic resolution is a resolution of singularities

$$\begin{array}{rc} \text{(holomorphic manifold)} & X \\ \text{(proper)} & \downarrow \\ \text{(affine Poisson variety)} & Y \end{array}$$

It includes **Springer resolutions**, **hypertoric varieties**, **Nakajima quiver varieties**, **affine Grassmannian slices** as examples. See [5].

Springer Resolution

Recall the **flag variety**

$$\mathcal{F}\ell_n = \left\{ 0 = \phi_0 \subset \phi_1 \subset \cdots \subset \phi_{n-1} \subset \phi_n = \mathbb{C}^n \,\middle|\, \dim \phi_i = i \right\}.$$

Denote the **nilpotent cone**

$$\mathcal{N} = \left\{ A \in \mathbb{M}_n(\mathbb{C}) \,\middle|\, A^n = \mathbf{0} \right\}.$$

The cotangent bundle of $\mathcal{F}\ell_n$

$$T^* \mathcal{F}\ell_n = \left\{ (A, \phi_\bullet) \in \mathcal{N} \times \mathcal{F}\ell_n \,\middle|\, A(\phi_i) \subset \phi_i \right\}.$$

forms a resolution of $\mathcal{N}$, i.e. **Springer resolution**.

Cotangent bundle of projective line

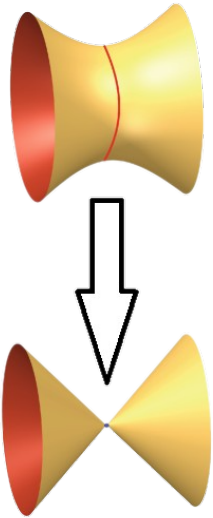
$$\mathcal{N} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{l} a+d=0, \\ ad-bc=0 \end{array} \right\}$$

- At the point 0, the fibre is

$$\mathcal{F}\ell_n = \mathbb{P}^1$$

- At a nonzero $A \in \mathcal{N}$, the fibre is

$$\{*\} = \left\{ (0 \subset \ker(A) \subset \mathbb{C}^2) \right\}.$$



Our main corollary

Theorem (Li, Su and Xiong)

$$QH^{\bullet}_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n) = \frac{\mathcal{O}[\hbar, x_1, \ldots, x_n]}{(\mathcal{E}_1(\chi), \ldots, \mathcal{E}_n(\chi))}$$

where $\mathcal{E}_i(x)$ is the coefficient of the characteristic polynomial of the matrix

$$\begin{bmatrix} \chi_1 & \frac{\hbar}{1-q_1/q_2} & \frac{\hbar}{1-q_1/q_3} & \cdots & \frac{\hbar}{1-q_1/q_n} \\ \frac{\hbar}{1-q_2/q_1} & \chi_2 & \frac{\hbar}{1-q_2/q_3} & \cdots & \frac{\hbar}{1-q_2/q_n} \\ \frac{\hbar}{1-q_3/q_1} & \frac{\hbar}{1-q_3/q_2} & \chi_3 & \cdots & \frac{\hbar}{1-q_3/q_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\hbar}{1-q_n/q_1} & \frac{\hbar}{1-q_n/q_2} & \frac{\hbar}{1-q_n/q_3} & \cdots & \chi_n \end{bmatrix}$$

$$\chi_i = x_i + \hbar \sum_{a<i} \frac{q_a/q_i}{1-q_a/q_i} - \hbar \sum_{i<b} \frac{q_i/q_b}{1-q_i/q_b}$$

Toda limit

Geometrically, we can recover $QH^\bullet(\mathcal{F}\ell_n)$ by taking the **Toda limit**, roughly speaking,

$$QH^\bullet_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n) \xrightarrow{\hbar\to\infty} QH^\bullet(\mathcal{F}\ell_n).$$

By taking entry-wise limit, we can recover the tridiagonal matrix for $QH^\bullet(\mathcal{F}\ell_n)$.

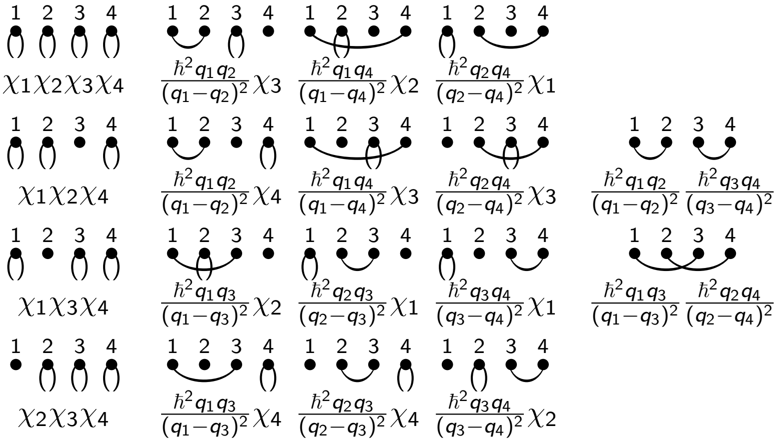
Similar description can be given for all types, and it agrees with [3] after taking Toda limit by case.

Question

Is there any **type-free** *connection between both descriptions?*

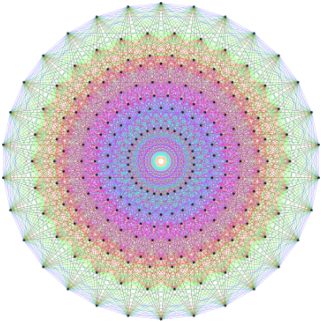
Combinatorial formulas

We give a combinatorial formula of $\mathcal{E}_k(\chi)$. For example, if $k = 3$ and $n = 4$



$\chi_1\chi_2\chi_3\chi_4$	$\frac{\hbar^2 q_1 q_2}{(q_1-q_2)^2}\chi_3$	$\frac{\hbar^2 q_1 q_4}{(q_1-q_4)^2}\chi_2$	$\frac{\hbar^2 q_2 q_4}{(q_2-q_4)^2}\chi_1$	
$\chi_1\chi_2\chi_4$	$\frac{\hbar^2 q_1 q_2}{(q_1-q_2)^2}\chi_4$	$\frac{\hbar^2 q_1 q_4}{(q_1-q_4)^2}\chi_3$	$\frac{\hbar^2 q_2 q_4}{(q_2-q_4)^2}\chi_3$	$\frac{\hbar^2 q_1 q_2}{(q_1-q_2)^2}\frac{\hbar^2 q_3 q_4}{(q_3-q_4)^2}$
$\chi_1\chi_3\chi_4$	$\frac{\hbar^2 q_1 q_3}{(q_1-q_3)^2}\chi_2$	$\frac{\hbar^2 q_2 q_3}{(q_2-q_3)^2}\chi_1$	$\frac{\hbar^2 q_3 q_4}{(q_3-q_4)^2}\chi_1$	$\frac{\hbar^2 q_1 q_3}{(q_1-q_3)^2}\frac{\hbar^2 q_2 q_4}{(q_2-q_4)^2}$
$\chi_2\chi_3\chi_4$	$\frac{\hbar^2 q_1 q_3}{(q_1-q_3)^2}\chi_4$	$\frac{\hbar^2 q_2 q_3}{(q_2-q_3)^2}\chi_4$	$\frac{\hbar^2 q_3 q_4}{(q_3-q_4)^2}\chi_2$	

quantum $\mathcal{D}$emazure - $\mathcal{L}$usztig operators



Steinberg varieties

Let us denote the **Steinbery variety**

$$\begin{aligned}\mathbf{St} &= T^* \mathcal{F}\ell_n \times_{\mathcal{N}} T^* \mathcal{F}\ell_n \\ &= \{(A, \phi_\bullet, \psi_\bullet) \mid A(\phi_i) \subset \phi_i, A(\psi_i) \subset \psi_i\}.\end{aligned}$$

Then

$$H^{BM}_{\mathbb{C}^*}(\mathbf{St}) \text{ acts on } H^\bullet_{\mathbb{C}^*}(\mathcal{F}\ell_n) \text{ via convolution.}$$

Moreover, classical **Springer theory** tells

$$\text{middle term of } H^{BM}_{\mathbb{C}^*}(\mathbf{St}) = \mathbb{C}[\mathfrak{S}_n].$$

Thus, there is a **symmetric group** action over $H^\bullet_{\mathbb{C}^*}(T^* \mathcal{F}\ell_n)$.

Demazure–Lusztig operators

The symmetric group action is given by a **Demazure–Lusztig type operator** defined by

$$s_i = 1 + (\hbar - (x_i - x_{i+1}))\partial_i,$$

which is never a ring automorphism except for the trivial case.

Definition (Quantum Demazure–Lusztig operators)

Let us denote **quantum Demazure–Lusztig operators** *for each* $w \in W$

$$T_w = w \otimes w$$

where the first w *is from Springer theory, and the second* w *permutes the quantum parameters.*

Weyl group action

Our main theorem is the following unexpected result.

Theorem (Li, Su, Xiong)

For any $w \in W$ and $\gamma_1, \gamma_2 \in QH^{\bullet}_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n)$,

$$T_w(\gamma_1 * \gamma_2) = T_w(\gamma_1) * T_w(\gamma_2).$$

As a result, T_w is a ring automorphism with respect to the quantum product.

Note that T_w would create a **pole** at the origin of the quantum variables, e.g.

$$T_{s_i} q^{\alpha_i^\vee} = q^{-\alpha_i^\vee},$$

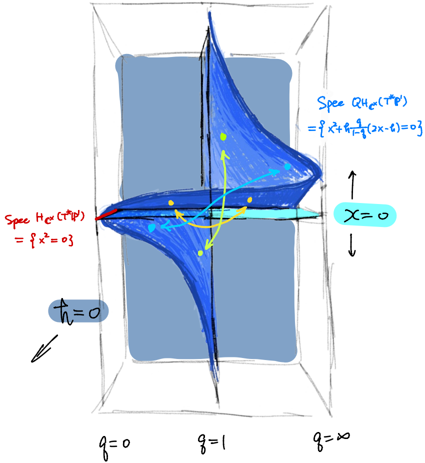
thus they cannot descent to the $H^{\bullet}_{\mathbb{C}^*}(T^*\mathcal{F}\ell_n)$.

Cotangent bundle of projective line

We have

$$QH_{\mathbb{C}^*}(T^*\mathbb{P}^1) = \frac{\mathcal{O}[x,\hbar]}{\left\langle x^2 + \hbar\frac{q}{1-q}(2x-\hbar)\right\rangle}.$$

The action of quantum Demazure–Lusztig operator is illustrated on the right.



Our proof is motivated by the following **suspicious identity**

$$\left.\begin{array}{rcl} 1 + q + q^2 + \cdots & = & \frac{1}{1-q} \\ q^{-1} + q^{-2} + \cdots & = & \frac{q^{-1}}{1-q^{-1}} \end{array}\right\} \Longrightarrow \sum_{n\in\mathbb{Z}} q^n = 0$$

Of course this does not make sense at all. But it tells that under the involution $q \mapsto q^{-1}$ over $\mathbb{Q}(q)$, we have

$$q + q^2 + \cdots \longmapsto -1 - q - q^2 - \cdots$$

This is what we need in the proof.

Stable basis

The main tool in the proof is the **stable basis** introduced by Maulik and Okounkov [6]. To be precise, we have a family of classes

$$\left\{ \mathrm{Stab}_{-}(w) \in H^{2\dim \mathcal{F}\ell_n}_{T\times\mathbb{C}^*}(T^*\mathcal{F}\ell_n) \,\middle|\, w \in W \right\}$$

which satisfies the following properties:

- $\mathrm{Stab}_{-}(w)|_u = 0$ unless $u \geq w$;
- $\mathrm{Stab}_{-}(w)|_w = \prod_{\alpha>0, w\alpha>0}(w\alpha - \hbar) \prod_{\alpha>0, w\alpha<0} w\alpha$;
- $\mathrm{Stab}_{-}(w)|_u$ is divisible by $\hbar$, for any $u > w$.

Moreover, the stable basis forms a basis of $H^*_{T\times\mathbb{C}^*}(T^*\mathcal{F}\ell_n)$ after inverting the equivariant parameters. See [8].

Chevalley formula

By [7], we have the following **Chevalley formula**

$$\begin{aligned} D_\lambda * \mathrm{Stab}_-(w) &= w(\lambda)\,\mathrm{Stab}_-(w) \\ &\quad - \hbar \textstyle\sum_{\alpha>0, w\alpha>0} \langle \lambda, \alpha^\vee \rangle\, \mathrm{Stab}_-(ws_\alpha) \\ &\quad - \hbar \textstyle\sum_{\alpha>0} \langle \lambda, \alpha^\vee \rangle\, \frac{q^{\alpha^\vee}}{1-q^{\alpha^\vee}} \big(\mathrm{Stab}_-(w) + \mathrm{Stab}_-(ws_\alpha) \big). \end{aligned}$$

where

$$\frac{q^{\alpha^\vee}}{1-q^{\alpha^\vee}} = q^{\alpha^\vee} + q^{2\alpha^\vee} + \cdots .$$

Thus the $\boxed{\sum q^n = 0}$ perfectly applies to the last term, so that we can prove the quantum Demazure–Lusztig operator is a ring automorphism.

Thank You

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