On Butler's Positivity Conjecture

(joint with P. Guo and M. Kang)

Rui Xiong

August 13, 2026

Macdonald Polynomials

Schur [1901]
$s_\lambda(X) \in \Lambda$
Schur function

$\Downarrow$

Littlewood [1961]
$P_\lambda(X;t) \in \Lambda_t$
Hall–Littlewood symmetric function

$\Downarrow$

Macdonald [1988]
$P_\lambda(X;q,t) \in \Lambda_{q,t}$
Macdonald symmetric function

There are different forms of Macdonald functions

$$P_\lambda = m_\lambda + (\text{lower term})$$
$$J_\lambda = P_\lambda \cdot \textstyle\prod_{\square\in\lambda}(1 - q^{a(\square)} t^{\ell(\square)+1})$$
$$H_\lambda = J_\lambda[\tfrac{X}{1-t}] = J_\lambda \begin{pmatrix} x_1 & tx_1 & t^2x_1 & \dots \\ x_2 & tx_2 & t^2x_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$
$$\widetilde{H}_\lambda = t^{n(\lambda)} H_\lambda|_{t\mapsto t^{-1}}$$

Theorem (Haiman, 2001)

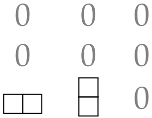
$$\widetilde{H}_\lambda \in \sum \mathbb{N}[q,t]\cdot s_\theta.$$

This was conjectured by Macdonald.

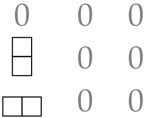
Example

$\widetilde{H}_{(1)} = s_{(1)}$

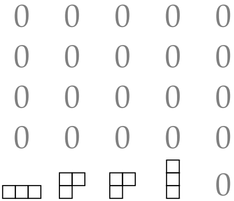
$\widetilde{H}_{(2)} = s_{(2)} + q s_{(1,1)}$



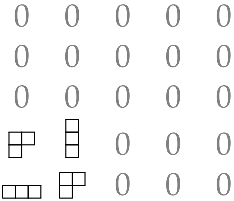
$\widetilde{H}_{(2)} = s_{(2)} + t s_{(1,1)}$



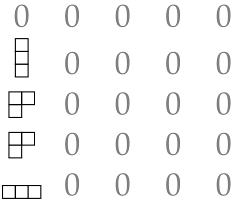
$\widetilde{H}_{(3)}$



$\widetilde{H}_{(2,1)}$



$\widetilde{H}_{(1,1,1)}$



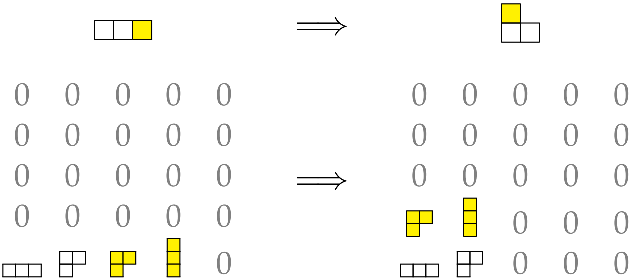
Butler's conjecture

Observing from the computation, Butler conjectured that

if μ is obtained by moving a box of λ, then

$\widetilde{H}_\mu$ is obtained by moving a summand of $\widetilde{H}_\lambda$.

For example,



Butler's conjecture

Note that if $\mu \vdash n$,

$$\widetilde{H}_\mu = 1 \cdot h_n + \cdots + T_\mu \cdot e_n,$$

where $T_\mu = \prod_{(i,j)\in\mu} q^i t^j = q^{n(\mu')} t^{n(\mu)}$. The precise conjecture can be stated as follows. we can write

$$\widetilde{H}_\lambda(X;q,t) = 1 \cdot I_{\lambda,\mu}(X;q,t) + T_\lambda \cdot J_{\lambda,\mu}(X;q,t)$$
$$\widetilde{H}_\mu(X;q,t) = 1 \cdot I_{\lambda,\mu}(X;q,t) + T_\mu \cdot J_{\lambda,\mu}(X;q,t),$$

such that

$$I_{\lambda,\mu}(X;q,t),\ J_{\lambda,\mu}(X;q,t) \in \sum \mathbb{N}[q,t] \cdot s_\theta(X).$$

Butler's conjecture

Since $T_\lambda \neq T_\mu$ if $\lambda \neq \mu$, we have

$$I_{\lambda,\mu}(X;q,t) = \frac{T_\lambda \widetilde{H}_\mu(X;q,t) - T_\mu \widetilde{H}_\lambda(X;q,t)}{T_\lambda - T_\mu}. \qquad (*)$$

Using the reciprocity identity:

$$\widetilde{H}_\mu(X;q,t) = t^{n(\mu)} q^{n(\mu')} \omega \widetilde{H}_\mu(X;q^{-1},t^{-1}),$$

it is easy to see $J_{\lambda,\mu}(X;q,t) = \omega I_{\lambda,\mu}(X;q^{-1},t^{-1})$. So the conjecture is equivalent to

$$(*) \in \mathbb{N}[q,t] \cdot s_\theta(X).$$

Geometry of Hilbert schemes

Recall

$$\begin{aligned} H_n &= \text{Hilbert scheme of } n\text{-points in } \mathbb{C}^2 \\ &= \{I \trianglelefteq \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}. \end{aligned}$$

By Fogarty, H_n is a nonsingular variety of dimension $2n$.

Example

When $n = 1$, any $I \in H_n$ is a maximal ideal, i.e. it takes the form $\langle x - x_0, y - y_0 \rangle$ for some $(x_0, y_0) \in \mathbb{C}^2$, so

$$H_n \cong \mathbb{C}^2.$$

Example

When $n = 2$, it is known that

$$H_n \cong \mathbb{C}^2 \times T^*\mathbb{P}^1$$

where $\mathbb{P}^1$ corresponds to I with $\mathbb{C}[x,y]/I \cong \mathbb{C}[\epsilon]/\epsilon^2$.

Geometry of Hilbert schemes

Consider

$$\mathbb{C}^{2n}/S_n = \operatorname{Spec}\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$$
$$= \{\text{muti-sets of } n\text{-points in } \mathbb{C}^2\}.$$

We have Hilbert–Chow map $\sigma : H_n \longrightarrow \mathbb{C}^{2n}/S_n$ by sending I to the support (with multiplicity) of $\mathbb{C}[x, y]/I$.

Example

When $n = 1$, a multi-set of one point is just one point, so

$$\mathbb{C}^{2n}/S_n = \mathbb{C}^2.$$

Then σ is identical.

Example

When $n = 2$, by linear transformation, we have

$$\mathbb{C}^{2n}/S_n \cong \mathbb{C}^2 \times (\mathbb{C}^2/\{\pm 1\}).$$

Then σ is a blow-up map.

Geometry of Hilbert schemes

The **isospectral Hilbert scheme** X_n is defined to be the reduced fiber product in the following diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\rho"'] & \mathbb{C}^{2n} \arrow[d] \\ H_n \arrow[r, "\sigma"] & \mathbb{C}^{2n}/S_n. \end{tikzcd}$$

Theorem (Haiman)

The map ρ is finite and flat.

The **Procesi bundle** is the sheaf

$$\mathcal{P} = \rho_* \mathcal{O}_{X_n}.$$

This is a $(T \times S_n)$-equivariant vector bundle, where $T = \mathbb{C}^\times \times \mathbb{C}^\times$ acts naturally on H_n, and S_n acts trivially.

Example ($n = 2$)

When $n = 2$, we can identify H_n with the variety of

- two distinct **unordered** points in $\mathbb{C}^2$;
- a point with a tangent direction in $\mathbb{C}^2$.

We can identify $H_n = \mathbb{C}^2 \times \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-2))$.

When $n = 2$, we can identify X_n with the variety of

- two distinct **ordered** points in $\mathbb{C}^2$;
- a point with a tangent direction in $\mathbb{C}^2$.

We can identify $X_n = \mathbb{C}^2 \times \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1))$.

The map $\rho : X_n \to H_n$ is by taking square

$$sq : \mathbb{C}^2 \times \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{C}^2 \times \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-2)).$$

Example ($n = 2$)

Let $L \subset \mathbb{C}^2$ be an element of $\mathbb{P}^1$. The map on each fiber is

$$sq : L \longrightarrow L^{\otimes 2}.$$

Choosing a nonzero $z \in L^*$, we have

$$\mathbb{C}[L] = \mathbb{C}[z], \qquad \mathbb{C}[L^{\otimes 2}] = \mathbb{C}[z^2].$$

The $-1 \in S_2$ acts by $z \mapsto -z$. Note that the zero of $L^{\otimes}$ is defined by $z^2 = 0$. In particular, we have

$$sq_* \mathcal{O}_L|_{\{0\}} = \mathbb{C}[z]/\langle z^2 \rangle \cong \mathbb{C} \oplus \mathbb{C}z = \mathbb{C} \oplus L^\vee.$$

The above computation glues to the following identification

$$\mathcal{P}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \otimes \mathsf{tri} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathsf{sgn}.$$

The connection

The isomorphism we computed in $n = 2$

$$\mathcal{P}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \otimes \mathsf{tri} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathsf{sgn}$$

is compatible with

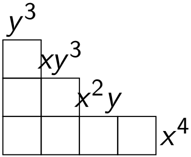
$$\widetilde{H}_{(2)} = 1 \cdot s_{(2)} + q \cdot s_{(1,1)},$$
$$\widetilde{H}_{(1,1)} = 1 \cdot s_{(2)} + t \cdot s_{(1,1)}.$$

This is obtained by localization and taking Frobenius character.

The connection

Let $\mu \vdash$ be a partition. We can define a monomial ideal

$$I_\mu = \langle x^a y^b : (a, b) \notin \mu \rangle \in H_n$$

$\mu =$ 

$$I_\mu = \langle x^4, x^2y, xy^3, y^3 \rangle.$$

Theorem (Haiman)

$$\mathrm{Frob}_{q,t}(\mathcal{P}|_{I_\mu}) = \widetilde{H}_\mu.$$

Here $\mathrm{Frob}_{q,t}$ is the bi-graded Frobenius character

$$\{(T \times S_n)\text{-representations}\} \xrightarrow{\mathrm{Frob}_{q,t}} \sum_\theta \mathbb{N}[q^{\pm 1}, t^{\pm 1}] s_\theta$$

The connection

Sketch of the proof of Butler's conjecture.

Let λ, μ differ by a box. Then we can find a T-equivariant curve $f : \mathbb{P}^1 \xhookrightarrow{} H_n$ joining I_λ and I_μ.

Now by Birkoff–Grothendieck theorem and some properties of $\mathcal{P}$, we can decompose

$$f^*\mathcal{P} \cong \mathcal{O}_{\mathbb{P}^1} \otimes M \oplus \mathcal{O}_{\mathbb{P}^1}(1) \otimes N.$$

The weight of $\mathcal{O}_{\mathbb{P}^1}(1)|_{I_\mu}$ is T_μ. We thus have

$$\widetilde{H}_\lambda = 1 \cdot \mathrm{Frob}_{q,t}(M) + T_\lambda \cdot \mathrm{Frob}_{q,t}(N),$$
$$\widetilde{H}_\mu = 1 \cdot \mathrm{Frob}_{q,t}(M) + T_\mu \cdot \mathrm{Frob}_{q,t}(N).$$

We are done. □

Thanks

We used ChatGPT 5.6 during the preparation of this work.