# Borel–Weil Theorem and Applications

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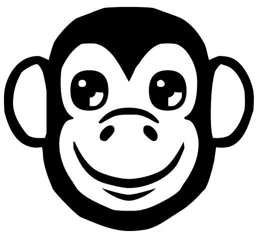

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# 1 Lecture 1 — Borel–Weil Theorem

**1.1.** Let $G$ be a reductive group over $\mathbb{C}$, and $B$ be its Borel subgroup. We call $G/B$ the **flag variety** of $G$.

- $G/B$ only depends on the Dynkin type of $G$.
- If $K$ is the compact form of $G$, then $G/B \cong K/S$ with $S = K \cap B$ the maximal torus of $K$.
- $G/B$ is a projective variety. An explicit embedding can be constructed by Plücker embedding.

For example, $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ has the same flag variety. One can also construct the flag manifold from $U(n)$ or $SU(n)$.

**1.2.** For type $A$, we take $\mathrm{GL}_n$, we take $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$ the group of invertible upper triangular matrices then we can identify $G/B$ with

$$\mathcal{F}\ell(n) = \mathcal{F}\ell(\mathbb{C}^n) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim V_i = i. \right\}$$

**1.3.** For other classic types, we take the symmetric form over $\mathbb{C}^n$ defining $\mathrm{SO}(n)$ to be

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{y}^{\mathsf{t}} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \mathbf{x} = x_1 y_n + \cdots + x_n y_1,$$

and the symplectic form over $\mathbb{C}^n$ defining $\mathrm{Sp}(n)$ to be

$$\omega(\mathbf{x}, \mathbf{y}) = \mathbf{y}^{\mathsf{t}} \begin{pmatrix} & & 1 \\ & \iddots & \\ -1 & & \end{pmatrix} \mathbf{x} = x_1 y_n + \cdots - x_n y_1$$

Then the Borel subgroup is exactly of the form $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$. In these cases, $G/B$ can be identifies with

$$\left\{ V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n : \dim V_i = i,\ V_i^{\perp} = V_{n-i}. \right\}.$$

**1.4.** Denote the maximal torus of $B$ to be $T$, and the unipotent radical of $B$ to be $U$. Recall that $B = U \rtimes T$, that is, we have a split short exact sequence of groups

$$0 \longrightarrow U \longrightarrow B \longrightarrow T \longrightarrow 0.$$

As a result, any representation of $T$ can be extended to $B$ (with trial $U$-action).

**1.5.** We denote $\mathbb{G}_m = \mathbb{C}^\times$ the algebraic group with natural multiplication. Let $T$ be a torus. An algebraic group homomorphism $\lambda : T \to \mathbb{G}_m$ is called a **character** of $T$. We denote $X(T)$ the group of all character, we will write them additively

$$(\lambda + \mu)(t) = \lambda(t)\mu(t), \qquad (-\lambda)(t) = \lambda(t)^{-1}.$$

Sometimes, we may write $e^\lambda$ to avoiding abuse of notations.

**1.6.** Let $\lambda$ be a character of $T$, that is an algebraic group homomorphism $T \to \mathbb{G}_m = \mathbb{C}^\times$. It corresponds to a one-dimenisonal representation $\mathbb{C}(\lambda)$ with $t \in T$ acts by $\lambda(t)^{-1}$. It naturally extended to $B$.

Consider the space $\xi(\lambda) = G \times_B \mathbb{C}(\lambda)$. It is a $G$-equivariant line bundle over $G/B$. Let us denote the corresponding sheaf to be $\mathcal{O}(\lambda)$.

Actually, all the $G$-equivariant line bundle over $G/B$ comes from this construction. since the fibre of $1 \cdot B/B$ is an one-dimensional representation of $B$ (thus factor through $T$).

**1.7.** For $G = \mathrm{GL}_n$, the maximal torus $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is the group of diagonal matrices. We denote $x_1, \ldots, x_n \in X(T)$ the coordinate of indices.

Let us denote the **tautological bundle** $\phi_k$ over $\mathcal{F}\ell(n)$ to be the $k$-dimensional vector bundle whose fibre at the flag $(V_0 \subseteq \cdots \subseteq V_n)$ is $V_k$. Then by explicit computation $\phi_k/\phi_{k-1} \cong \mathcal{O}(-x_k)$.

In particular, for $n = 2$, $\mathcal{F}\ell(2) = \mathbb{P}^1$, $\mathcal{O}(x_1) = \mathcal{O}(1)$.

**1.8.** **Borel–Weil Theorem** For any character $\lambda \in X(T)$,

$$H^0(G/B; \mathcal{O}(\lambda))^* = \begin{cases} L(\lambda) & \lambda \text{ is dominant} \\ 0 & \text{otherwise} \end{cases}$$

where $L(\lambda)$ the the finite dimensional representation of $G$ with the highest weight $\lambda$.

**Proof** We have a $G$-bimodule decomposition

$$\mathbb{C}[G] = \bigoplus_{\lambda \text{ dominant}} L(\lambda)^* \otimes L(\lambda).$$

Since $\mathrm{Hom}_G(V(\lambda), \mathbb{C}[G]) \cong \mathrm{Hom}_{\mathbb{C}}(V(\lambda), \mathbb{C})$. On the other hand, a section of $\mathcal{O}(\lambda)$ is exactly a map $f : G \to \mathbb{C}$ with $f(g) = \lambda^{-1}(b)f(gb) = \lambda^{-1}(b)(r_b f)(g)$ where $r_b$ is the right multiplication by $b$.

$$\begin{array}{ccc} G \times \mathbb{C}(\lambda) & \longrightarrow & G \times_B \mathbb{C}(\lambda) \\ \uparrow\downarrow & & \uparrow\downarrow \\ G & \longrightarrow & G/B \end{array}$$

As a result, there only rest $L(\lambda)^*$. Q.E.D.

**1.9.** The tangent bundle of $G/B$ is given by $G \times_B \mathfrak{g}/\mathfrak{b}$ with the action by adjoint action. Note that $U$ does not acts $\mathfrak{g}/\mathfrak{b}$ trivially, but there is a filtration, such that

$$\mathrm{gr}\, \Omega^1_{G/B} = \bigoplus_{\alpha_i \in \Delta^+} \mathcal{O}(-\alpha_i)$$

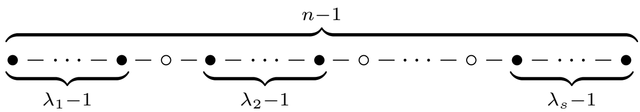
where $\Delta^+$ the set of positive roots. In particular, the canonical bundle $\omega = \mathcal{O}(-2\rho)$ where $\rho$ is the half sum of positive roots. By Serre duality,

$$H^{N-i}(G/B; \mathcal{O}(-2\rho - \lambda)) = H^i(G/B; \mathcal{O}(\lambda))^*,$$

where $N = \dim G/B$. The dual is the dual of $G$-representation when $G$ is semi-simple.

**1.10.** Let $P$ be a standard parabolic subgroup. That is, there is a subset $J \subseteq \mathbb{I}$ such that $P = \bigcup_{w \in W_J} BwB$, where $W_J$ is the Weyl group generated by $\{s_j : j \in J\}$. We denote $P_i = B \cup Bs_iB$ the **minimal parabolic subgroup**.

**1.11.** For the case of type $A$. A subset of $\mathbb{I} = \{1, \ldots, n-1\}$ cuts the Dynkin diagram into pieces. Assume it is



Then $n = \lambda_1 + \cdots + \lambda_s$, and the corresponding

$$P = \begin{pmatrix} \mathrm{GL}_{\lambda_1} & * & \cdots & * \\ & \mathrm{GL}_{\lambda_2} & \cdots & * \\ & & \ddots & \vdots \\ & & & \mathrm{GL}_{\lambda_s} \end{pmatrix}.$$

Furthermore, $G/P$ is identified with the partial flag variety

$$\mathcal{F}\ell_\lambda(n) = \mathcal{F}\ell_\lambda(\mathbb{C}^n) = \Big\{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_s : \dim V_i/V_{i-1} = \lambda_i. \Big\}.$$

In particular, $G/P_i$ is identified with

$$\Big\{ 0 \subseteq V_1 \subseteq \cdots \widehat{V_i} \cdots \subseteq V_n : \dim V_i = i. \Big\}$$

For the case $n = k + (n-k)$, then $G/P$ is identified with the **Grassmannian**

$$\mathcal{G}r(k,n) = \Big\{ V \subseteq \mathbb{C}^n : \dim V = k \Big\}.$$

**1.12. Plücker Embedding** Let $\rho$ be the half sum of simple roots. Let $L(\rho)$ be the finite dimensional representation of $G$ with the highest vector $v_0$. The orbit map

$$G \longrightarrow \mathbb{P}(L(\rho)) \qquad g \longmapsto g[v_0]$$

factors through an embedding of $G/B$. This is called the **Plücker embedding**. In general, for any $\lambda \in X(T)$,

$$G \longrightarrow \mathbb{P}(L(\lambda)) \qquad g \longmapsto g[v_0]$$

factors through an embedding of $G/P$ for $P$ the stablizer of $[v_0]$.

**1.13.** For example, when $\lambda = \omega_i$ the fundamental weight, then the corresponding $P$ is maximal parabolic. In $\mathrm{GL}_n$, for $\lambda = \omega_k = x_1 + \ldots + x_k$, $L(\omega_k) = \Lambda^k \mathbb{V}$ where $\mathbb{V}$ is the natural representation. It gives the classic Plücker embedding for $\mathcal{G}r(k,n)$.

**1.14.** For each $i$, we have a natrual map $\mathrm{SL}_2 \to G$ with image in $P_i$. This inducing an isomorphism $\mathbb{P}^1 \cong \mathrm{SL}_2 / \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \cong P_i/B$. The restriction of $\mathcal{O}(\lambda)$ to $P_i/B$ corresponds to $\mathcal{O}(d)$ over $\mathbb{P}^1$ with $d = \langle \alpha_i^\vee, \lambda \rangle$.

The natrual projection $G/B \to G/P$ is a fibre bundle with fibre $P/B$. In particular, when $P = P_i$, it is a $\mathbb{P}^1$ bundle.

**1.15.** Recall that over $\mathbb{P}^1$, we have

| $\mathcal{O}(n)$ | $\cdots$ | $\mathcal{O}(-4)$ | $\mathcal{O}(-3)$ | $\mathcal{O}(-2)$ | $\mathcal{O}(-1)$ | $\mathcal{O}(0)$ | $\mathcal{O}(1)$ | $\mathcal{O}(2)$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $\dim H^0$ | $\cdots$ | 0 | 0 | 0 | 0 | 1 | 2 | 3 | $\cdots$ |
| $\dim H^1$ | $\cdots$ | 3 | 2 | 1 | 0 | 0 | 0 | 0 | $\cdots$ |

Actually the pairing

$$H^i(\mathbb{P}^1;\mathcal{O}(-1+d)) \times H^{1-i}(\mathbb{P}^1;\mathcal{O}(-1-d)) \to H^1(\mathbb{P}^1;\mathcal{O}(-2))$$

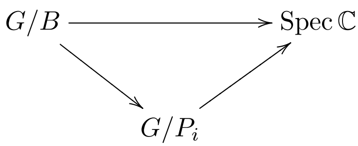
is a perfect pairing.

**1.16. Borel–Weil Theorem** When $\langle \alpha_i^\vee, \lambda \rangle \geq -1$,

$$H^i(G/B;\mathcal{O}(\lambda)) = H^{i+1}(G/B;\mathcal{O}(s_i \bullet \lambda)).$$

Recall: for $w \in W$ and $\lambda \in X(T)$, we denote $w \bullet \lambda = w(\lambda+\rho) - \rho$.

**1.17. Proof of the case $\langle \alpha_i^\vee, \lambda \rangle = -1$** Consider the Serre–Leray spectral sequence for




Since $G/B \to G/P_i$ is a fibre bundle, it suffices to see the cohomology of the fibre. But by the computation of $\mathbb{P}^1$, it is identical zero. Q.E.D.

**1.18. Proof of the case $\langle \alpha_i^\vee, \lambda \rangle = 0$** Denote $p : G/B \to G/P$. Consider the natural map

$$p^* p_* \mathcal{O}(\lambda+\rho) \longrightarrow \mathcal{O}(\lambda+\rho).$$

This is surjective by fibrewise computation. The kernel of this map is $\mathcal{O}(s_i(\lambda+\rho))$ by direct computation. So we get

$$0 \longrightarrow \mathcal{O}(s_i \bullet \lambda) \longrightarrow p^* p_* \mathcal{O}(\lambda+\rho) \otimes \mathcal{O}(-\rho) \longrightarrow \mathcal{O}(\lambda) \longrightarrow 0.$$

Use the spectral sequence argument again, we get from the long exact sequence that

$$H^i(G/B;\mathcal{O}(s_i \bullet \lambda) = H^{i+1}(G/B;\mathcal{O}(\lambda)).$$

We get the assertion. Q.E.D.

**1.19. Proof of the general case** The general case is similar, but technical. We can construct a filtration of $p^*p_*\mathcal{O}(\lambda+\rho)$ with subquotients

$$\mathcal{O}(s_i(\lambda+\rho)), \quad p^*p_*\mathcal{O}(\lambda-\alpha_i+\rho), \quad \mathcal{O}(\lambda+\rho).$$

By the spectral sequence argument, we can ignore $p^*p_*\mathcal{O}(\cdots)$ after tensoring with $-\rho$. Q.E.D.

**1.20. Principal Block** Assume $G$ is semisimple. We denote $\mathcal{O}(w)=\mathcal{O}(w\bullet 0)$, then

$$\dim H^i(G/B;\mathcal{O}(w)) = \begin{cases} 1 & i=\ell(w) \\ 0 & \text{otherwise} \end{cases}$$

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**1.21.** Assume the smooth projective variety $X$ is acted by algebraic torus $T$ with discrete fixed points $X^T$. For a $T$-equivariant vector bundle $\mathcal{F}$ over $X$, we have the **Atyiah–Bott Localization** for $t\in T$,

$$\sum(-1)^i \operatorname{tr}(t;H^i(X;\mathcal{F})) = \sum_{x\in X^T} \frac{\operatorname{tr}(t;\mathcal{F}|_x)}{\det(1-t|_{T_x^*X})}$$

where $T_x^*X$ is the cotangent space of $X$ at $x$, and $\mathcal{F}|_x=\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ is the fibre at $x$.

**1.22.** At any point $xB/B\in G/B$, the tangent space is naturally identified with $\operatorname{ad}_x\mathfrak{g}/\mathfrak{b}$. We know at point $1\cdot B/B$, $T_x^*=\bigoplus_{\alpha\in\Delta^+}\mathbb{C}(-\alpha)$ as $T$-space. So

$$\det(1-t|_{T_x^*X}) = w\cdot\prod_{\alpha\in\Delta^+}(1-e^{\alpha_i}).$$

Similarly, $\operatorname{tr}(t;\mathcal{O}(\lambda)|_x)=w\cdot e^{-\lambda}$. Thus

$$\operatorname{tr}(t;H^i(X;\mathcal{O}(\lambda))) = \sum_{w\in W} w\frac{e^{-\lambda}}{\prod_{\alpha\in\Delta^+}(1-e^{\alpha_i})} = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(-\lambda-\rho)}}{\prod_{\alpha\in\Delta^+}(e^{-\alpha_i/2}-e^{\alpha_i/2})}.$$

Then taking the dual, we get

$$\operatorname{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha_i/2}-e^{-\alpha_i/2})}.$$

We get the **Weyl character formula**.

**1.23.** In the case $\mathrm{GL}_n$. We denote $X_i = e^{x_i}$. Then the Weyl character formula gives

$$\mathrm{ch}(L(\lambda)) = \frac{\sum(-1)^{\ell(w)} X^{w(\lambda+\rho)}}{\prod_{i<j}(X_i - X_j)} = \frac{\det(X_i^{\lambda_j+n-j})}{\det(X_i^{n-j})}$$

the Schur polynomial.

## References


- Knutson. Lie groups. [notes]
- Sepanski. Compact Lie Groups.


# 2 Lecture 2 — Demazure Character Formula

**2.1.** Let $w_0$ be the longest word in Wely group. Then the opposite Borel subgroup $B^-$ is $w_0 B w_0$. We denote the **Schubert variety** to be

$$\Sigma_w = \overline{BwB/B} \subseteq G/B, \qquad \Sigma^w = \overline{B^- wB/B} \subseteq G/B.$$

Then $\dim \Sigma_w = \operatorname{codim} \Sigma^w = \ell(w)$. In particular, $\Sigma_{s_i} = P_i/B$, $\Sigma_{\mathrm{id}} = \Sigma^{w_0}$ is the point $1 \cdot B/B$, and $\Sigma_{w_0} = \Sigma^{\mathrm{id}} = G/B$.

**2.2.** For standard parabolic subgroup $P$ defined by $J \subseteq \mathbb{I}$, define the **Schubert variety** for $w$ which is shortest among $wW_J \in W/W_J$

$$\Sigma_w = \overline{BwP/P} \subseteq G/P, \qquad \Sigma^w = \overline{B^- wP/P}.$$

Then $\dim \Sigma_w = \operatorname{codim} \Sigma^w = \ell(w)$.

**2.3.** Denote $K_G(G/B)$ the $G$-equivariant K-theory. It is naturally isomorphic to the group algebra of $X(T)$. We denote the class of $\mathcal{O}(\lambda)$ by $e^\lambda$.

Assume $P$ is standard parabolic corresponding to $J \subseteq \mathbb{I}$. Then $K_G(G/P)$ is the $W_J$-invariant subalgebra of $K_G(G/B)$.

**2.4.** Let $p_i : G/B \to G/P_i$ be the natural projection. We define the **Demazure operator** $\pi_i$ to be the composition

$$K_G(G/B) \xrightarrow{(p_i)_*} K_G(G/P_i) \xrightarrow{(p_i)^*} K_G(G/B).$$

We denote the class of $\mathcal{O}(\lambda)$ by $e^\lambda$. By the compuation in cohomology and Grothendieck–Riemann–Roch, we have

$$\forall f \in K_G(G/B), \qquad \pi_i f = \frac{f - e^{-\alpha_i} s_i f}{1 - e^{-\alpha_i}}.$$

By direct computation, $\pi_i$ satisfies Braid relations and $\pi_i^2 = \pi_i$. Thus we can define $\pi_w$ for any element $w \in W$ by

$$\pi_w = \pi_{i_1} \circ \cdots \circ \pi_{i_r}, \qquad w = s_{i_1} \cdots s_{i_r} \quad \text{(any reduced word)}$$

**2.5. Demazure Character Formula** For dominant $\lambda \in X(T)$,

$$\operatorname{ch}\left(H^0(\Sigma_w; \mathcal{O}(\lambda))^*\right) = \pi_w e^\lambda,$$

and

$$\forall i \geq 1, \quad H^i(\Sigma_w; \mathcal{O}(\lambda)) = 0.$$

The proof is difficult and will not be given here.

**2.6.** Roughly speaking, push forward is "taking global section along fibres". Actually, when $\ell(ws_i) = \ell(w) + 1$, $p : \Sigma_w \to \Sigma_w^P$ is birational, and $\Sigma_{ws_i} = p^{-1}(\Sigma_w^P)$. But K-theory is very sensitive with respect to birational morphisms.

- When $w = \mathrm{id}$, $H^0(1 \cdot B/B; \mathcal{O}(\lambda))$ is nothing but $\mathbb{C}(\lambda)$. So the character is $e^\lambda$.
- When $w = w_0$, one can compute

$$\pi_{w_0} f = \frac{\sum (-1)^{\ell(w)} w(f e^\rho)}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

  Thus with Borel–Weil theorem, we have

$$\operatorname{ch}(L(\lambda)) = \frac{\sum (-1)^{\ell(w)} w(e^{\lambda+\rho})}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

  So this recovers Weyl character formula.

**2.7.** For the case $\mathrm{SL}_2$, denote $e = e^{\omega_1} = e^{x_1}$, then $e^{\alpha_1} = e^2$.

$$\begin{array}{ccccccccccc} -n & -n{+}2 & & n{-}2 & n & & -n & -n{+}2 & & n{-}2 & n \\ 0 & 0 & \cdots & 0 & 1 & \xmapsto{\pi_1} & 1 & 1 & \cdots & 1 & 1 \end{array}$$

$$e^n \quad \xmapsto{\pi_1} \quad \frac{e^n - e^{-2} e^{-n}}{1 - e^{-2}} = e^{-n} + \cdots + e^n$$

$$\begin{array}{cccccccccccc} {\scriptstyle -n} & {\scriptstyle -n+2} & & {\scriptstyle n-2} & {\scriptstyle n} & & {\scriptstyle -n} & {\scriptstyle -n+2} & & {\scriptstyle n-2} & {\scriptstyle n} \\ 1 & 1 & \cdots & 1 & 1 & \xmapsto{\pi_1} & 1 & 1 & \cdots & 1 & 1 \end{array}$$

$$e^{-n} + \cdots + e^{n} \qquad \xmapsto{\pi_1} \qquad e^{-n} + \cdots + e^{n}$$

Then for example,

$$\begin{array}{ccccccccccccc} {\scriptstyle -5} & {\scriptstyle -3} & {\scriptstyle -1} & {\scriptstyle 1} & {\scriptstyle 3} & {\scriptstyle 5} & & {\scriptstyle -5} & {\scriptstyle -3} & {\scriptstyle -1} & {\scriptstyle 1} & {\scriptstyle 3} & {\scriptstyle 5} \\ 0 & 0 & 1 & 2 & 1 & 1 & \xmapsto{\pi_1} & 1 & 2 & 4 & 4 & 2 & 1 \end{array}$$

since we can decompose $(001211) = (001100) + (000100) + (000010) + (000001)$.

**2.8.** Consider the case $\mathrm{SL}_3$, see Figure 1.

**2.9.** For $\mathrm{GL}_n$ a series (i.e. composition) $\lambda = (\lambda_1, \ldots, \lambda_n)$, we can define the **Key polynomial** by

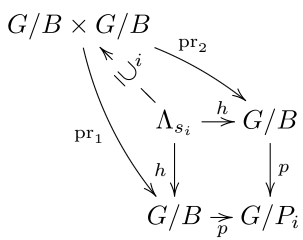
$$\kappa_\lambda = X_1^{\lambda_1} \cdots X_n^{\lambda_n} \qquad \text{if } \lambda_1 \geq \cdots \geq \lambda_n$$
$$\kappa_{s_i\lambda}(X) = \pi_i \kappa_\lambda(X) \qquad \text{if } \lambda_i \geq \lambda_{i+1}$$

This is essentially the Demazure character formula $\pi_w e^\lambda$. Note that in this case, $\pi_i f = \frac{X_i f - s_i(X_i f)}{X_i - X_{i+1}}$.

**2.10.** Let us also lift everything to $G$-version. The $G$-orbit of $G/B \times G/B$ are one-to-one corresponding to $B$-orbit of $G/B$. Let us denote

$$\Lambda_w = \overline{\{(xB, yB) : xy^{-1} \in BwB\}} \subseteq G/B \times G/B.$$

Note that when $w = s_i$, we have a pull back square




$$\begin{aligned} \pi_i \alpha &= p^* p_* \alpha = h_* h^* \alpha \\ &= (\mathrm{pr}_1)_* i_* i^* \,\mathrm{pr}_2^* \,\alpha \\ &= (\mathrm{pr}_1)_* \big( [\mathcal{O}_{\Lambda_{s_i}}] \cdot \mathrm{pr}_2^* \,\alpha \big) \\ &:= [\mathcal{O}_{\Lambda_{s_i}}] * \alpha \end{aligned}$$

So the Demazure operator $\pi_i : K_G(G/B) \longrightarrow K_G(G/B)$ is actually given by convolution with $[\mathcal{O}_{\Lambda_{s_i}}] \in K_G(G/B)$. In general, the Demazure operator $\pi_w$ is given by convlution with $[\mathcal{O}_{\Lambda_w}]$.

1 1 1 1 1
0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0
0 0 0 0
0 0 0

$s_2$

0 0 0 0 1
0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0
0 0 0 0
0 0 0

id

1 1 1 1 1
1 1 1 1 1 1
1 1 1 1 1 1 1
0 0 0 0 0 0
0 0 0 0 0
0 0 0 0
0 0 0

$s_2 s_1$

0 0 0 0 1
0 0 0 0 0 1
0 0 0 0 0 0 1
0 0 0 0 0 0
0 0 0 0 0
0 0 0 0
0 0 0

$s_1$

1 1 1 1 1
1 2 2 2 2 1
1 2 3 3 3 2 1
1 2 3 3 2 1
1 2 3 2 1
1 2 2 1
1 1 1

$s_1 s_2 s_1$

1 1 1 1 1
0 1 1 1 1 1
0 0 1 1 1 1 1
0 0 1 1 1 1
0 0 1 1 1
0 0 1 1
0 0 1

$s_1 s_2$

Figure 1: Example of $\mathrm{SL}_3$

**2.11.** In the case of $\mathrm{GL}_n$,

$$\Lambda_{s_i} = \left\{ 0 \subseteq V_1 \subseteq \cdots \begin{array}{c} \rotatebox{-45}{$\subseteq$} \quad V_i^1 \quad \rotatebox{45}{$\subseteq$} \\ \rotatebox{45}{$\subseteq$} \quad V_i^2 \quad \rotatebox{-45}{$\subseteq$} \end{array} \cdots \subseteq V_n : \dim V_i^{\cdots} = i \right\}$$

**2.12. Tits system** Recall Tits system

$$Bs_iB \cdot BwB = \begin{cases} Bws_iB & \ell(ws_i) = \ell(w) + 1 \\ BwB \cup Bws_iB & \text{otherwise} \end{cases}$$

Actually, we can say more that if $\ell(uv) = \ell(u) + \ell(v)$,

$$BuB \times_B BvB \longrightarrow BuvB$$

is an isomorphism.

**2.13.** For an element $w \in W$, we pick a reduced word $\underline{w} = (s_{i_1}, \ldots, s_{i_r})$ for $w$. Define the **Bott–Samelson variety** to be

$$\mathrm{BS}(\underline{w}) = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r}/B.$$

- Note that $\mathrm{BS}(\underline{w})$ is smooth, since it is iterated $\mathbb{P}^1$ bundle over $\mathbb{P}^1 \cong P_{i_r}/B$.
- the map $\mu : \mathrm{BS}(\underline{w}) \longrightarrow \Sigma_w$ induced by multiplication is birational by Tits system.

**2.14.** When $\ell(ws_i) = \ell(w) + 1$, then we have the following pull back square

$$\begin{array}{ccccccc} \cdots \times P_\bullet \times_B P_i/B & = & \mathrm{BS}(\underline{w} \oplus s_i) & \longrightarrow & \mathrm{BS}(\underline{w}) & = & \cdots \times P_\bullet/B \\ & & \downarrow & & \downarrow & & \\ & & G/B & \longrightarrow & G/P_i & & \end{array}$$

**2.15.** We may also consider

$$\begin{aligned} \widehat{\mathrm{BS}}(\underline{w}) &= G/B \underset{G/P_{i_1}}{\times} G/B \underset{G/P_{i_r}}{\times} \cdots \underset{G/P_{i_r}}{\times} G/B \\ &= P_{i_1} \times_B G/B \underset{G/P_{i_r}}{\times} \cdots \underset{G/P_{i_r}}{\times} G/B = \cdots \\ &= P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r} \times_B G/B \end{aligned}$$

So $\mathrm{BS}(\underline{w})$ is the fibre at $1 \cdot B/B$ of

$$\widehat{\mathrm{BS}(\underline{w})} \longrightarrow G/B.$$

**2.16.** We can also define the line bundle $\mathcal{O}(\lambda)$ on $\mathrm{BS}(\underline{w})$ by pull back from $G/B$. Actually, its total space is

$$P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r} \times_B \mathbb{C}(\lambda).$$

**2.17. Demazure Character Formula** For any reduced word $\underline{w}$ for $w$, for dominant $\lambda \in X(T)$,

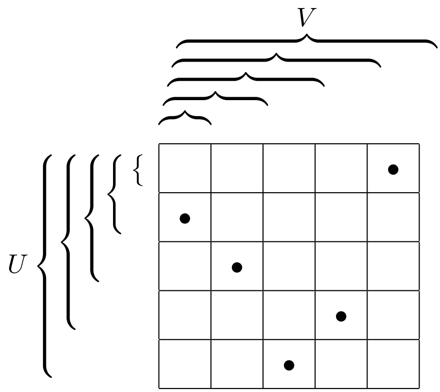
$$\operatorname{ch}\left(H^0(\mathrm{BS}(\underline{w}); \mathcal{O}(\lambda))^*\right) = \pi_w e^\lambda,$$

and

$$\forall i \geq 1, \quad H^i(\mathrm{BS}(\underline{w}); \mathcal{O}(\lambda)) = 0.$$

**2.18. Sketch of the Proof** Actually, the second assertion can be proved by spectral sequence argument as before. The first argument follows from the definition of Demazure operator — Bott–Samelson variety is the variety-theoretical composition of push forward and pull back.

**2.19.** For two flags $(0 \subseteq V_1 \subseteq \cdots \subseteq V_n)$ and $(0 \subseteq U_1 \subseteq \cdots \subseteq U_n)$, we can assume a permutation $w(U,V)$ as follows. There exists a set of basis $v_1, \ldots, v_n$ such that $V_i = \operatorname{span}(v_1, \ldots, v_i)$, and $U_i = \operatorname{span}(v_{w^{-1}(1)}, \ldots, v_{w^{-1}(i)})$. See Figure 2. Equivalently, $w(U,V)$ is the unique permutation $w$ with




$$\begin{pmatrix} 12345 \\ 23541 \end{pmatrix}$$

Figure 2: Relative Position

$$\dim \frac{V_i + U_{j+1} \cap V_{i+1}}{V_i + U_j \cap V_{i+1}} = \begin{cases} 1, & w(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Besides, it is also equivalent to the condition

$$\dim(U_j \cap V_i) = \#\{b \leq j, a \leq i : w(a) = b\}.$$

We pick a standard flag $(0 \subseteq V_1^0 \subseteq \cdots \subseteq V_n^0)$. Then

$$BwB/B = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : w(U, V^0) = w \right\}.$$

Its closure

$$\Sigma_w = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : \dim(U_j \cap V_i^0) \geq \#\{b \leq j, a \leq i : w(a) = b\} \right\}.$$

If we pick the opposite standard flag $(0 \subseteq V_1' \subseteq \cdots \subseteq V_n')$, then

$$\Sigma^w = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : (\dim U_j \cap V_i') \geq \#\{b \leq j, a \leq i : w_0 w(a) = b\} \right\}.$$

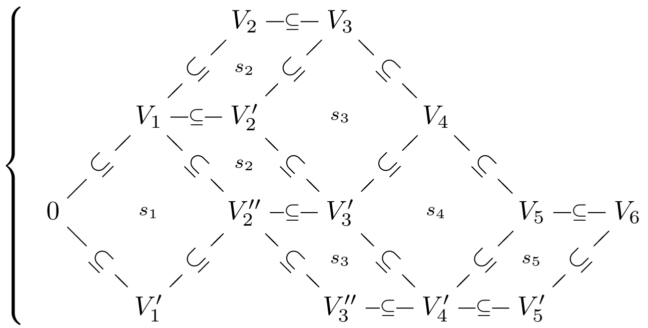
**2.20.** For the case $\mathcal{G}r(k,n)$, the shortest representive are in one-to-one correspondence with Young diagrams inside $k \times (n-k)$. To be exact, for a partition $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, the map $i \mapsto \lambda_{k+1-i} + i$ naturally extends to a permutation which is monotonous on $\{k+1, \ldots, n\}$. In this case,

$$\Sigma_\lambda = \left\{ V \in \mathcal{G}r(k,n) : \dim(V \cap V^0_{\lambda_{k+1-i}+i}) \geq i \right\},$$

$$\Sigma^\lambda = \left\{ V \in \mathcal{G}r(k,n) : \dim(V \cap V'_{n-k+i-\lambda_i}) \geq i \right\}.$$

See Figure 3

**2.21.** In the case $\mathrm{GL}_n$, we may regard $\widehat{\mathrm{BS}}(\underline{w})$ as flags of a given shape. For example, for $\mathrm{GL}_6$, for $\underline{w} = s_5 s_3 s_4 s_1 s_2 s_3 s_2$, $\widehat{\mathrm{BS}}(\underline{w})$ is




$: \dim V_i^{\cdots} = i$

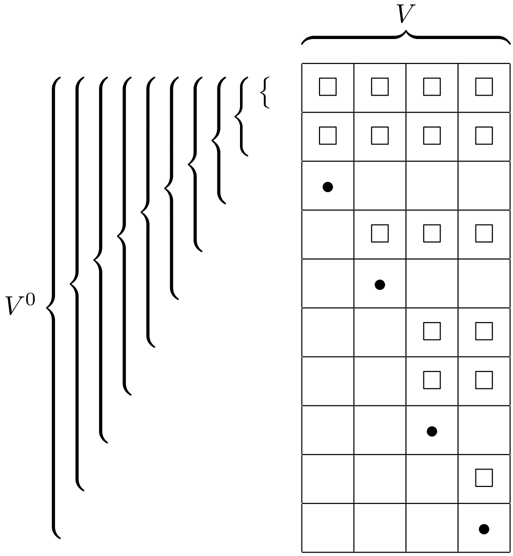
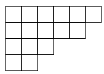



Figure 3: Schubert Cells 

The map $\widehat{\mathrm{BS}(\underline{w})} \to G/B$ corresponds to the topmost flag.

## References


- Kumar. Kac-Moody Groups, their Flag Varieties and Representation Theory.


# 3 Lecture 3 — Schur–Weyl Modules

**3.1.** Let $\mathcal{G}r(k,n)$ be the Grassmaniann. There is a line bundle $\mathcal{O}(1)$ defining plücker embedding. Let $\mathcal{V}$ be the tautological bundle of $\mathcal{G}r(k,n)$, that is, the fibre at $V \in \mathcal{G}r(k,n)$ is $V$ itself. Then $\mathcal{O}(1) = \Lambda^k \mathcal{V}^*$.

Denote the natural map $p : G/B \longrightarrow G/P$, that is, $p : \mathcal{F}\ell(n) \longrightarrow \mathcal{G}r(k,n)$ by picking $k$-th space. Then $p^*\mathcal{V}$ is exactly the $k$-th tautological bundle $\phi_k$. Thus it is not hard to $p^*\mathcal{O}(1) = \mathcal{O}(\omega_k)$, with $\omega_k = x_1 + \cdots + x_k$.

**3.2.** There is a

$$H^0(\mathcal{G}r(k,n);\mathcal{O}(d)) \longrightarrow H^0(\mathcal{F}\ell(n);\mathcal{O}(d\omega_k)).$$

It is injective, $G$-equivariant, and nonzero. So

$$H^0(\mathcal{G}r(k,n);\mathcal{O}(d))^* = L(d\omega_k)$$

This can also be seen from the proof of Borel–Weil theorem.

**3.3.** We have another point view of Grassmannian, that

$$\mathcal{G}r(k,n) = \mathrm{St}(k,n)/\operatorname{GL}_k$$

where $\mathrm{St}(k,n)$ is Stiefel variety, the space of $n \times k$ full rank matrices (geometrically, the space of all $k$-frames in $\mathbb{C}^n$). Then the total space of $\mathcal{O}(1)$ is

$$\mathrm{St}(k,n) \times_{\mathrm{GL}_k} \mathbb{C}(\det),$$

where $\mathbb{C}(\det)$ is the one-dimensional representation with $g \in \mathrm{GL}_k$ acts by $(\det g)^{-1}$.

**3.4.** Let us fix the coordinate

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \in \mathrm{St}(k,n).$$

A section of $\mathcal{O}(1)$ is then a map $f : \mathrm{St}(k,n) \to \mathbb{C}$ with $f(Xg) = \det(g)f(X)$. Such $f$ has to be linear combination of $\Delta^I := \det\left(x_{ij} : {}^{i\in I}_{j\in[k]}\right)$ for a subset $I \in \binom{[n]}{k}$. That is, $f$ can be view as alternating $k$-linear map on $\mathbb{C}^n$, sending $\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k$ to $f(X)$ where the $j$-th column of $X$ is $\mathbf{x}_j$. Thus, taking the dual, we get

$$L(\omega_k) = \Lambda^k \mathbb{C}^n,$$

as we expected.

**3.5.** In general, a section of $\mathcal{O}(d)$ is linear combination of $\Delta^{I_1}\Delta^{I_2}\cdots\Delta^{I_d}$ for $I_1,\dots,I_d\in\binom{[n]}{k}$. Thus

$$L(d\omega_k)^* = \operatorname{span}(\Delta^{I_1}\Delta^{I_2}\cdots\Delta^{I_d}) \subseteq \mathbb{C}\big[x_{ij} : {\scriptstyle 1\le i\le n \atop 1\le j\le k}\big] \subseteq \mathbb{C}[\mathrm{St}(k,n)]$$

Note that $\Delta^I$'s are not linear independent, the relation defining them is Plücker relations. So formally,

$$L(d\omega_k) = \mathsf{S}^d\big(\Lambda^k\mathbb{C}^n\big)/\text{Plücker relations}.$$

This is also as we expected.

**3.6.** Then consider the diagonal embedding

$$\mathcal{F}\ell(n) \longrightarrow \mathcal{G}r(1,n)\times\cdots\times\mathcal{G}r(n,n) =: \mathcal{G}r(1,\dots,n).$$

The the pull back $\mathcal{O}(c_1,\dots,c_n) := \mathcal{O}(c_1)\boxtimes\cdots\boxtimes\mathcal{O}(c_n)$ is exactly $\mathcal{O}(c_1\omega_1+\cdots+c_n\omega_n)$. Denote $\lambda = c_1\omega_1+\cdots+c_n\omega_n$. We get a restriction map

$$H^0(\mathcal{G}r(1,\cdots,n);\mathcal{O}(c_1,\dots,c_n)) \longrightarrow H^0(\mathcal{F}\ell(n);\mathcal{O}(\lambda)).$$

This is nonzero, $G$-equivariant, thus is surjective. Then we can compute over a dense subset. One choice is $\mathfrak{n}^- \cong w_0Bw_0B/B$. But it turns out, the next choice is the most convenient.

**3.7.** We use the map for the dense orbit

$$\kappa : B \longrightarrow G/B \qquad b \longmapsto bw_0\cdot B/B.$$

Then clear

$$H^0(\mathcal{F}\ell(n);\mathcal{O}(\lambda)) \xrightarrow{\kappa^*} H^0(B;\kappa^*\mathcal{O}(\lambda))$$

is injective. We use the coordinate

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn}\end{pmatrix} \in B.$$

Then the map $B\to\mathcal{F}\ell(n)\to\mathcal{G}r(k,n)$ factor through $\mathrm{St}(k,n)$ by

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn}\end{pmatrix} \longmapsto \begin{pmatrix} x_{1n} & \cdots & x_{1,n-k} \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & x_{n-k,n-k} \\ \vdots & \iddots & \\ x_{nn} & & \end{pmatrix}$$

Thus as $T$-module,

$$L(\lambda)^* \cong \operatorname{span}\left(\Delta^{I_1}(x_{ij}w_0)\cdots\Delta^{I_h}(x_{ij}w_0)\right)\Big|_{i>j\Rightarrow x_{ij}=0} \subseteq \mathbb{C}\big[x_{ij}\big]_{1\le i\le j\le n} \subseteq \mathbb{C}[B],$$

where $h = c_1 + \ldots + c_{n-1}$, $I_1, \ldots, I_{c_1} \in \binom{[n]}{1}$, $I_{c_1+1}, \ldots, I_{c_1+c_2} \in \binom{[n]}{2}$, etc.

**3.8.** For example, consider the case $\lambda = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2x_1 + x_2$ whose $c_1 = c_2 = 1$. Then the two maps factor through $\mathrm{St}(1,3)$ and $\mathrm{St}(2,3)$ is $\left(\begin{smallmatrix} x_{13} \\ x_{23} \\ x_{33}\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} x_{13} & x_{12} \\ x_{23} & x_{22} \\ x_{33} & \end{smallmatrix}\right)$, hence

$$L(\lambda)^* = \operatorname{span}\begin{pmatrix} x_{13}\cdot(x_{13}x_{22}-x_{12}x_{23}) & x_{13}\cdot x_{22}x_{33} & x_{13}\cdot x_{12}x_{33} \\ x_{23}\cdot(x_{13}x_{22}-x_{12}x_{23}) & x_{23}\cdot x_{22}x_{33} & x_{23}\cdot x_{12}x_{33} \\ x_{33}\cdot(x_{13}x_{22}-x_{12}x_{23}) & x_{23}\cdot x_{22}x_{33} & x_{23}\cdot x_{12}x_{33} \end{pmatrix}$$

Note that we have one relation

$$(x_{33})(x_{13}x_{22}-x_{12}x_{23}) = (x_{13})(x_{22}x_{33}) - (x_{23})(x_{12}x_{33}).$$

The dimension is $9-1=8$. The action of $T$ is on left, so $t\cdot X = t^{-1}X$. The character of is just the row number,

$$\mathrm{ch}(L(\lambda)) = (e^{x_1}+e^{x_2}+e^{x_3})(e^{x_1}e^{x_2}+e^{x_2}e^{x_3}+e^{x_1}e^{x_3}) - e^{x_3}(e^{x_1}e^{x_2})$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

**3.9.** If we use the natural map $G \to G/B$, with coordinate

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in G$$

Then as $G$-module,

$$L(\lambda)^* \cong \operatorname{span}\left(\Delta^{I_1}(x_{ij}w_0)\cdots\Delta^{I_h}(x_{ij}w_0)\right) \subseteq \mathbb{C}\big[x_{ij}\big]_{\substack{1\le i\le n \\ 1\le j\le n}} \subseteq \mathbb{C}[G],$$

where $h = c_1 + \ldots + c_{n-1}$, $I_1, \ldots, I_{c_1} \in \binom{[n]}{1}$, $I_{c_1+1}, \ldots, I_{c_1+c_2} \in \binom{[n]}{2}$, etc. As a result, taking the dual, we get

$$L(\lambda) \cong \mathsf{S}^{c_1}\left(\Lambda^1\mathbb{C}^n\right) \otimes \cdots \otimes \mathsf{S}^{c_n}\left(\Lambda^n\mathbb{C}^n\right) \Big/ \text{Plücker relations.}$$

This is exactly how Weyl construct representations.

**3.10.** Lastly, let us consider the analogy for Demazure characrer. In this case, we only need to exchange $w_0$ by any $w$. Note $\Sigma_w \longrightarrow \mathcal{G}r(1,\ldots,n-1)$ factor through $\mathcal{F}\ell(n)$, and by Demazure character formula,

$$H^0(G/B;\mathcal{O}(\lambda)) \longrightarrow H^0(\Sigma_w;\mathcal{O}(\lambda))$$

is also surjective (by induction). So theoretically, there is no problem.

**3.11.** But to be general, assume $D=(D_1,\ldots,D_h)$ is a series of subsets of $[n]$. Let us denote **flagged Weyl module**

$$M_D = \operatorname{span}\left(\prod_{i=1}^{h} \Delta_{D_i}^{C_i} : C_i \subseteq \binom{[n]}{\# D_i}\right) \subseteq \mathbb{C}[x_{ij}]_{1\le i\le j\le n},$$

where $\Delta_D^I$ is the determinant of sub-matrix $I \times D$ in

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn} \end{pmatrix}.$$

We define the **character** to be

$$\operatorname{ch}(M_D) = \sum_{\lambda\in X(T)} \dim(M_D)_\lambda e^\lambda, \qquad \operatorname{ch}^*(M_D) = \overline{\operatorname{ch}(M_D)}.$$

**3.12.** A hint:

$$\operatorname{ch}^*(\mathbb{C}\cdot x_{i_1 j_1}\cdots x_{i_r j_r}) = e^{x_{i_1}}\cdots e^{x_{i_r}} = X_{i_1}\cdots X_{i_r}.$$

**3.13.** For a composition $\lambda$ (i.e. a series of numbers ), define the **skyline diagram**

$$D(\lambda) : D(\lambda)_j = \{\bullet : \alpha_\bullet \ge j\}.$$

For example, $\lambda=(3,2,1,0,1)$,

| | | | |
|---|---|---|---|
| 3 | 1 | 1 | 1 |
| 2 | 2 | 2 | |
| 1 | 3 | | |
| 0 | | | |
| 1 | 5 | | |

$D_1 D_2 D_3 D_4 D_5$

Then $\kappa_\lambda(X) = \mathrm{ch}^*(M_{D(\lambda)})$. It involves some careful combinatorial translation which is left to readers.

**3.14.** For example $D = \{1, 2, 3, 5\}$,

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline \\ \hline 5\\ \hline\end{array} \qquad \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15}\\ & x_{22} & x_{23} & x_{24} & x_{25}\\ & & x_{33} & x_{34} & x_{35}\\ & & & x_{44} & x_{45}\\ & & & & x_{55} \end{pmatrix}$$

Its maximal minor (i.e. $4 \times 4$ minors), i.e. $\Delta_D^C$ has only two nonzero values,

$$x_{11}x_{22}x_{23}x_{45}, \qquad x_{11}x_{22}x_{23}x_{55}.$$

**3.15.** For example, when $D = [i] = \{1, \ldots, i\}$, then

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \vdots\\ \hline i\\ \hline\end{array} \qquad \begin{pmatrix} x_{11} & \cdots & x_{1i} & x_{1,i+1} & \cdots\\ & \ddots & \vdots & \vdots & \vdots\\ & & x_{ii} & x_{i,i+1} & \cdots\\ & & & x_{i+1,i+1} & \cdots\\ & & & & \ddots \end{pmatrix}$$

$$\operatorname{span}\left(\Delta_D^C : C \in \binom{[n]}{i}\right) = \mathbb{C} \cdot x_{11} \cdots x_{ii}$$

**3.16.** When $\lambda$ is weakly decreasing, each member of $D$ is of the form $[i]$. So $\kappa_\lambda = X^\lambda$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1\\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 \\ \cline{1-1}\end{array} \qquad \begin{aligned} &M_D = \mathbb{C} \cdot x_{11}^6 x_{22}^4 x_{33}.\\ &\kappa_{(6,4,1)} = X_1^6 X_2^4 X_3 \end{aligned}$$

**3.17.** When $\lambda = (0, 1, 2) = (1,2)$, the skyline diagram is

$$\begin{array}{c|c|c|}0 \\ \cline{2-2} 1 & 2 \\ \cline{2-3} 2 & 3 & 3 \\ \cline{2-3}\end{array}$$

Thus, $M_D$ is spanned by

$$\text{maxmial minors of } \begin{pmatrix} x_{12} & x_{13}\\ x_{22} & x_{23}\\ & x_{33}\end{pmatrix} \cdot \text{maxmial minors of } \begin{pmatrix} x_{13}\\ x_{23}\\ x_{33}\end{pmatrix}$$

It is essentially the same as we did before (up to a permutation of row indices).

**3.18.** The modern way to deal with vanishing of higher cohomology in representation and combinatorics is **Frobenius splitting**.

## References


- Peter Magyar. Schubert Polynomials and Bott-Samelson Varieties.


# 4 Lecture 4 — Horn's Problem

**4.1. Horn's problem** concerns the eigenvalues of sum of two Hermitian matrices with given eigenvalues. To be exact, given three Hermitian matrices $A, B, C$ with $A + B = C$, with eigenvalues

$$\begin{gathered}\lambda_1(A) \geq \cdots \geq \lambda_n(A)\\ \lambda_1(B) \geq \cdots \geq \lambda_n(B)\\ \lambda_1(C) \geq \cdots \geq \lambda_n(C)\end{gathered}$$

Horn's problem is to characterize these $3n$ values that appears in this way.

**4.2.** Let $M$ be a symplectic manifold. For each $f \in \mathcal{C}^\infty(M)$, we can define the **Hamitonian vector field** $\mathfrak{X}_f \in \mathfrak{X}(M)$ with

$$\omega(-, \mathfrak{X}_f) = df.$$

Then $\mathcal{C}^\infty$ has a **Poisson structure** defined by

$$\{f, g\} := \omega(\mathfrak{X}_f, \mathfrak{X}_g).$$

**4.3.** Let $M$ be a symplectic manifold with Hamitonian $G$-action. That is, the inducing map $\mathfrak{g} \to \mathfrak{X}(M)$ factor through

$$\mathfrak{g} \xrightarrow{H} \mathcal{C}^\infty(X) \xrightarrow{\mathfrak{X}} \mathfrak{X}(M),$$

with $H$ is a Lie algebra homomorphism. In this case, we can define the **moment map** $\mu : M \to \mathfrak{g}^*$ by dualizing $H$. That is,

$$^{M\ni}x \longmapsto \left[^{\mathfrak{g}\ni} X \mapsto H_X(x)_{\in\mathbb{R}}\right]_{\in\mathfrak{g}^*}.$$

**4.4. Atiyah, Guillemin and Sternberg Theorem** In the case $G = T$ is a compact torus and $X$ is compact, the image of moment map is a polytope with vertices $\mu(X^T)$ the image of fixed points of $X$. Actually, for any point $p$ on a $k$-face of this polytope, and $x \in \mu^{-1}(p)$, the orbit of $x$ is of dimension $k$.

**4.5.** For a Lie subgroup $H \subseteq G$, the restriction of $H$ is also Hamitonian, with moment map

$$\mu_H : M \to \mathfrak{h}^*$$

obtained by composition $\mu_G : M \to \mathfrak{g}^*$ with the restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$.

**4.6.** For two spaces $X, Y$ with Hamiltonian $G$-action. Then so is $X \times Y$. The the moment map $\mu_{X\times Y}$ satisfies

$$\mu_{X\times Y}(x, y) = \mu_X(x) + \mu_Y(y).$$

**4.7. Kirillov-Kostant-Souriau symplectic structure** For any Lie group $G$, Each coadjoint orbit $\mathbb{O}$ of $\mathfrak{g}^*$ can be equipped with a symplectic structure with moment map the natural inclusion of $\mathbb{O} \to \mathfrak{g}^*$.

**4.8.** Let $\mathfrak{h}_n$ be the space of Hermitian matrices. Consider $\mathfrak{u}_n$ the space of skew-Hermitian matrices. The pairing

$$\mathfrak{u}_n \times \mathfrak{h}_n \to \mathbb{R} \qquad (A, B) \longmapsto \operatorname{tr}(\mathbf{i} \cdot AB)$$

is perfect. So we can identify $\mathfrak{u}_n^* = \mathfrak{h}_n$.

**4.9.** Denote $\mathfrak{t} \cong \mathbf{i} \cdot \mathbb{R}^n$ the diagonal subalgebra of $\mathfrak{u}_n$. Then we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$. The restriction map $\mathfrak{u}_n^* \to \mathfrak{t}^*$ is then given by taking diagonal entries

$$\mathfrak{h}_n \longrightarrow \mathbb{R}^n \qquad (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \mapsto (a_{ii})_{i=1}^n.$$

**4.10.** Each coadjoint orbit is isomorphic to a partial flag veriety $G/P_\lambda$ where $P_\lambda$ is a parabolic subgroup.

Note that the $T$-fixed point of $\mathfrak{h}_n$ is exactly the diagonal matrices. As a result, the orbit of $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is exactly all the permutations of it.

**4.11. Schur–Horn's Theorem** There exists a Hermitian matrix $A$, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and diagonal entries $d_1 \geq \ldots \geq d_n$ if and only if

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \lambda_i, \quad (1 \leq k \leq n-1), \qquad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i.$$

Actually, this is equivalent to $(d_i)$ lies in the polytope spanned by permutations of $(\lambda_i)$.

**4.12.** For example, $\lambda_1 = 2$ and $\lambda_2 = 1$. Then $(d_1, d_2)$ should lies on the segment between $(1,2)$ and $(2,1)$. That is,

$$d_1 \leq 2, \qquad d_1 + d_2 = 3.$$

Actually, in this case $d_1 d_2 \geq 2$, thus some $z \in \mathbb{C}$ such that $|z|^2 = d_1 d_2 - 2$, then $\left(\begin{smallmatrix} d_1 & z \\ \bar{z} & d_2 \end{smallmatrix}\right)$ is the Hermitian matrix desired.

**4.13. Horn's conjecture** There exists three Hermitian matrices $A, B, C$ with $A + B = C$, with eigenvalues

$$\begin{matrix} \lambda_1(A) \geq \cdots \geq \lambda_n(A) \\ \lambda_1(B) \geq \cdots \geq \lambda_n(B) \\ \lambda_1(C) \geq \cdots \geq \lambda_n(C) \end{matrix}$$

if and only if

$$\sum_{i=1}^{n} \lambda_i(A) + \sum_{i=1}^{n} \lambda_i(B) = \sum_{i=1}^{n} \lambda_i(C)$$

for any $k$ and $I, J, K \subseteq \binom{[n]}{k}$ with $c_{\lambda(J)}^{\lambda(I)\lambda(J)} \neq 0$

$$\sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B) \geq \sum_{k \in K} \lambda_k(C).$$

Here, $\lambda(I) = \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ is the partition with $I = \{\lambda_1 + k, \ldots, \lambda_k + 1\} \subseteq [n]$, and $c_{\lambda\mu}^{\nu}$ the Littlewood–Richardson coefficient.

**4.14. Proof of "⇒"** For a Hermintian matrix $A$, we define the Rayleigh trace over $\mathcal{G}r(k,n)$ by

$$R_A : \mathcal{G}r(d,n) \longrightarrow \mathbb{R} \qquad V \longmapsto \sum_{i=1}^{k} \mathbf{x}_i^{\mathrm{t}} A \mathbf{x}_i$$

with $\mathbf{x}_1, \dots, \mathbf{x}_k$ a choice of orthogonal normal basis of $V$. Then

$$\sum_{i \in I} \lambda_i(A) = \min_{x \in \Sigma_{\lambda(I)}(A)} R_A(x),$$

where $\Sigma_{\lambda(I)}(A)$ is the Schubert variety corresponding to the flag $(0 \subseteq V_1 \subseteq \cdots V_n)$ with $V_i$ spanned by the first $k$ eigenvectors (with eigenvalues weakly decreasing).

Note that $\Sigma_{\lambda(I)}(A) \cap \Sigma_{\lambda(I)}(B) \cap \Sigma_{\lambda(I)^c}(C) = \varnothing$ implies $c^{\lambda(K)}_{\lambda(I)\lambda(J)} = 0$. Thus we get the condition stated in the theorem.

**4.15. Sketch of "$\Leftarrow$"** We should use some convex properties for non-torus. Let $C \subseteq \mathfrak{t}^*$ be any Weyl chamber. We have a map

$$\phi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^* / \operatorname{ad} G \cong \mathfrak{t}^*/W \cong C$$

where $C$ is a closed Weyl chamber. Kirwan's theorem claims that the image $\phi \circ \mu_G$ is convex.

**4.16.** Let $X$ be a projective variety with an very ample line bundle $\mathcal{L}$. Denote

$$\Gamma^\bullet(\mathcal{L}) = \bigoplus_{n \geq 0} H^0(X; \mathcal{L}^{\otimes n}).$$

Then $\operatorname{Proj} \Gamma^\bullet(\mathcal{L}) = X$. Actually, $\Gamma^\bullet(\mathcal{L})$ is the projective coordinate ring for $X$ in $\mathbb{P}^N$.

**4.17.** For example, when $X = \mathcal{F}\ell(n)$, and $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$ with $\lambda_1 > \cdots > \lambda_n$, then

$$\Gamma^\bullet(\mathcal{O}(\lambda)) = \bigoplus_{d \geq 0} L(d\lambda)^*.$$

In general, if $\lambda_1 \geq \cdots \geq \lambda_n$, we can find a partial flag variety $G/P_\lambda$ with the same property.

**4.18.** Over $\mathbb{P}^{n-1}_{\mathbb{C}}$, there is a natural Fubini–Study symplectic structure

$$\omega = \frac{|dx_1|^2 + \cdots + |dx_n|^2}{|x_1|^2 + \cdots + |x_n|^2}$$

The action of $U(n)$ on $\mathbb{P}^1$ is Hamiltonian with moment map

$$\mu(\ell) = \text{rank one Hermitian matrix projecting to } \ell \in \mathfrak{h}_n.$$

That is, view $\mathbb{P}^{n-1}$ as the orbit of $\operatorname{diag}(1, \ldots, 0) \in \mathfrak{h}_n$. Thus the $T$ action moment map is

$$\mu = \left( \frac{|x_1|^2}{|x_1|^2 + \cdots + |x_n|^2}, \ \ldots, \ \frac{|x_n|^2}{|x_1|^2 + \cdots + |x_n|^2} \right).$$

**4.19. Kirwan–Ness Theorem** If a compact group $K \subseteq U(N)$ acts on a smooth closed subvariety $X$ of $\mathbb{P}^N_{\mathbb{C}}$. Denote the moment map

$$\mu : X \xrightarrow{\subseteq} \mathbb{P}^N \longrightarrow \mathfrak{u}^* \longrightarrow \mathfrak{k}^*$$

Denote the complexification of $K$ by the reductive group $G$. Then

$$\mu^{-1}(0)/K \longrightarrow X//G \quad \text{(GIT quotient)}.$$

**4.20.** Denote $\mathbb{O}(\lambda)$ the space of Hermitian matrices with eigenvalue $\lambda_1, \ldots, \lambda_n$ for a weakly decreasing integer sequence. We then apply above theorem to

$$\mathbb{O}(\lambda) \times \cdots \times \mathbb{O}(\mu)$$

with $\lambda, \ldots, \mu$ weakly decreasing integer sequences. Actually, the moment map cooresponds to the Plücker embedding (by computation)

$$\begin{array}{ccc}
\mathbb{O}(\lambda) \times \cdots \times \mathbb{O}(\mu) & \longrightarrow & \mathfrak{h}_n \\
\downarrow & & \uparrow \\
\mathbb{P}(L(\lambda)) \times \cdots \times \mathbb{P}(L(\mu)) & \longrightarrow & \mathfrak{h}_{??} \times \cdots \times \mathfrak{h}_{??} \\
\downarrow & & \uparrow \\
\mathbb{P}(L(\lambda) \otimes \cdots \otimes L(\mu)) & \longrightarrow & \mathfrak{h}_{??}
\end{array}$$

Thus $\mu^{-1}(0)$ is nonempty if and only if

$$\operatorname{Proj} \bigoplus_{d \geq 0} \Big( L(d\lambda) \otimes \cdots \otimes L(d\mu) \Big)^{\mathrm{GL}_n}$$

is nonempty. Equivalently, $\Big( L(d\lambda) \otimes \cdots \otimes L(d\mu) \Big)^{\mathrm{GL}_n} \neq 0$ for some $d \geq 1$.

**4.21.** There exist Hermitian matrices $A, \ldots, B$ with $A + \cdots + B = 0$ with eigenvalues $\lambda(A), \ldots, \lambda(B)$ if and only if

$$\Big( L(d\lambda(A)) \otimes \cdots \otimes L(d\lambda(B)) \Big)^{\mathrm{GL}_n} \neq 0$$

for some $d \geq 1$. By a limit argument, this method also solves Horn's problem.

**4.22.** Since the Littlewood–Richardson coefficients has a combinaotrial model by honeycomb due to Knutson and Tao, see Figure 4.

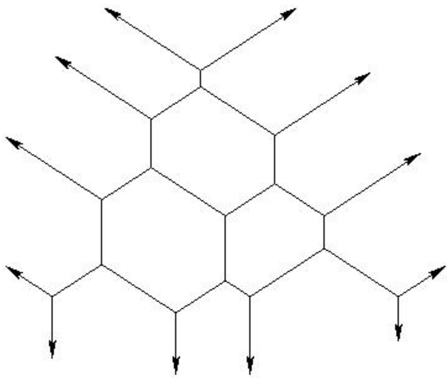


Figure 4: Honeycomb

# References


- Knutson. The symplectic and algebraic geometry of Horn's problem. [arXiv]

- Fulton. Eigenvalues, invariant factors, highest weights, and Schubert calculus. [AMS]
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# 5 Appendix: Schubert Calculus

**5.1.** Actually, by an affine paving argument

$$K_B(G/P) = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\mathcal{O}^w]_B = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\mathcal{O}_w]_B$$

where $\mathcal{O}^w = i_* \mathcal{O}_{\Sigma^w}$ the push forward of regular ring of $\Sigma^w$ and similarly notation for $\mathcal{O}_w$. It turns out

$$\pi_i[\mathcal{O}^w] = \begin{cases} [\mathcal{O}^{ws_i}] & \ell(ws_i) = \ell(w) - 1 \\ [\mathcal{O}^w] & \text{otherwise} \end{cases}$$

or

$$\pi_i[\mathcal{O}_w] = \begin{cases} [\mathcal{O}_{ws_i}] & \ell(ws_i) = \ell(w) + 1 \\ [\mathcal{O}_w] & \text{otherwise} \end{cases}$$

Note that the second case follows from the first, since $\pi_i^2 = \pi_i$. The first case follows from the fact that the push forward induced by $\mathrm{BS}(\underline{w}) \longrightarrow \Sigma_w$ sending $[\mathcal{O}_{\mathrm{BS}(\underline{w})}]$ to $[\mathcal{O}_{\Sigma_w}]$. To be exact, it has no higher cohomology by a spectral sequence argument, and preserves structure sheaf by applying Zariski connected theorem on Stein decomposition).

**5.2.** Assume $P = \bigcup_{w \in W_J} BwB$ for $J \subseteq \mathbb{I}$. If we denote

$$R_T = \mathbb{Q}[e^\lambda]_{\lambda \in X(T)}, \qquad R_G = R_T^W, \qquad R_P = R_T^{W_J},$$

then

$$K_B(G/B; \mathbb{Q}) = R_T \otimes_{R_G} R_T, \qquad K_B(G/P; \mathbb{Q}) \cong R_T \otimes_{R_G} R_P.$$

For $G/B$, the class of $\mathcal{O}(\lambda)$ is presented by $1 \otimes e^\lambda \in R_T \otimes_{R_G} R_T$. The class of pull back of $e^\lambda \in K_B(\mathrm{pt}; \mathbb{Q}) = R_T$ is $e^\lambda \otimes 1$.

**5.3.** The natural map $G/B \longrightarrow G/P$ induces

$$K_B(G/P) \longrightarrow K_B(G/B) \qquad [\mathcal{O}^w] \longmapsto [\mathcal{O}^w]$$

thus an injection. The corresponding $\mathbb{Q}$-efficient map is just the inclusion

$$R_T \otimes_{R_G} R_P \xrightarrow{\subseteq} R_T \otimes_{R_G} R_T.$$

**5.4. Atiyah–Bott–Berline–Vergne** Let $X$ be a smooth projective variety algebraically acted by an algebraic torus $T$. Then the localization, i.e. the restriction to the fixed points

$$K_T(X) \longrightarrow K_T(X^T)$$

is an isomorphism after tensoring with $\operatorname{Frac} R_T$.

In particular, if $K_T(X)$ is a free $K_T(\mathrm{pt})$-module, then the localization map is injective.

**5.5.** The class of $[\mathcal{O}_{\Sigma^w}]_B$ in $K_B(G/B) = R_T \otimes_{R_G} R_T$ is called the **double Grothendieck polynomial** $\mathfrak{G}_w(x,t)$. Here we take the convention that

$$e^{\lambda(t)} = e^\lambda \otimes 1, \qquad e^{\lambda(x)} = 1 \otimes e^\lambda.$$

Then by localization

$$\forall u \not\leq w, \quad \mathfrak{G}_w(ut,t) = 0.$$

Actually, $\mathfrak{G}_w(x,t)$ is uniquely determined by

- $\mathfrak{G}_{\mathrm{id}}(x,t) = 1$;
- $\pi_i \mathfrak{G}_w(x,t) = \mathfrak{G}_{ws_i}(x,t)$ when $\ell(ws_i) = \ell(w) - 1$;
- $\mathfrak{G}_w(t,t) = \delta_{w=\mathrm{id}}$.

**5.6.** In type $A$, recall that we denote $X_i = e^{x_i}$, and

$$\pi_i f = \frac{X_i f - s_i(X_{i+1} f)}{X_i - X_{i+1}}.$$

We have the stable choice

$$\mathfrak{G}_{w_0}(X,Y) = \prod_{i+j\leq n} \left(1 - \frac{Y_i}{X_i}\right).$$

**5.7.** Denote $T_1, \dots, T_{n-1}$ the symbols with

$$T_i^2 = -T_i, \qquad \begin{cases} T_iT_j = T_jT_i & |i-j| \geq 2 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{cases}$$

Thus $T_w$ can be defined. We consider the generating function

$$\mathfrak{G}(X,Y) = \sum \mathfrak{G}_w(X,Y)T_w,$$

It is amazing that it factors into

$$\begin{array}{ccccc} h_{n-1}(X_1,Y_{n-1}) & h_{n-2}(X_1,Y_{n-2}) & \cdots & h_1(X_1,Y_2) & h_1(X_1,Y_1) \\ & h_{n-1}(X_2,Y_{n-2}) & \cdots & h_3(X_2,Y_2) & h_2(X_1,Y_1) \\ & & \ddots & \vdots & \vdots \\ & & & h_{n-1}(X_{n-2},Y_2) & h_{n-2}(X_{n-2},Y_1) \\ & & & & h_{n-1}(X_{n-1},Y_1) \end{array}$$

where $h_k(X,Y) = 1 + \big(1 - \frac{X}{Y}\big)T_k$.

**5.8.** The cohomological version is similar. In this case, the cohomological Demazure operator

$$\partial_i : H_G^\bullet(G/B) \xrightarrow{(p_i)_*} H_G^\bullet(G/P_i) \xrightarrow{(p_i)^*} H_G^\bullet(G/B)$$

is given by

$$\partial_i f = \frac{f - s_i f}{\alpha_i},$$

where $\alpha_i = c_1(\mathcal{O}(\alpha_i))$. It satisfies $\partial_i^2 = 0$ and braid relations.

**5.9.** In the cohomological case, we need to replace

$$R_T^\bullet = \mathsf{S}^\bullet(X(T)_\mathbb{Q}), \qquad R_G = R_T^W, \qquad R_P = R_T^{W_J}.$$

then

$$H_B^\bullet(G/B;\mathbb{Q}) = R_T \otimes_{R_G} R_T, \qquad H_B^\bullet(G/P;\mathbb{Q}) \cong R_T \otimes_{R_G} R_P.$$

For $G/B$, $c_1(\mathcal{O}(\lambda)$ is presented by $1 \otimes \lambda \in R_T \otimes_{R_G} R_T$. The class of pull back of $\lambda \in H_B^\bullet(\mathrm{pt};\mathbb{Q}) = R_T$ is $\lambda \otimes 1$.

**5.10.** By a similar affine paving argument

$$H_B^\bullet(G/P) = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\Sigma^w]_B = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\Sigma_w]_B$$

It turns out

$$\partial_i[\Sigma^w] = \begin{cases} [\Sigma^{ws_i}] & \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$\partial_i[\Sigma_w] = \begin{cases} [\Sigma_{ws_i}] & \ell(ws_i) = \ell(w) + 1 \\ 0 & \text{otherwise} \end{cases}$$

**5.11.** The natural map $G/B \longrightarrow G/P$ induces

$$H_B^\bullet(G/P) \longrightarrow H_B^\bullet(G/B) \qquad [\Sigma^w] \longmapsto [\Sigma^w]$$

thus an injection. The corresponding $\mathbb{Q}$-efficient map is just the inclusion

$$R_T \otimes_{R_G} R_P \xrightarrow{\subseteq} R_T \otimes_{R_G} R_T.$$

**5.12.** The class of $[\Sigma^w]_B$ in $K_B(G/B) = R_T \otimes_{R_G} R_T$ is called the **double Schubert polynomial** $\mathfrak{S}_w(x,t)$. Here we take the convention that

$$\lambda(t) = \lambda \otimes 1, \qquad \lambda(x) = 1 \otimes \lambda.$$

Actually, $\mathfrak{S}_w(x,t)$ is uniquely determined by

- $\mathfrak{S}_{\mathrm{id}}(x,t) = 1$;
- $\partial_i \mathfrak{S}_w(x,t) = \mathfrak{S}_{ws_i}(x,t)$ when $\ell(ws_i) = \ell(w) - 1$;
- $\mathfrak{S}_w(t,t) = \delta_{w=\mathrm{id}}$.

**5.13.** For $\mathcal{G}r(k,n)$, the case $w$ is shortest, $\mathfrak{S}_w(x,t)$ is the corresponding double Schur polynomial.

**5.14.** In type $A$,

$$\pi_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

We have the stable choice

$$\mathfrak{S}_{w_0}(x,y) = \prod_{i+j \le n} (x_i - y_j).$$

Denote $T_1, \ldots, T_{n-1}$ the symbols with

$$T_i^2 = 0, \qquad \begin{cases} T_i T_j = T_j T_i & |i-j| \geq 2 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \end{cases}$$

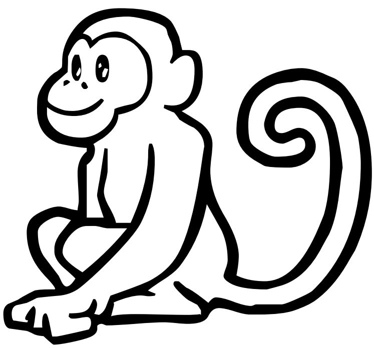
Thus $T_w$ can be defined. We consider the generating function

$$\mathfrak{S}(x,y) = \sum \mathfrak{S}_w(x,y) T_w,$$

It is amazing that it factors into

$$\begin{array}{ccccc} h_{n-1}(x_1, y_{n-1}) & h_{n-2}(x_1, y_{n-2}) & \cdots & h_1(x_1, y_2) & h_1(x_1, y_1) \\ & h_{n-1}(x_2, y_{n-2}) & \cdots & h_3(x_2, y_2) & h_2(x_1, y_1) \\ & & \ddots & \vdots & \vdots \\ & & & h_{n-1}(x_{n-2}, y_2) & h_{n-2}(x_{n-2}, y_1) \\ & & & & h_{n-1}(x_{n-1}, y_1) \end{array}$$

where $h_k(X,Y) = 1 + (x-y)T_k$.



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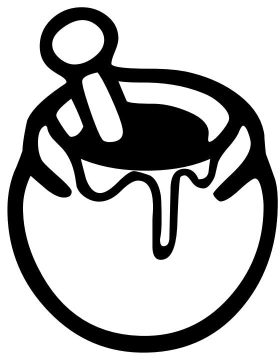

Honey