Pieri Rules over Grassmannian and Applications
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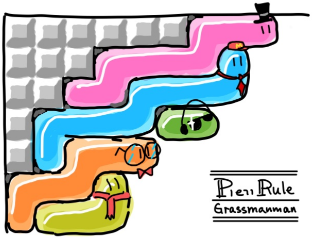
with Neil J.Y. Fan, Peter L.Guo and Changjian Su

Rui Xiong



Grassmannian

Recall that **Grassmannian manifold**

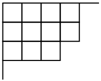
$$\mathrm{Gr}(k,n) = \{V \subseteq \mathbb{C}^n : \dim V = k\}.$$

We have the following **Bruhat decomposition**

$$\mathrm{Gr}(k,n) = \bigcup_{\lambda \subseteq (n-k)^k} Y(\lambda)^\circ \quad \text{(disjoint)},$$

where $Y(\lambda)^\circ$ is the **opposite Schubert cell**.

Description

For example, $V \in Y\Big($  $\Big)^{\circ} \iff$

codim=4: $\phi_1 \cap V = \phi_0 \cap V$

$\cap$

codim=3: $\phi_2 \cap V$

$\cap$

codim=2: $\phi_4 \cap V = \phi_3 \cap V$

$\cap$

codim=1: $\phi_8 \cap V = \phi_7 \cap V = \phi_6 \cap V = \phi_5 \cap V$

$\cap$

codim=0: $\phi_9 \cap V$

where $\phi_i = \text{span}(\mathbf{e}_n, \ldots, \mathbf{e}_{n-i+1})$.

Bruhat Decomposition

Denote **Schubert variety**

$$Y(\lambda) = \overline{Y(\lambda)^\circ} = \bigcup_{\mu \supseteq \lambda} Y(\mu)^\circ.$$

The **cohomology group** and **K-group** can be computed to be

$$H^*\big(\mathrm{Gr}(k,n)\big) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [Y(\lambda)].$$

$$K\big(\mathrm{Gr}(k,n)\big) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [\mathcal{O}_{Y(\lambda)}]$$

Chern Classes

Let $\mathcal{V}$ be the **tautological bundle** over $\mathrm{Gr}(k, n)$. We denote

$$c_r = c_r(\mathcal{V}^\vee) = \text{the } r\text{-th equivariant } \textbf{Chern classes} \text{ of } \mathcal{V}^\vee.$$

It is known that

$$H^*\big(\,\mathrm{Gr}(k, n)\big) = \mathbb{Q}[c_1, \ldots, c_k]\big/\text{some ideal.}$$

$$K\big(\,\mathrm{Gr}(k, n)\big) = \mathbb{Q}[c_1, \ldots, c_k]\big/\text{some ideal.}$$

Geometry of cohomology

Roughly speaking **cohomology**

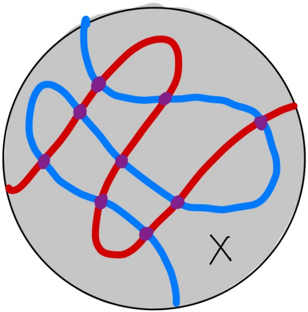
$$H^\bullet(X) = \bigoplus_{Y \text{ closed } \subseteq X} \mathbb{Q} \cdot [Y] \Big/ \text{HOMOTOPY EQUIVALENCE}$$

with product **the transversal intersection**

$$[Y_1] \cdot [Y_2] = [Y_1 \pitchfork Y_2].$$

Over $\mathrm{Gr}(k, n)$, we have **Schubert class**

$$[Y(\lambda)] \in H^{2\ell(\lambda)}(\mathrm{Gr}(k, n)).$$



Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

We have

$$Y(\square) = \text{the point } \infty,$$

$$Y(\varnothing) = \text{the entire } \mathbb{P}^1.$$

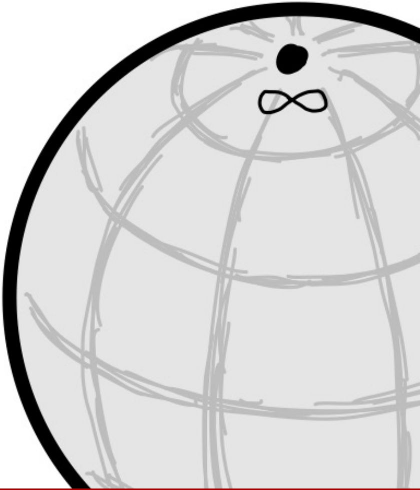
The intersection

$\pitchfork$	pt	$\mathbb{P}^1$
pt	$\varnothing$	pt
$\mathbb{P}^1$	pt	$\mathbb{P}^1$

The cohomology

$$H^*(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = [Y(\square)]$.

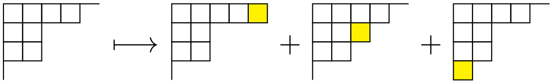


Classical Chevalley Formula

Theorem (Chevalley Formula)

$$c_1(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{\mu = \lambda + \square} [Y(\mu)].$$

Example:



Classical Pieri Rule

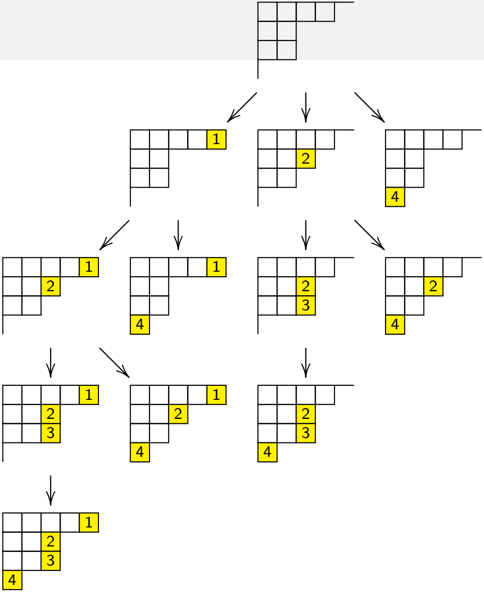
Let us denote **Schur operators**

$$[i] \to [Y(\lambda)] = \begin{cases} [Y(\mu)], & \mu = \lambda + \Box \text{ in the } i\text{-th row}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Pieri Rule)

$$c_r(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{1 \le i_1 < \cdots < i_r \le k} [i_r] \to \cdots [i_1] \to [Y(\lambda)].$$

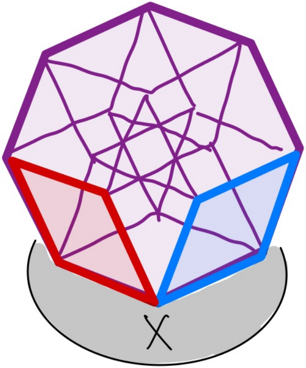
Example



Geometry of K-theory

Roughly speaking, the **K-theory**

$$K(X) = \bigoplus_{\mathcal{F} \in \mathcal{C}\mathrm{oh}\, X} \mathbb{Q} \cdot [\mathcal{F}] \Big/ \begin{matrix}\text{EXACT} \\ \text{SEQUENCES}\end{matrix}$$



with product **the tensor product**

$$[\mathcal{F}_1] \cdot [\mathcal{F}_2] = [\mathcal{F}_1 \otimes \mathcal{F}_2].$$

Over $\mathrm{Gr}(k, n)$, we have **structure sheaves**

$$[\mathcal{O}_{Y(\lambda)}] \in K(\mathrm{Gr}(k, n)).$$

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

We have

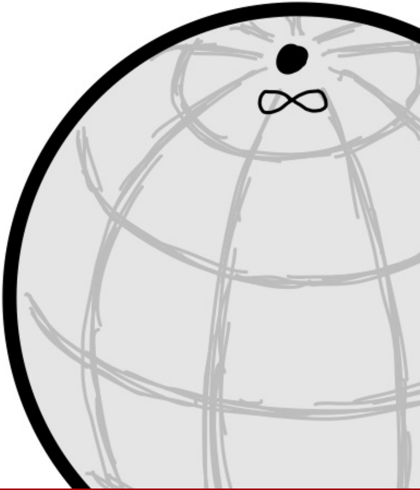
$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_\infty \to 0.$$

Thus $[\mathcal{O}_\infty] = 1 - \mathcal{O}(-1)$.

The K-theory

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = 1 - \mathcal{O}(-1) = [\mathcal{O}_{Y(\square)}]$.

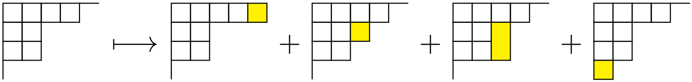


K-theory Chevalley Formula

Theorem (Lenart [1])

$$c_1(\mathcal{V}^\vee)\cdot[\mathcal{O}_{Y(\lambda)}] = \sum_{\mu=\lambda+\square}[\mathcal{O}_{Y(\mu)}].$$

Example:



K-theory Pieri Rule

Let us denote **Schur operators**

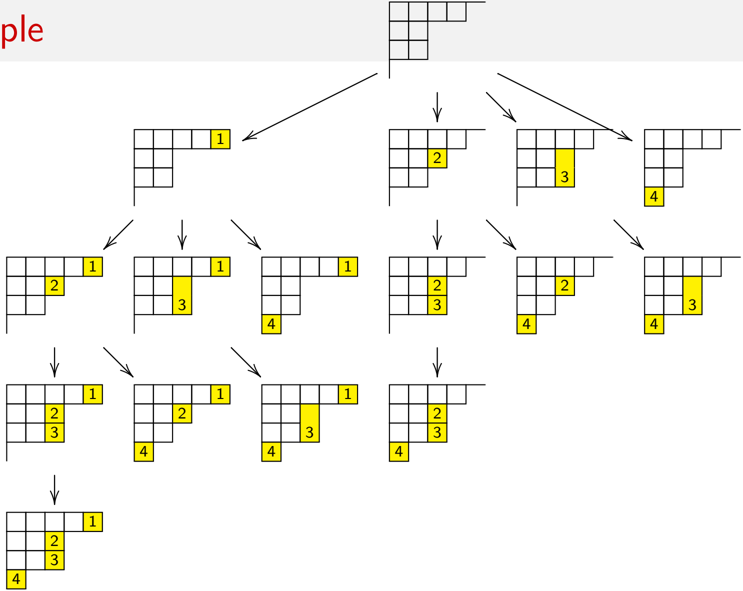
$$[i] \to [\mathcal{O}_{Y(\lambda)}] = [\mathcal{O}_{Y(\mu)}]$$

where $\mu = \lambda +$ a vertical strip with its tail at the i-th row.

Theorem (Lenart [1])

$$c_r(\mathcal{V}^\vee) \cdot [\mathcal{O}_{Y(\lambda)}] = \sum_{1 \le i_1 < \cdots < i_r \le k} [i_r] \to \cdots [i_1] \to [\mathcal{O}_{Y(\lambda)}].$$

Example



Constructible Functions

Consider

$$\begin{aligned}\mathrm{Fun}(X) &= \{\text{constructible functions over } X\} \\ &= \mathrm{span}(\mathbf{1}_A : A \subseteq X \text{ closed}).\end{aligned}$$

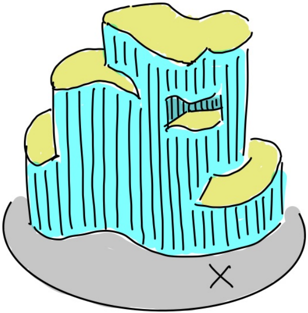
For any proper map $f : X \to Y$, we have a **push-forward**

$$f_* : \mathrm{Fun}(X) \to \mathrm{Fun}(Y)$$

defined such that

$$(f_*(\mathbf{1}_A))(y) = \chi_c(A_y)$$

the Euler characteristic of fibre A_y.



CSM classes

By MacPherson [2], there is a natural transform (wrt push-forward) called **Chern–Schwartz–MacPherson classes**

$$c_{\mathsf{SM}} : \mathsf{Fun}(-) \to H_\bullet(-),$$

such that when X is smooth

$$c_{\mathsf{SM}}(X) = \text{total Chern class of the tangent bundle of } X.$$

Over $\mathrm{Gr}(k,n)$, we have **CSM classes**

$$c_{\mathsf{SM}}(Y(\lambda)^\circ) := c_{\mathsf{SM}}(\mathbf{1}_{Y(\lambda)^\circ}) \in H^*(\mathrm{Gr}(k,n)).$$

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

Recall

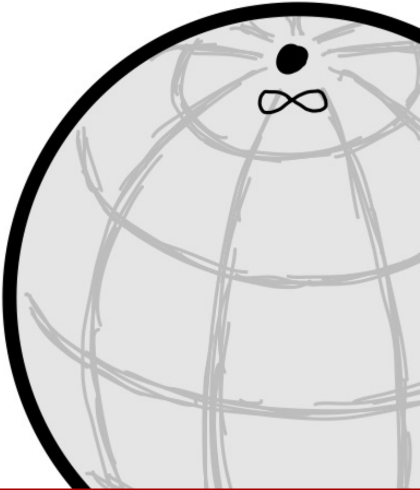
$$Y(\square)^\circ = \text{the point } \infty,$$
$$Y(\varnothing)^\circ = \mathbb{P}^1 \setminus \{\infty\}.$$

So by definition,

$$c_{\mathrm{SM}}(Y(\square)^\circ) = [Y(\square)] = x.$$

Since $\mathscr{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{array}{r} \text{total Chern class} = 1 + 2x \\ \hline c_{\mathrm{SM}}(Y(\square)^\circ) = \phantom{1 + {}} x \\ c_{\mathrm{SM}}(Y(\varnothing)^\circ) = 1 + x \end{array}$$

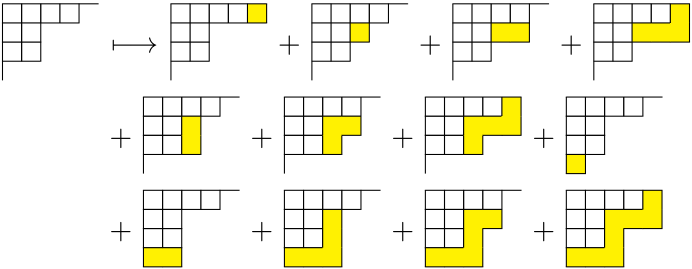


CSM Chevalley formula

Theorem (Aluffi, Mihalcea, Schürmann and Su [3])

$$c_1(\mathcal{V}^\vee)\cdot c_{\mathrm{SM}}(Y(\lambda)^\circ) = \sum_{\mu=\lambda+} c_{\mathrm{SM}}(Y(\mu)^\circ).$$

Example:



CSM Pieri Rule

Let us denote **ribbon Schubert operators**

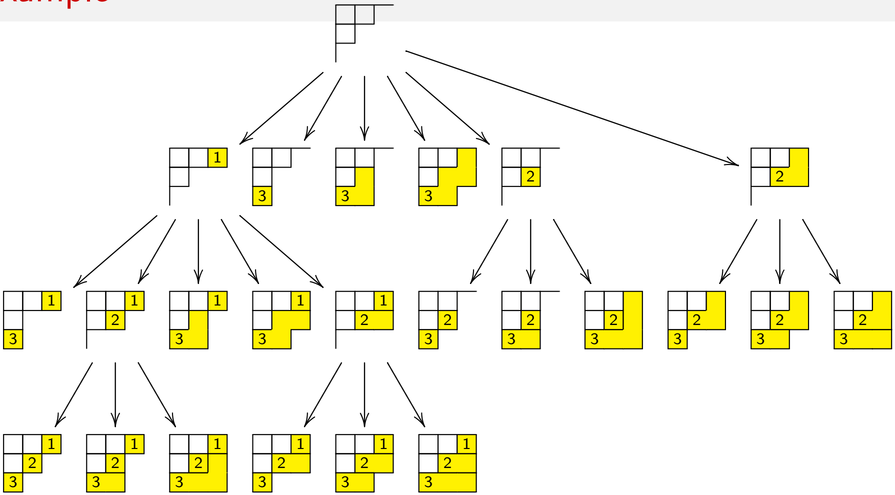
$$[i\,|\to c_{\mathrm{SM}}(Y(\lambda)^\circ)=\sum_{\mu}c_{\mathrm{SM}}(Y(\mu)^\circ)$$

where the sum over $\mu=\lambda+$ a ribbon strip with its tail at the i-th row.

Theorem (Fan, Guo and Xiong [4])

$$c_r(\mathcal{V}^\vee)\cdot c_{\mathrm{SM}}(Y(\lambda)^\circ)=\sum_{1\le i_1<\cdots<i_r\le k}[i_r\,|\to\cdots[i_1\,|\to c_{\mathrm{SM}}(Y(\lambda)^\circ).$$

Example



Grothendieck group

Consider the **Grothendieck group of varieties**

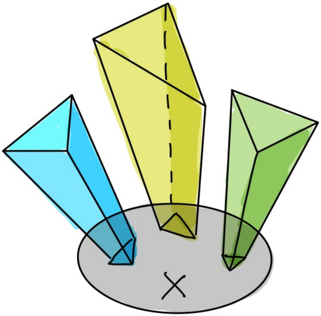
$$\mathrm{G}(X) = \bigoplus_{\text{variety } Z\to X} \mathbb{Z}\cdot[Z\to X] \Big/ [U\to X]+[Z\setminus U\to X]=[Z\to X].$$

For any proper map $f: X\to Y$, we have a **push-forward**

$$f_*: \mathrm{G}(X)\to\mathrm{G}(Y)$$

with

$$f_*[Z\to X] = [Z\to X\to Y].$$



Motivic Chern classes

By Brasselet, Schürmann and Yokura [5], there is a natural transform (wrt push-forward) called **motivic Chern classes**

$$\mathrm{MC}_y : \mathrm{G}(-) \to K(-)[y],$$

such that when X is smooth,

$$\mathrm{MC}_y(X) = \lambda\text{-class} = \sum_{k=1}^{\dim X} y^k[\Lambda^k \mathscr{T}_X^\vee].$$

Over $\mathrm{Gr}(k,n)$, we have **motivic Chern classes**

$$\mathrm{MC}_y(Y(\lambda)^\circ) := \mathrm{MC}_y([Y(\lambda)^\circ \to \mathrm{Gr}(k,n)]) \in K(\mathrm{Gr}(k,n)).$$

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

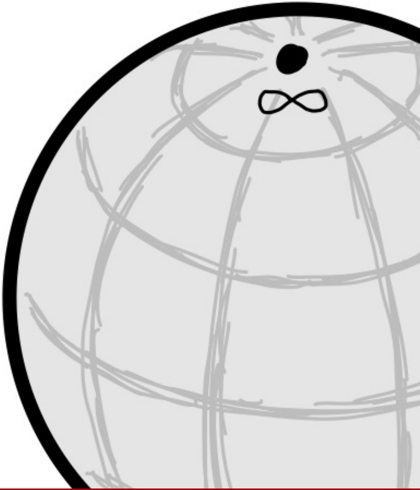
Similarly,

$$MC_y(Y(\square)^\circ) = [\mathcal{O}_{Y(\square)}] = x.$$

Recall that $x = 1 - \mathcal{O}(-1)$.

Since $\mathscr{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{aligned} \lambda\text{-class} &= 1 + y\mathcal{O}(-2) \\ &= (1+y) - 2yx \\ \hline MC_y(Y(\varnothing)^\circ) &= (1+y) - (2y+1)x \\ MC_y(Y(\square)^\circ) &= x \end{aligned}$$

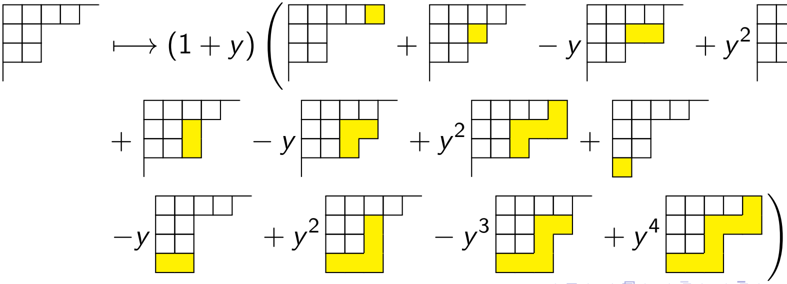


MC Chevalley formula

Theorem (Fan, Guo, Su and Xiong)

$$c_1(\mathcal{V}^\vee)\cdot \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y) \sum_{\mu=\lambda+\,\square} (-y)^{\mathrm{wd}(\mu/\lambda)-1}\, \mathrm{MC}_y(Y(\mu)^\circ).$$

Example:



MC Pieri Rule

Let us denote **ribbon Schubert operators**

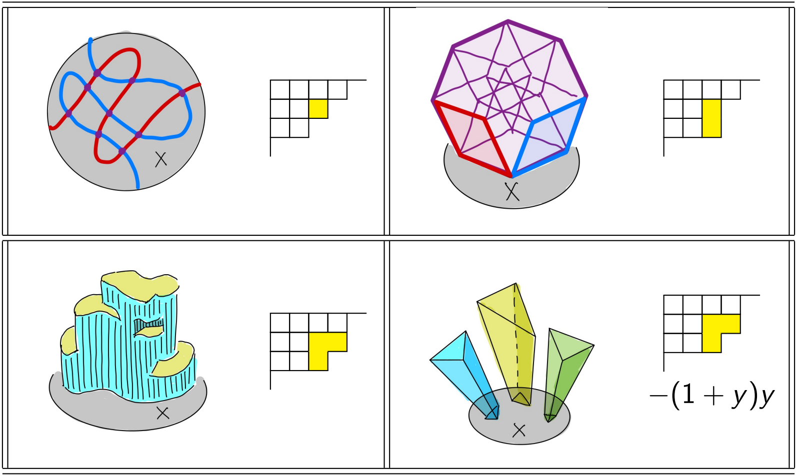
$$[i\,| \to \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu}(-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ)$$

where the sum over $\mu = \lambda +$ a ribbon strip with its tail at the i-th row.

Theorem (Fan, Guo, Su and Xiong)

$$c_r(\mathcal{V}^\vee)\cdot\mathrm{MC}_y(Y(\lambda)^\circ) = \sum_{1\le i_1<\cdots<i_r\le k} [i_r\,| \to \cdots [i_1\,| \to \mathrm{MC}_y(Y(\lambda)^\circ).$$

Summary



Affine Hecke algebra

Our approach is by introducing a version of affine Hecke algebra of three parameters

$$T_i^2 = -(p-q)T_i + pq$$

$$T_iT_j = T_jT_i, \quad |i-j| > 1,$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},$$

$$x_ix_j = x_jx_i,$$

$$T_ix_j = x_jT_i, \quad j \neq i, i+1,$$

$$T_ix_i = x_{i+1}T_i + (\hbar - (p-q)x_i),$$

$$T_ix_{i+1} = x_iT_i - (\hbar - (p-q)x_i).$$

Rôles of $p, q, \hbar$

It turns out that $p, q, \hbar$ control the following ribbon statistics

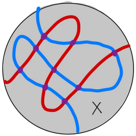
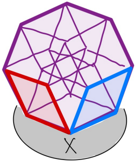
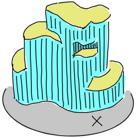
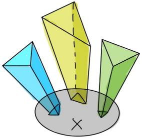
$$p : \text{height} - 1, \qquad q : \text{width} - 1, \qquad \hbar : \text{number of ribbons.}$$

We have the following table

classes	$(p, q, \hbar)$	Pieri rule
$[Y(\lambda)]$	$(0, 0, 1)$	adding boxes □
$[\mathcal{O}_{Y(\lambda)}]$	$(1, 0, 1)$	adding vertical strips 
$c_{\mathrm{SM}}(Y(\lambda)^\circ)$	$(1, 1, 1)$	adding ribbons 
$\mathrm{MC}_y(Y(\lambda)^\circ)$	$(1, -y, 1 + y)$	adding ribbons  and counting width

Dual Basis

In all four cases, we have another choice of basis

Theory				
basis	$[Y(\lambda)]$	$[\mathcal{I}_{\partial Y(\lambda)}]$	$c_{\mathrm{SM}}(Y(\lambda)^\circ)$	$\mathrm{MC}_y(Y(\lambda)^\circ)$
opposite dual basis	$[Y(\lambda)]$	$[\mathcal{O}_{Y(\lambda)}]$	$s_{\mathrm{SM}}(Y(\lambda)^\circ)$	$\mathrm{SMC}_y(Y(\lambda)^\circ)$

Theorem (Fan, Guo, Su and Xiong)

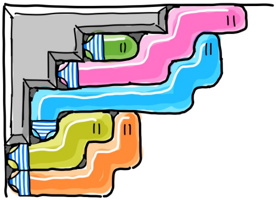
The **opposite dual basis** *has the same Pieri rule as* **basis**.

Discussion of the proof

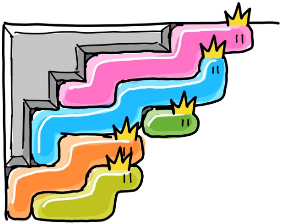
A priori, the Pieri rule for the opposite dual basis is given by $|\, i\,]$, the adjoint operator on the 180° rotated complement.

$[\, i\,|$ . . . with its **tail** at the i-th row . . .	$\longleftrightarrow$	$|\, i\,]$ . . . with its **head** at the i-th row . . .

But they are equivalent:



v.s.



Equivariant version

All the basis are defined in equivariant cohomology/K-theory.

Theorem (Fan, Guo, Su and Xiong)

The equivariant classes

$$[Y(\lambda)], \qquad [\mathcal{I}_{\partial Y(\lambda)}], \qquad c_{\mathrm{SM}}(Y(\lambda)^\circ), \qquad \mathrm{MC}_y(Y(\lambda)^\circ)$$

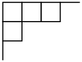
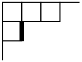
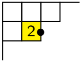
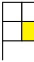
satisfy the **head-valued Pieri rule**, *i.e.* $|\, i\,]\!\bullet$ *or equivalently* $[\, i \,|\!\bullet$.

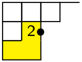
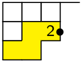
Theorem (Fan, Guo, Su and Xiong)

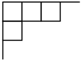
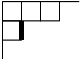
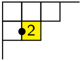
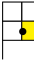
The equivariant classes

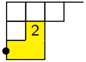
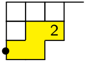
$$[Y(\lambda)], \qquad [\mathcal{O}_{Y(\lambda)}], \qquad s_{\mathrm{SM}}(Y(\lambda)^\circ), \qquad \mathrm{SMC}_y(Y(\lambda)^\circ)$$

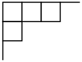
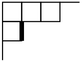
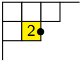
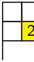
satisfy the **tail-valued Pieri rule**, *i.e.* $\bullet\!|\, i\,]$ *or equivalently* $\bullet\![\, i \,|$.

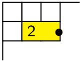
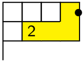
$|2]\!\bullet \to$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_4) \cdot$  $+ (\hbar - (p-q)t_5) \cdot$ 

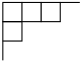
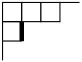
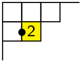
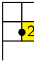
$+ (\hbar - (p-q)t_4)pq \cdot$  $+ (\hbar - (p-q)t_5)pq^2 \cdot$  ,

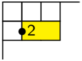
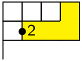
$\bullet\!|2] \to$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_3) \cdot$  $+ (\hbar - (p-q)t_3) \cdot$ 

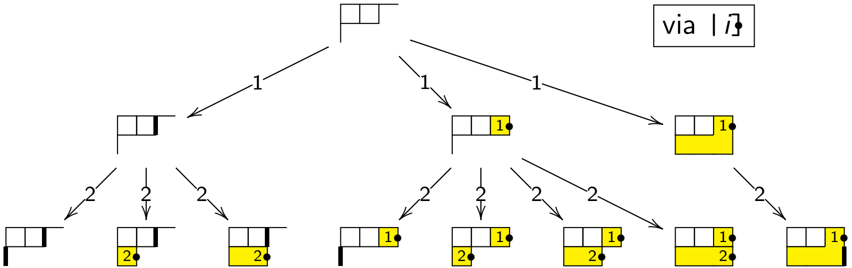
$+ (\hbar - (p-q)t_1)pq \cdot$  $+ (\hbar - (p-q)t_1)pq^2 \cdot$  ,

$[2|\!\bullet \to$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_4) \cdot$  $+ (\hbar - (p-q)t_5) \cdot$ 

$+ (\hbar - (p-q)t_5)q \cdot$  $+ (\hbar - (p-q)t_7)pq^2 \cdot$  ,

$\bullet\![2| \to$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_3) \cdot$  $+ (\hbar - (p-q)t_3) \cdot$ 

$+ (\hbar - (p-q)t_3)q \cdot$  $+ (\hbar - (p-q)t_3)pq^2 \cdot$  .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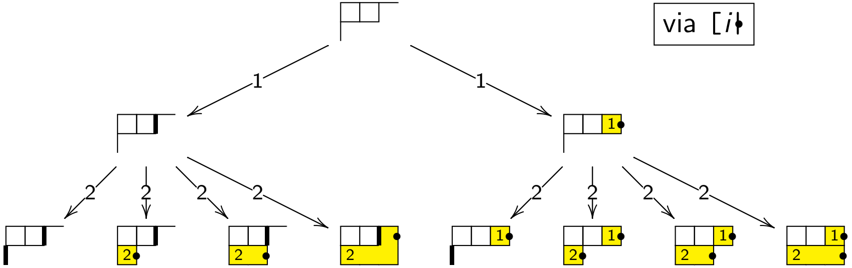

via

Application A

There is a classic relation between ideal sheaves and structure sheaves over Grassmannian.

Theorem (Buch [6], see also [7, Prop. 4.2])

$$(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}] = [\mathcal{I}_{\partial Y(\lambda)}].$$

This can be generalized to equivariant K-theory.

$$\frac{(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}]}{1 - [\mathcal{O}_{Y(\square)}]|_\lambda} = [\mathcal{I}_{\partial Y(\lambda)}] \in K_T(\mathrm{Gr}(k,n)).$$

Relation between MC and SMC

Using our Pieri rule, we can prove the following analogy for MC and SMC classes.

Theorem (Fan, Guo, Su and Xiong)

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)}) \cdot (1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^\circ) = \mathrm{MC}_y(Y(\lambda)^\circ).$$

This can be generalized to equivariant K-theory.

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)}) \cdot \frac{(1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^\circ)}{1 - [\mathcal{O}_{Y(\square)}]|_\lambda} = \mathrm{MC}_y(Y(\lambda)^\circ).$$

If we set $y = 0$, we will recover the result in the previous page.

Discussion of the proof

The proof is by one sentence:

> both sides have the same Pieri rule and
> they agree after certain specialization.

Precisely:

- the factor

$$1 - [\mathcal{O}_{Y(\square)}]|_\lambda$$

  intertwines $\bullet\!\!\![\, i \,|$ and $[\, i \,|\!\!\bullet$;
- the factor rest gives normalization by looking at localization.

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

Recall

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2), \qquad x = 1 - [\mathcal{O}(-1)].$$

We have

$$1 + \frac{y}{1+y}x = \mathrm{SMC}_y(Y(\varnothing)^\circ) \xrightarrow{(1-x)\big((1+y)-2yx\big)} \mathrm{MC}_y(Y(\varnothing)^\circ) = (1+y) - (2y+1)x$$

dual basis

$$\frac{1}{1+y}x = \mathrm{SMC}_y(Y(\square)^\circ) \xrightarrow{(1-x)\big((1+y)-2yx\big)} \mathrm{MC}_y(Y(\square)^\circ) = x$$

Application B

Recall the **stable grothendieck polynomial** is defined using set-valued tableaux:

$$\tilde{G}_\lambda = \sum_{T \in \mathrm{SVT}(\lambda)} x^T, \quad \text{e.g.}$$

1	123	35	6
234	46		
5			

filled by nonempty sets
strictly increasing in column
weakly increasing in row

Theorem (Buch [6])

$$(-1)^{|\lambda|}\tilde{G}_\lambda(-x_1, \cdots, -x_k, 0, \ldots) = [\mathcal{O}_{Y(\lambda)}] \in K(\mathrm{Gr}(k, n)).$$

Dualizing Sheaves

In Lam and Pylyavskyy [8], the omega involution of $\tilde{G}_\lambda$ was studied. It is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T \in \mathrm{WSVT}(\lambda)} x^T, \quad \text{e.g.}$$

11	334	55	6
12	4		
223			

$\left\{\begin{array}{l}\text{filled by nonempty multi-sets} \\ \text{strictly increasing in row} \\ \text{weakly increasing in column}\end{array}\right.$

Theorem (Fan, Guo, Su and Xiong)

$$\big((1 - G_\square)^n J_{\lambda'}\big)(x_1, \ldots, x_k, 0, \ldots) = [\omega_{Y(\lambda)}] \in K(\mathrm{Gr}(k, n))$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of $Y(\lambda)$.

Discussion of the proof

By [9],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\dim}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree}).$$

In the Pieri rule of motivic Chern classes, only the horizontal strip ▭ contributes the highest y-degree. Thus

$$\text{Pieri rule of } [\omega_{Y(\lambda)}] = \text{adding horizontal strips ▭}.$$

Compare:

$$\text{Pieri rule of } [\mathcal{O}_{Y(\lambda)}] = \text{adding vertical strips ▯}.$$

The omega involution switches two kind of strips.

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

From the (weak) set-tableaux model, $\tilde{G}_\square = J_\square = 1$, and

$$\tilde{G}_\square = \sum_A x^A = -1 + \prod_{i=1}^{\infty}(1+x_i), \quad \boxed{A} \quad \text{nonempty sets } A \text{ of positive integers,}$$

$$J_\square = \sum_B x^B = -1 + \prod_{i=1}^{\infty}\frac{1}{1-x_i}, \quad \boxed{B} \quad \text{nonempty multisets } B \text{ of positive integers.}$$

Thus for $\mathbb{P}^1$

$$\begin{aligned}
(-1)^0\,\tilde{G}_\varnothing(-x,0,\cdots) &= 1 = [\mathcal{O}_{Y(\varnothing)}],\\
(-1)^1\,\tilde{G}_\square(-x,0,\cdots) &= x = [\mathcal{O}_{Y(\square)}],\\
\big((1-G_\square)^2\cdot J_\varnothing\big)(x,0,\ldots) &= 1-2x = [\omega_{Y(\varnothing)}],\\
\big((1-G_\square)^2\cdot J_\square\big)(x,0,\ldots) &= x = [\omega_{Y(\square)}].
\end{aligned}$$

Bonus

A fast algorithm for computing the **Hodge diamond** of the smooth Plücker hyperplane section of Grassmannian.

Note that

$$h^{pq}(X) = \dim H^{pq}(X) = \dim H^q(X, \Omega_X^p).$$

As a result, by definition,

$$\chi(X, \lambda_y(X)) = \sum_{p,q} y^p (-1)^q h^{pq}(X) := \chi_y(X).$$

Motivic Chern class of smooth divisor

For a hypersurface $Y \subset X$, by Lefschetz theorem, only the middle dimension of Hodge diamond of Y cannot directly read from the Hodge diamond of X. To determine the middle dimension, it suffices to compute $\chi_y(Y)$. We have

$$\mathrm{MC}_y(Y) = \lambda_y(X) \cdot \frac{1 - \mathcal{O}(-Y)}{1 + y\mathcal{O}(-Y)} \in K(X)[\![y]\!].$$

Thus we can compute the

$$\chi_y(Y) = \chi\left(X, \lambda_y(X)\frac{1 - \mathcal{O}(-Y)}{1 + y\mathcal{O}(-Y)}\right).$$

Algorithm

Now let us consider a smooth Plücker hyperplane $Y \subset \mathrm{Gr}(k,n)$. Let us write

$$\lambda_y(\mathrm{Gr}(k,n)) = \sum_{\lambda \subseteq (n-k)^k} \mathrm{MC}_y(Y(\lambda)^\circ).$$

Using our Pieri rule, we can determine the expansion

$$\lambda_y(\mathrm{Gr}(k,n)) \frac{1-\det}{1+y\det} = \sum_{\lambda \subseteq (n-k)^k} ?\, \mathrm{MC}_y(Y(\lambda)^\circ).$$

Then we can compute $\chi_y(Y)$.

Example: $k = 3$

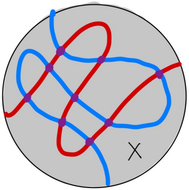
$Y \subset$	$\mathrm{Gr}(3,5)$	$\mathrm{Gr}(3,6)$	$\mathrm{Gr}(3,7)$	$\mathrm{Gr}(3,8)$	$\mathrm{Gr}(3,9)$	$\mathrm{Gr}(3,10)$
$\diamondsuit$	1 1 2 2 1 1	1 1 2 3 4 3 2 1 1	1 1 2 3 4 4 4 4 3 2 1 1	1 1 2 3 4 5 6 7 6 5 4 3 2 1 1	1 1 2 3 4 5 7 7 8 2 2 8 7 7 5 4 3 2 1 1	1 1 2 3 4 5 7 8 9 10 1 30 1 10 9 8 7 5 4 3 2 1 1

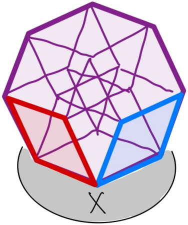
The case of $\mathrm{Gr}(3,10)$ was studied in [10, Theorem 2.2] using Griffiths' description of the vanishing cohomology.

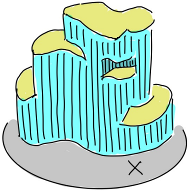
Example: $n = 12$

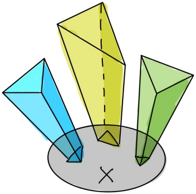
	middle dimension
$Y \subset \mathrm{Gr}(1,12)$	1
$Y \subset \mathrm{Gr}(2,12)$	0
$Y \subset \mathrm{Gr}(3,12)$	1 77 365 77 1
$Y \subset \mathrm{Gr}(4,12)$	1 351 21308 310168 1172951 1172951 31
$Y \subset \mathrm{Gr}(5,12)$	1 648 82225 3037969 37876409 169351908 278364056 16935
$Y \subset \mathrm{Gr}(6,12)$	1 780 121693 5729219 95625310 608266232 1524047370 152404737
$Y \subset \mathrm{Gr}(7,12)$	1 648 82225 3037969 37876409 169351908 278364056 16935
$Y \subset \mathrm{Gr}(8,12)$	1 351 21308 310168 1172951 1172951 31
$Y \subset \mathrm{Gr}(9,12)$	1 77 365 77 1
$Y \subset \mathrm{Gr}(10,12)$	0
$Y \subset \mathrm{Gr}(11,12)$	1

Thank You!









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