Pieri Rules over Grassmannian and Applications

Neil. J.Y. Fan[1], Peter L. Guo[2], Changjian Su[3] and Rui Xiong[4]

[1]Sichuan University [2]Nankai University Tsinghua University [4]University of Ottawa

1 Introduction

We prove a Pieri formula for motivic Chern classes of Schubert cells in the equivariant K-theory of Grassmannians, which is described in terms of ribbon operators on partitions. Our approach is to convert the Schubert calculus over Grassmannians into the calculation in a certain affine Hecke algebra. As a consequence, we derive a Pieri formula for Segre motivic classes of Schubert cells in Grassmannians. We apply the Pieri formulas to discover a relation between motivic Chern classes and Segre motivic classes, extending a well-known relation between the classes of structure sheaves and ideal sheaves. As another application, we find a symmetric power series representative for the class of the dualizing sheaf of a Schubert variety.

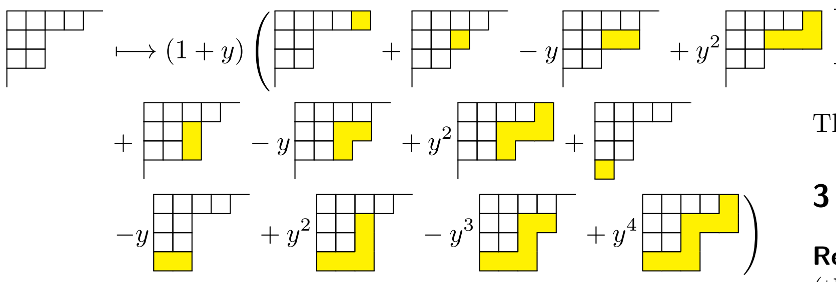
2 The Pieri rules

Our result is a Pieri rule for motivic Chern classes, a common generalization of Grothendieck polynomial and Chern–Schwartz–MacPherson classes over Grassmannians.

Chevalley formula The Chevalley formula for motivic Chern classes is given by adding a ribbon and counting width

$$c_1(\mathcal{V}^\vee)\cdot \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu=\lambda+\text{ribbon}} (-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ).$$

Example:



Pieri formula Let us denote ribbon Schubert operators

$$[i| \to \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu}(-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ)$$

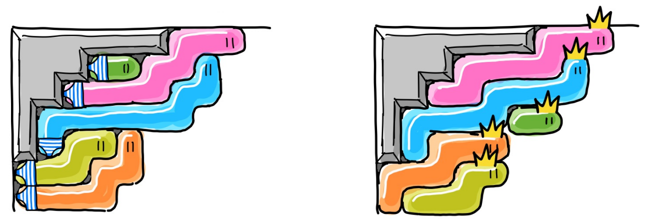
where the sum over $\mu=\lambda+$ a ribbon strip with its tail at the i-th row. Then our Pieri formula can be stated as follows.

$$c_r(\mathcal{V}^\vee)\cdot\mathrm{MC}_y(Y(\lambda)^\circ) = \sum_{1\le i_1<\cdots<i_r\le k} [i_r| \to \cdots [i_1| \to \mathrm{MC}_y(Y(\lambda)^\circ).$$

We also proved the equivalence of the following two operators

$[i|$ ... with its tail at the i-th row ... $\longleftrightarrow$ $|i]$ ... with its head at the i-th row ...

Example:



Affine Hecke Algebra Our approach is by introducing a version of affine Hecke algebra of three parameters. It turns out that $p,q,\hbar$ control the following ribbon statistics

$$p:\text{height}-1,\qquad q:\text{width}-1,\qquad \hbar:\text{number of ribbons}.$$

We have the following table

classes	$(p,q,\hbar)$	Pieri rule
$[Y(\lambda)]$	$(0,0,1)$	adding boxes $\square$
$[\mathcal{O}_{Y(\lambda)}]$	$(1,0,1)$	adding vertical strips
$c_{\mathsf{SM}}(Y(\lambda)^\circ)$	$(1,1,1)$	adding ribbons
$\mathrm{MC}_y(Y(\lambda)^\circ)$	$(1,-y,1+y)$	adding ribbons & width

This unifies many results [1–3].

3 Applications

Relations with SMC classes We proved the Segre motivic class (the opposite dual basis) has the same Pieri rule.

Since they have the same Pieri rule, we arrive a surprizing result on their relations

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)})\cdot(1-[\mathcal{O}_{Y(\square)}])\cdot\mathsf{SMC}_y(Y(\lambda)^\circ) = \mathrm{MC}_y(Y(\lambda)^\circ).$$

This generalizes the famous relation between ideal sheaves and structure sheaves $(1-[\mathcal{O}_{Y(\square)}])\cdot[\mathcal{O}_{Y(\lambda)}] = [\mathcal{I}_{\partial Y(\lambda)}]$ by Buch [4].

Representatives for dualizing sheaves By [5],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\dim}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree})$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of the Schubert variety. In the Pieri rule of motivic Chern classes, only the horizontal strip $\square\!\square$ contributes the highest y-degree.

Using this fact and Pieri rule, we proved

$$\big((1-G_\square)^n J_{\lambda'}\big)(x_1,\ldots,x_k,0,\ldots) = [\omega_{Y(\lambda)}]\in K(\mathrm{Gr}(k,n))$$

where J_λ be its omega involution of the stable grothendieck polynomial (without sign). By Lam and Pylyavskyy [6], J_λ is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T\in\mathrm{WSVT}(\lambda)} x^T,$$

e.g.

11	334	55	6
12	4		
223			

filled by nonempty multi-sets
strictly increasing in row
weakly increasing in column

Hodge diamond of a smooth Plücker surface Using our Pieri rule, we get a fast algorithm of computing the Hodge diamond of a smooth Plücker surface in Grassmannian. For example

	middle dimension
$Y\subset\mathrm{Gr}(1,12)$	1
$Y\subset\mathrm{Gr}(2,12)$	0
$Y\subset\mathrm{Gr}(3,12)$	1 77 365 77 1
$Y\subset\mathrm{Gr}(4,12)$	1 351 21308 310168 1172951 1172951 310168 21
$Y\subset\mathrm{Gr}(5,12)$	1 648 82225 3037969 37876409 169351908 278364056 169351908 3
$Y\subset\mathrm{Gr}(6,12)$	1 780 121693 5729219 95625310 608266232 1524047370 1524047370 60826
$Y\subset\mathrm{Gr}(7,12)$	1 648 82225 3037969 37876409 169351908 278364056 169351908 3
$Y\subset\mathrm{Gr}(8,12)$	1 351 21308 310168 1172951 1172951 310168 21
$Y\subset\mathrm{Gr}(9,12)$	1 77 365 77 1
$Y\subset\mathrm{Gr}(10,12)$	0
$Y\subset\mathrm{Gr}(11,12)$	1

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