Pieri Rules over Grassmannians — two more applications

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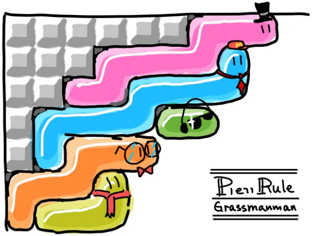


Ribbon Schubert operators

Fix $0 \leq k \leq n$. Let us consider the space

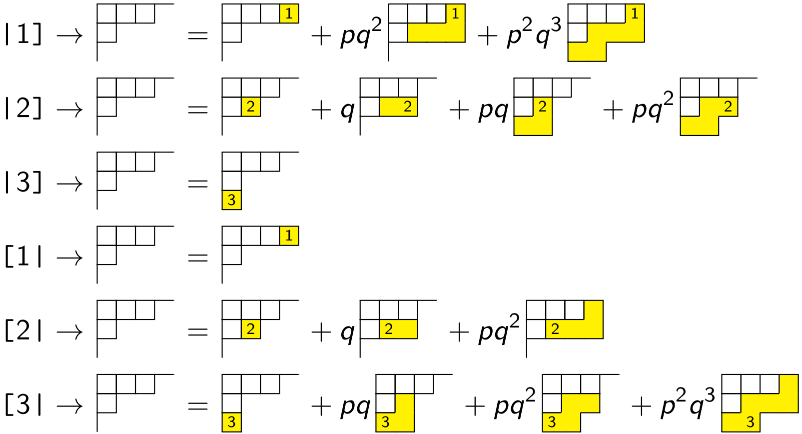
$$\bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q}[p, q] \cdot \lambda.$$

Let us define the **ribbon Schubert operator** to be the linear operator for $1 \leq i \leq k$

$$| i] \to \lambda = \sum p^{\mathrm{ht}(\mu/\lambda)-1} q^{\mathrm{wd}(\mu/\lambda)-1} \mu$$
$$[i | \to \lambda = \sum p^{\mathrm{ht}(\mu/\lambda)-1} q^{\mathrm{wd}(\mu/\lambda)-1} \mu$$

where the sum is taken over all $\mu \subseteq (n-k)^k$ such that μ/λ is a ribbon with **head**/**tail** in row i.

Example



Pieri rules

These operators naturally arise from the Pieri rule of **motivic Chern classes** over Grassmannian. Precisely, we have

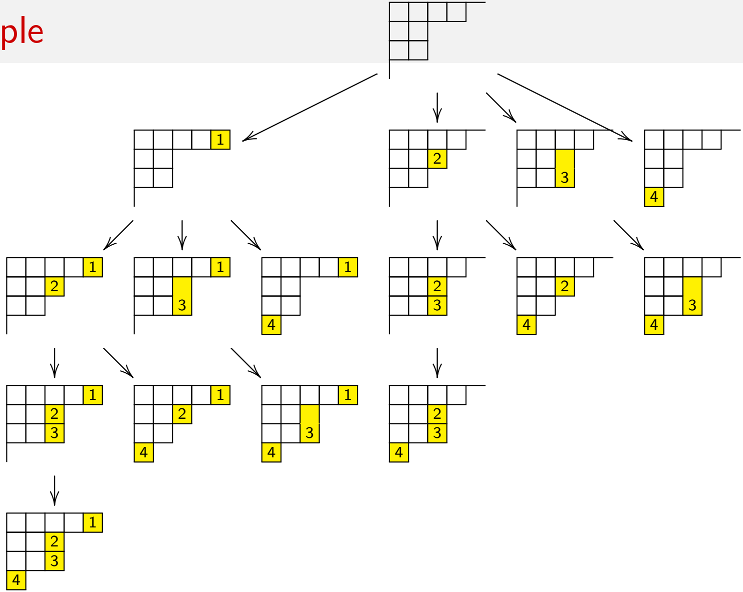
Theorem (Fan, Guo, Su, Xiong)

Set $(p,q)=(1,-y)$. *Over* $K(\mathrm{Gr}(k,n))[y]$,

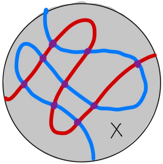
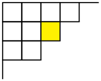
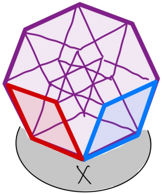
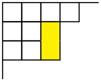
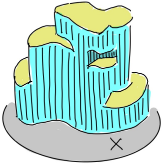
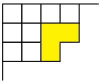
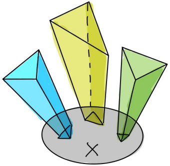
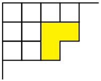
$$
\begin{aligned}
& c_r(\mathcal{V}^\vee)\cdot \mathrm{MC}_y(\lambda) \\
={}& (1+y)^r \sum_{1\le i_1<\cdots<i_r\le k} |\, i_r] \to \cdots |\, i_1] \to \mathrm{MC}_y(\lambda) \\
={}& (1+y)^r \sum_{1\le i_1<\cdots<i_r\le k} [\, i_r\,| \to \cdots [\, i_1\,| \to \mathrm{MC}_y(\lambda).
\end{aligned}
$$

This is derived from the equivariant version.

Example



Summary

  cohomology	  K-theory
  CSM/SSM classes	  $-(1+y)y^{\text{width}}$ MC/SMC classes

$\mathscr{A}$pplication

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simple relation between MC and SMC

SMC classes

Motivic Chern classes admit a family of dual basis called **Segre motivic classes**. They specialize

$$\mathrm{MC}_y(\lambda)\big|_{y=0} = \text{ideal sheaf } \mathcal{I}_{Y(\lambda)} \in K(\mathrm{Gr}(k,n)),$$
$$\mathrm{SMC}_y(\lambda)\big|_{y=0} = \text{structure sheaf } \mathcal{O}_{Y(\lambda)} \in K(\mathrm{Gr}(k,n)).$$

Note that $\mathcal{O}_{Y(\lambda)}$ is represented by the **symmetric Grothendieck polynomial**. We proved that

Theorem (Fan, Guo, Su, Xiong)

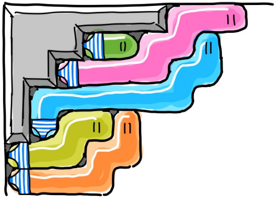
The Pieri rule of $\mathrm{SMC}_y(\lambda)$ *is the same as the rule of* $\mathrm{MC}_y(\lambda)$.

Discussion of the proof

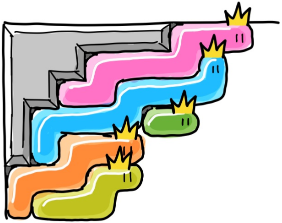
A priori, the Pieri rule for the opposite dual basis is given by $|\,i\,]$, the adjoint operator on the 180° rotated complement.

$[\,i\,|$ ...with its **tail** at the i-th row ... $\longleftrightarrow$ $|\,i\,]$ ...with its **head** at the i-th row ...

But they are equivalent:



v.s.



Relation between MC and SMC

The similarity between the Pieri formulas indicates that there should be some relation between motivic Chern classes and Segre motivic classes.

Theorem (Fan, Guo, Su and Xiong)

$$\lambda_y(\mathscr{T}^{\vee}_{\mathrm{Gr}(k,n)}) \cdot (1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^{\circ}) = \mathrm{MC}_y(Y(\lambda)^{\circ}).$$

If we set $y = 0$, we will recover the result of Buch [1]

$$(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}] = [\mathcal{I}_{\partial Y(\lambda)}].$$

$\mathscr{A}$pplication

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polynomial representatives of dualizing sheaves

Grothendieck polynomial

Recall the **symmetric Grothendieck polynomials** are defined using set-valued tableaux:

$$\tilde{G}_\lambda = \sum_{T \in \mathrm{SVT}(\lambda)} x^T, \quad \text{e.g.} \quad \begin{array}{|c|c|c|c|} \hline 1 & 123 & 35 & 6 \\ \hline 234 & 46 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \quad \left\{ \begin{array}{l} \text{filled by nonempty sets} \\ \text{strictly increasing in column} \\ \text{weakly increasing in row} \end{array} \right.$$

Theorem (Buch [1])

$$(-1)^{|\lambda|} \tilde{G}_\lambda(-x_1, \cdots, -x_k, 0, \dots) = [\mathcal{O}_{Y(\lambda)}] \in K(\mathrm{Gr}(k, n)).$$

Dualizing Sheaves

In Lam and Pylyavskyy [2], the omega involution of $\tilde{G}_\lambda$ was studied. It is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T \in \mathrm{WSVT}(\lambda)} x^T, \quad \text{e.g.}$$

11	334	55	6
12	4		
223			

$\left\{\text{filled by nonempty multi-sets; strictly increasing in row; weakly increasing in column}\right.$

Theorem (Fan, Guo, Su and Xiong)

$$\left((1 - G_\square)^n J_{\lambda'}\right)(x_1, \ldots, x_k, 0, \ldots) = [\omega_{Y(\lambda)}] \in K(\mathrm{Gr}(k,n))$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of $Y(\lambda)$.

Discussion of the proof

By [3],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\dim}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree}).$$

In the Pieri rule of motivic Chern classes, only the horizontal strip contributes the highest y-degree. Thus

$$\text{Pieri rule of } [\omega_{Y(\lambda)}] = \text{adding horizontal strips}.$$

Compare:

$$\text{Pieri rule of } [\mathcal{O}_{Y(\lambda)}] = \text{adding vertical strips}.$$

The omega involution switches two kind of strips.

$\mathscr{A}$*pplication*

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Hodge diamond of smooth Plücker hyperplane

Hodge diamond

We get a fast algorithm for computing the **Hodge diamond** of the smooth Plücker hyperplane section of Grassmannian.

Note that

$$h^{pq}(X) = \dim H^{pq}(X) = \dim H^q(X, \Omega_X^p).$$

As a result, by definition,

$$\chi(X, \lambda_y(X)) = \sum_{p,q} y^p(-1)^q h^{pq}(X) := \chi_y(X).$$

Algorithm

Now let us consider a smooth Plücker hyperplane $Y \subset \mathrm{Gr}(k,n)$. Let us write

$$\lambda_y(\mathrm{Gr}(k,n)) = \sum_{\lambda \subseteq (n-k)^k} \mathrm{MC}_y(Y(\lambda)^\circ).$$

Using our Pieri rule, we can determine the expansion

$$\lambda_y(\mathrm{Gr}(k,n)) \frac{1-\det}{1+y\det} = \sum_{\lambda \subseteq (n-k)^k} ?\,\mathrm{MC}_y(Y(\lambda)^\circ).$$

Then we can compute $\chi_y(Y)$.

Example: $k = 3$

$Y \subset$	$\mathrm{Gr}(3,5)$	$\mathrm{Gr}(3,6)$	$\mathrm{Gr}(3,7)$	$\mathrm{Gr}(3,8)$	$\mathrm{Gr}(3,9)$	$\mathrm{Gr}(3,10)$
$\diamondsuit$	1 1 2 2 1 1	1 1 2 3 4 3 2 1 1	1 1 2 3 4 4 4 4 3 2 1 1	1 1 2 3 4 5 6 7 6 5 4 3 2 1 1	1 1 2 3 4 5 7 7 8 2 2 8 7 7 5 4 3 2 1 1	1 1 2 3 4 5 7 8 9 10 1 30 1 10 9 8 7 5 4 3 2 1 1

The case of $\mathrm{Gr}(3,10)$ was studied in [4, Theorem 2.2] using Griffiths' description of the vanishing cohomology.

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a new family of symmetric functions

Ribbon operators

Consider the operator with parameter x

$$v(x) = \cdots (1 + x \mid 2] \!\rightarrow)(1 + x \mid 1] \!\rightarrow)$$
$$= \sum_{r=0}^{\infty} x^r \sum_{1 \le i_1 < \cdots < i_r} \mid i_r] \rightarrow \cdots \mid i_1] \rightarrow .$$

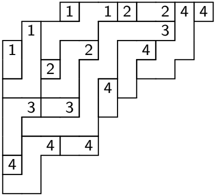
By our Pieri rule, we have $v(x)v(y) = v(y)v(x)$. It would be more convenient to work with its omega involution $u(x) = v(-x)^{-1}$. For a skew shape λ/μ, we can define a symmetric function

$$c_{\lambda/\mu} = \text{coefficient of } \lambda \text{ in } \cdots u(x_2)u(x_1)\mu$$

We call it by **Chern polynomial**.

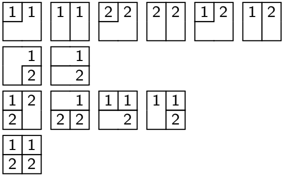
Combinatorial formula

The definition implies $c_{\lambda/\mu}$ admits a monomial expansion of **semi-standard ribbon tableaux**



For example,

$$\begin{aligned} c_{\boxplus} &= (pq+p^2)(x_1^2+x_1x_2+x_2^2) \\ &+ (qp+q^2)x_1x_2 \\ &+ (p+q)(x_1^2x_2+x_1x_2^2) \\ &+ x_1^2x_2^2. \end{aligned}$$



Properties

- We have $c_{\lambda/\mu}|_{p=q=0} = s_{\lambda/\mu}$ the skew Schur function;
- We have $c_{\lambda/\mu}|_{p=1,q=0} = g_{\lambda/\mu}$ the skew dual Grothendieck polynomial.
- The polynomial

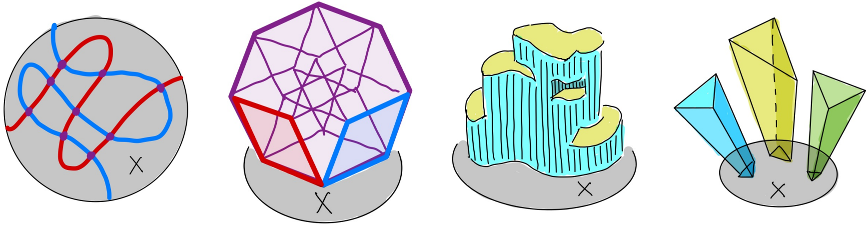
$$(x_1 \cdots x_k)^{n-k} c_{\lambda/\mu}(x_1^{-1}, \ldots, x_k^{-1})|_{p=q=1}$$

represents the CSM class of an open Richardson variety over $\mathrm{Gr}(k, n)$.

Conjecture

Chern polynomial $c_{\lambda/\mu}$ is Schur positive.

Thank You!



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