An ADE type classification of Hodge–Tate hyperplanes in Grassmannians [arXiv:2509.01101]

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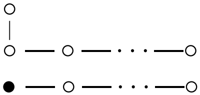
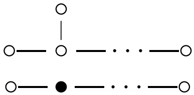
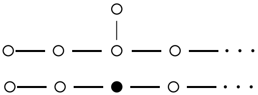
Rui Xiong



$\mathbb{A}_n$: Gr$(1, n)$



$\mathbb{D}_n$: Gr$(2, n)$

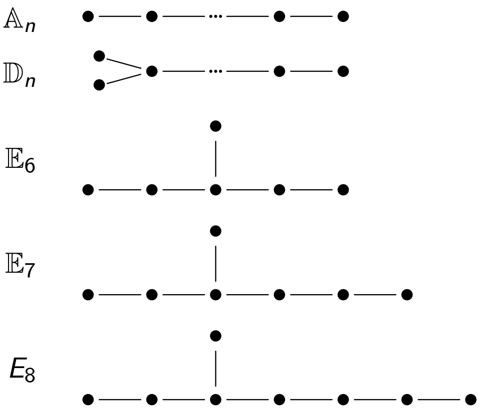


$\mathbb{E}_n$: Gr$(3, n)$

ADE CLASSIFICATIONS

Simply-laced Dynkin diagrams

The following is the list of simply-laced Dynkin diagrams.



ADE classifications

The ADE classification is a phenomenon in mathematics where certain kinds of objects are classified by simply-laced Dynkin diagrams.

- simple Lie algebras;
- quivers of finite types;
- finite subgroups in $SU(2)$.



[1]images from irasutoya

Simple Lie algebras

The picture illustrates Grothendieck's vision of a pinned reductive group: the body is a maximal torus T, the wings are the opposite Borel subgroups B, and the pins rigidify the situation.

The corresponding Lie algebra is

$\mathbb{A}$	$\mathbb{D}$	$\mathbb{E}$
$\mathfrak{sl}_2, \mathfrak{sl}_3, \ldots$	$\mathfrak{so}_3, \mathfrak{so}_5, \ldots$	$\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$

The Dynkin diagram records all the information.

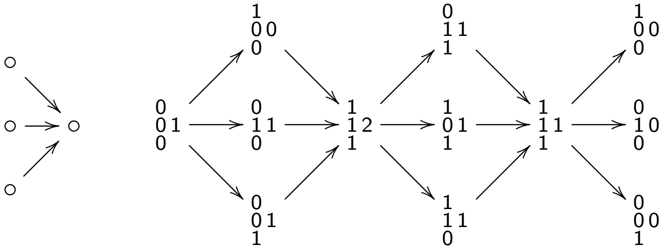
[2]Blue Morpho butterfly. Credit: LPETTET/Digital Vision Vectors/Getty Images. From Milne: Books — Algebraic Groups.

Quivers of finite type

A quiver is of finite type if it has only finitely many isomorphism classes of indecomposable representations.

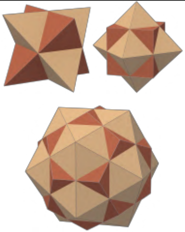
Gabriel theorem gives an ADE classification of quivers of finite type.

For example,



Finite Subgroups in $SU(2)$

There is an ADE classification of finite subgroups in $SU(2)$.

$\mathbb{A}$	$\mathbb{D}$	$\mathbb{E}$
		
cyclic groups	dihedral groups	A_4, S_4, A_5

We represent the image under the two-to-one map $SU(2) \to SO(3)$, as the group of symmetries.

[3] from wiki:cyclic_groups, wiki:Hosohedron, wiki:Platonic_solid

Hodge diamonds of Hyperplanes

Hyperplanes in Grassmannians

Consider the **Grassmannian variety**

$$X = \mathrm{Gr}(k,n) = \{\text{subspaces } V \leq \mathbb{C}^n : \dim V = k\}.$$

We have the **Plücker embedding**

$$\mathrm{Gr}(k,n) \xrightarrow{\subset} \mathbb{P}(\Lambda^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}.$$

Consider a smooth hyperplane

$$Y = X \cap \text{a generic hyperplane}.$$

Question

When the cohomology $H^*(Y)$ is of Hodge–Tate type?

Examples (I)

When $k = 1$, $\mathsf{Gr}(1, n) = \mathbb{P}^{n-1}$ is a projective space. So $Y \subset \mathsf{Gr}(1, n)$ is a projective space of smaller dimension.

Note that $\mathsf{Gr}(2, 4) \subset \mathbb{P}^5$ is a quadrics. So $Y \subset \mathsf{Gr}(2, 4)$ is a quadrics of smaller dimension.

Generally, $Y \subset \mathsf{Gr}(2, 2n)$ is isomorphic to $\mathsf{SG}(2, 2n)$ a homogeneous variety of $\mathsf{Sp}(2\mathrm{n})$.

As noticed by Semenov [Sem08], there is a torus action on $Y \subset \mathsf{Gr}(3, 6)$ with discrete fixed points.

Semenov, N. Motivic decomposition of a compactification of a Merkurjev-Suslin variety. J. Reine Angew. Math. 617 (2008), 153–167.

Examples (II)

The Hodge diamond of $Y \subset \mathrm{Gr}(3, 10)$ was computed by Debarre and Voisin [DV10].

$$\begin{array}{c} \vdots \\ 20 \\ \cdots\ 0\ 1\ 30\ 1\ 0\ \cdots \\ 20 \\ \vdots \end{array}$$

More general, when $n > 3k$ and $k > 2$, Bernardara, Fatighenti, Manivel [BFM21] showed that $Y \subset \mathrm{Gr}(k, n)$ is not of Hodge–Tate type.

O. Debarre and C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. 649 (2010), 63–87.

Nested varieties of K3 type. M. Bernardara, E. Fatighenti, L. Manivel. Journal de l'École polytechnique - Mathématiques, Tome 8 (2021), pp. 733-778.

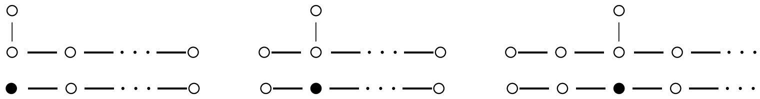
Classification

Theorem (GLLX)

The cohomology $H^*(Y)$ *is of Hodge–Tate type if and only if*

- $Y \subset \mathrm{Gr}(1, n) \simeq \mathrm{Gr}(n-1, n)$ *where* $n \geq 1$;
- $Y \subset \mathrm{Gr}(2, n) \simeq \mathrm{Gr}(n-2, n)$ *where* $n \geq 4$;
- $Y \subset \mathrm{Gr}(3, n) \simeq \mathrm{Gr}(n-3, n)$ *where* $n = 6, 7, 8$.

The result can be viewed as an ADE classification



$\mathbb{A}_n : \mathrm{Gr}(1, n)$ $\mathbb{D}_n : \mathrm{Gr}(2, n)$ $\mathbb{E}_n : \mathrm{Gr}(3, n)$

It reflects all exceptional isomorphisms.

Example ($k = 3$)

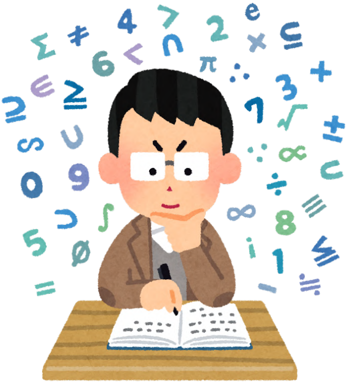
$Y \subset$	$\mathrm{Gr}(3,5)$	$\mathrm{Gr}(3,6)$	$\mathrm{Gr}(3,7)$	$\mathrm{Gr}(3,8)$	$\mathrm{Gr}(3,9)$	$\mathrm{Gr}(3,10)$
$\diamondsuit$	1 1 2 2 1 1	1 1 2 3 4 3 2 1 1	1 1 2 3 4 4 4 4 3 2 1 1	1 1 2 3 4 5 6 7 6 5 4 3 2 1 1	1 1 2 3 4 5 7 7 8 2 2 8 7 7 5 4 3 2 1 1	1 1 2 3 4 5 7 8 9 10 1 30 1 10 9 8 7 5 4 3 2 1 1

Try it online: cubicbear.github.io/PluckerHodge.html.

Example ($k = 4$)

$Y \subset$	$\mathrm{Gr}(4,6)$	$\mathrm{Gr}(4,7)$	$\mathrm{Gr}(4,8)$	$\mathrm{Gr}(4,9)$	$\mathrm{Gr}(4,10)$
$\diamondsuit$	1 1 2 2 2 2 2 1 1	1 1 2 3 4 4 4 4 3 2 1 1	1 1 2 3 5 5 7 7 7 7 5 5 3 2 1 1	1 1 2 3 5 6 8 9 11 11 11 11 9 8 6 5 3 2 1 1	1 1 2 3 5 6 9 10 13 14 16 16 16 16 14 13 10 9 6 5 3 2 1 1
			3 7 3	1 45 11 45 1	1 111 1220 16 1220 111 1

Explanations



Finiteness of orbits

Let us consider the following well-known question.

Question

When $\Lambda^k\mathbb{C}^n$ has finite GL_n-orbits?

The answer gives the same ADE classification.

Similar as the Gabriel theorem for quivers of finite type, we should first do the dimension counting. Say, if it has finite many orbits, then it is necessary to have

$$\dim \Lambda^k\mathbb{C}^n = \binom{n}{k} \leq n^2 = \dim \mathsf{GL}_n.$$

It is not hard to solve all possible (n, k).

Example

When $k = 1$, there are only two orbits

$$\mathbb{C}^n = \{0\} \cup \{\text{nonzero vectors}\}.$$

When $k = 2$, there are $\lfloor n/2 \rfloor$ orbits

$$\Lambda^2\mathbb{C}^n = \{0\} \cup \left\{ \begin{array}{c} \text{anti-symmetric} \\ \text{forms of rank 2} \end{array} \right\} \cup \left\{ \begin{array}{c} \text{anti-symmetric} \\ \text{forms of rank 4} \end{array} \right\} \cup \cdots.$$

When $k > 2$,

(k, n)	$(3,6)$	$(3,7)$	$(3,8)$	$(3,9)$	$(4,8)$
$\dim \Lambda^k\mathbb{C}^n = \binom{n}{k}$	20	35	56	84	70
$\dim \mathrm{GL}_n = n^2$	36	49	64	81	64

Classification of trivectors

Can we classify the GL_n-orbits of $\Lambda^3\mathbb{C}^n$?

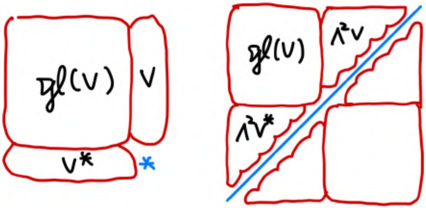
> *In contrast to the classification of linear operators and bivectors, a classification of trivectors depends essentially on the dimension n of the base space. It is trivial for $n \leq 5$ and was obtained for $n = 6$ by Reichel in 1907, for $n = 7$ by Schouten in 1931, and for $n = 8$ by Gurevich [8], [9] in 1935. In this paper we present a classification of trivectors for $n = 9$. For larger n such a classification, if at all possible, is significantly more complicated.*

E. B. Vinberg and A. G. Elasvili, Classification of trivectors of a nine-dimensional space, Trudy Sem. Vekt. Tenz. Analizu, M.G.U. No. XVIII (1978), 197-233. [link]

Graded Lie algebras

The classification of trivectors is closely related to graded Lie algebras. They are very special nilpotent elements in $\mathfrak{g}$. We can decompose

$$\begin{aligned}
\mathfrak{sl}_{n+1} &\cong \overset{-1}{V^*} \oplus \overset{0}{\mathfrak{gl}_n} \oplus \overset{1}{V} \\
\mathfrak{so}_{2n} &\cong \overset{-1}{\Lambda^2 V^*} \oplus \overset{0}{\mathfrak{gl}_n} \oplus \overset{1}{\Lambda^2 V} \\
\mathfrak{e}_n &\cong \cdots \oplus \overset{-1}{\Lambda^3 V^*} \oplus \overset{0}{\mathfrak{gl}_n} \oplus \overset{1}{\Lambda^3 V} \oplus \cdots
\end{aligned}$$



Hessenberg varieties

After identifying $\mathrm{Gr}(k,n) = \mathrm{GL}_n/P$, we can reformulate Y as the generic fiber of

$$\begin{array}{ccc} G \times_P N & = & \{(gP, \tau) \in \mathrm{Gr}(k,n) \times \Lambda^k\mathbb{C}^n : \tau \in gN\} \\ \downarrow & & \\ \Lambda^k\mathbb{C}^n & & \text{where } N = \Lambda^k\mathbb{C}^n \ominus \text{lowest weight space} \end{array}$$

This is also known as a (generalized) **Hessenberg varieties**, introduced to study affine Springer fibers, see [Yun, §1.6.4].

Question

Does this provide a non-pure affine Springer fiber in type A?

Z. Yun. Lectures on Springer theories and orbital integrals.

Affine Dynkin diagrams

Representation theorists would ask the appearance of affine Dynkin diagrams. It turns out its analogy is the hyperplane of (co)adjoint varieties, studied by Benedetti and Perrin [BP22].

For example, for an adjoint variety, the Plücker emebdding

$$G/P \xrightarrow{\subset} \mathbb{P}(\mathfrak{g}).$$

A generic hyperplane corresponds to a regular semisimple element in $\mathfrak{g}^* \cong \mathfrak{g}$. In particular, $Y \subset G/P$ admits a T-action, and $Y^T = (G/P)^T$, so $H^*(Y)$ is always of Hodge–Tate type.

Benedetti, V.; Perrin, N. Cohomology of hyperplane sections of (co)adjoint varieties. *arXiv:2207.02089*.

Cluster algebras

The same classification appears in cluster algebras. Precisely, there is an ADE classification of

cluster algebra of $\mathbb{C}[\mathsf{Gr}(k,n)]$ being of finite type.

See [Sco06].

	$\mathsf{Gr}(1,n)$	$\mathsf{Gr}(2,n)$	$\mathsf{Gr}(3,6)$	$\mathsf{Gr}(3,7)$	$\mathsf{Gr}(3,8)$
Quiver	$\varnothing$	$\mathbb{A}_{n-3}$	$\mathbb{D}_4$	$\mathbb{E}_6$	$\mathbb{E}_8$

J. Scott, Grassmannians and cluster algebras, Proc. London Math. Soc., 92 (2006), 345–380.

Prove via Tableaux Combinatorics

How to compute the Hodge diamonds (I)

By hard Lefschetz theorem, if we know the Hodge diamond of X and the y-characteristic

$$\chi_y(Y) = \sum_{p \geq 0} y^p \chi(Y, \Omega_Y^p) \in \mathbb{Z}[y],$$

then we are able to compute the Hodge diamond of Y. There are several ways of doing it.

- Using Riemann-Roch theorem (very slow);
- Using Localization theorem (slow);
- Using the Pieri formulas [FGSX24] (fast).

N. Fan, P. Guo, C. Su and R. Xiong, *A Pieri type formula for motivic Chern classes of Schubert cells in Grassmannians*, arXiv: math.CO/2402.04500, 2024.

How to compute Hodge diamonds (II)

Griffiths' theory [Voi, §6.1.2] provides a way of computing the Hodge filtration of a hyperplane section assuming certain cohomology vanishing.

The vanishing could be evaluated by Bott's parabolic version of Borel–Weil theorem [Bo57]. Luckily, over Grassmanian, there is a combinatorial formula exists.

C. Voisin, *Hodge theory and complex algebraic geometry. II*, volume 77 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2003.

R. Bott, *Homogeneous vector bundles*, Ann Math. 66 (1957), 203–248.

Tableaux combinatorics

Theorem (Snow)

For $\ell \geq 0$*, we have*

$$H^j(Gr(k,n), \Omega^p(\ell)) \neq 0$$

if and only if

there exists $\lambda \in \mathcal{P}_{k,n}$ *of* p *cells with no cell of hook length* ℓ *such that* j *equals the number of cells in* λ *of hook length larger than* ℓ*.*

6	5	3	2	1
2	1			

$$H^2(X, \Omega^7(4)) \neq 0$$

$7 =$ number of cells
$4 =$ not a hook length
$2 =$ number of hook length > 4

D. Snow, *Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces*, Math. Ann. 276.1 (1986): 159-176.

Example

The key step is the compute the $(\dim X - 1)$-th hypercohomology of

$$\Omega_X^{\dim X - n + 1}(1) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{\dim X - 1}(n-1) \xrightarrow{d} K_X(n) \to 0.$$

Consider $\mathrm{Gr}(3,9)$. The spectral sequences is

	10	11	12	13	14	15	16	17	18
8	0		0						0
7		0	0						0
6			1						0
5			0	0					0
4			0		0				0
3			0			0			0
2			0				0		0
1			0					0	0
0			0						1

8	7	5	4	2	1
5	4	2	1		
2	1				

$$H^6(\mathrm{Gr}(3,9), \Omega^{12}(3)) \neq 0$$

$$\boxed{h^{9,8} = 2}$$

Example

Consider Gr$(4,8)$. The spectral sequences is

	9	10	11	12	13	14	15	16
7	0	0		0				0
6		1		0				0
5		0	0	0				0
4		0		1				0
3		0		0	0			0
2		0		0		0		0
1		0		0			0	0
0		0		0				1

$$h^{8,7} = 3$$

7	5	3	1
5	3	1	
3	1		
1			

$$H^6(\mathrm{Gr}(4,8), \Omega^{10}(2)) \neq 0$$

7	6	3	2
6	5	2	1
3	2		
2	1		

$$H^4(\mathrm{Gr}(4,8), \Omega^{12}(4)) \neq 0$$

Precise statement

Actually, in the application of Griffiths' argument, there is no other exceptional non-vanishing for $k \geq 3$ and $n \geq 2k$.

Theorem (GLLX)

Assume $Y \subset \mathrm{Gr}(k,n)$ is not in the Hodge–Tate classification. When $p < n$, the Hodge diamond $h^{\dim X - p, p-1}(Y) = 0$ and

$$h^{\dim X - n, n-1}(Y) = \begin{cases} 2, & Y \subset \mathrm{Gr}(3,9) \simeq \mathrm{Gr}(6,9), \\ 3, & Y \subset \mathrm{Gr}(4,8), \\ 1, & \textit{otherwise}. \end{cases}$$

The condition for the case "otherwise" is called "N-Calabi–Yau".

THANKS